NTNU | Norwegian University of Science and Technology

RELATIVISTIC CORRECTIONS TO THE BINARY INSPIRAL

Using Feynman diagrams in classical gravity

A master's thesis presentation by Vegard Undheim

Supervised by Assoc. Prof. Alex B. Nielsen (UiS)
& Prof. Jens O. Andersen (NTNU)

Outline

The binary problem - The problem of relativistic binaries

In Newtonian theory

- it is modelled by the Newtonian gravitational potential $V_{\text{Newt}} = -\frac{Gm_1m_2}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}$.
- it is analytically solvable, with Keplerian orbits as solutions (ellipses).

In general relativity

- it has no known analytical solution.
- it produces *gravitational waves*, dissipating energy and angular momentum out of the binary.
  - This in turn makes the binary fall inn, and eventually merge.
  - The evolution can be separated into three phases:

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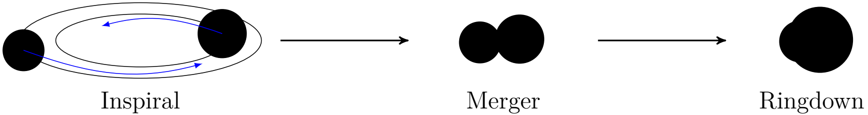

Separation of scale

In order to approximate its evolution, a non-relativistic binary can be separated into 2 time scales.

1. A short time scale of only a few periods of the orbit.
   - GW radiation can be neglected.
   - The motion can be approximated by conservative orbits.
   - At leading order $E_{\text{Newt}} = -\frac{1}{2}\mu v^2$ and $\mathcal{F}_{\text{quad}} = \frac{32}{5}\frac{\eta^2}{Gc^5}v^{10}$ for circular, conservative motion.
2. A long time scale of many periods.
   - GW radiation reduces the orbital energy, making the binary spiral in.

Post-Newtonian expansion

In the short time scale, slow moving / far separated binaries are well described by Newtonian gravity. Adding relativistic corrections increases the accuracies for faster moving orbits. Expanding quantities in factors of (v/c) is called *post-Newtonian expansion.*

Computing the PN expansion of the orbital energy E, and the energy flux $\mathcal{F}$, in the short time scale provides information that approximate the evolution in the long time scale.

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How does gravitational waves affect the binary dynamics?

Using energy conservation

$$-\mathcal{F} = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}\omega}\frac{\mathrm{d}\omega}{\mathrm{d}t} \qquad (1)$$

$$\Rightarrow \quad \mathrm{d}t = -\mathcal{F}^{-1}\frac{\mathrm{d}E}{\mathrm{d}\omega}\mathrm{d}\omega\,.$$

Thus, knowing $E(\omega)$ and $\mathcal{F}(\omega)$ provides the diff.eq. dictating the time evolution of $\omega(\tau) = \omega(t_c - t)$.

Using Kepler's third law ω, r, and $v = \omega r$ can be related to each other, assuming circular orbits.

$$\omega^2 = \frac{GM}{r^3} \Leftrightarrow v^2 = \frac{GM}{r}. \qquad (2)$$

$$\omega(\tau) = \frac{5}{8}\left(\frac{5GM}{c^3}\right)^{-5/8}\tau^{-3/8}\left\{1 + \left(\frac{743}{2688} + \frac{11}{32}\eta\right)\left(\frac{5GM}{c^3\eta}\right)^{1/4}\tau^{-1/4}\right\}. \qquad (3)$$

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Using Feynman diagrams to compute orbital energy - Gravity as a field theory

In the 60's R. Feynman investigated gravity, using field theory techniques. He determined the graviton Lagrangian to be

$$\mathcal{L}_{(2)} + \mathcal{L}_{\text{int}} = -\frac{1}{2} h_{\mu\nu,\rho} h^{\mu\nu,\rho} + \frac{1}{4} h_{,\mu} h^{,\mu} + \frac{\lambda}{2} h_{\mu\nu} T^{\mu\nu}, \tag{4}$$

with equation of motion

$$\Box h_{\mu\nu} = -\frac{\lambda}{2} \frac{1}{2} \left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta} \right) T^{\alpha\beta} \equiv -\frac{\lambda}{2} P_{\mu\nu:\alpha\beta} T^{\alpha\beta} \tag{5}$$

$$\Rightarrow \quad h_{\mu\nu}(x) = -\frac{\lambda}{2} P_{\mu\nu:\alpha\beta} \int \Delta_{\text{ret}}(x-y) T^{\alpha\beta}(y) \, \mathrm{d}^4 y \,. \tag{6}$$

Here $\Delta_{\text{ret}}(r) = \frac{-\delta(ct - |\boldsymbol{r}|)}{4\pi |\boldsymbol{r}|}$ is the retarded Green's function of the d'Alembertian operator $\Box = \partial_\sigma \partial^\sigma$.

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Using Feynman diagrams to compute orbital energy - The action of classical point particles

This makes the action of two point particles interacting via the graviton field

$$S_{pp} = \int \frac{\mathrm{d}^4 x}{c} \sum_{a=1}^{2} \left[\frac{1}{2} m_a \dot{\boldsymbol{x}}^2 \delta^3(\boldsymbol{x} - \boldsymbol{x}_a) + \frac{\lambda}{2} h_{\mu\nu}(x) \overbrace{T^{\mu\nu}_{pp\,a}(x)}^{\gamma_a^{-1} m_a \dot{x}^\mu \dot{x}^\nu \delta^3(\boldsymbol{x} - \boldsymbol{x}_a)} \right] + \mathcal{L}_{(2)}. \tag{7}$$

The graviton field can be extremized using the EoM (6) yielding

$$S_{pp} = \int \frac{\mathrm{d}^4 x}{c} \sum_{a=1}^{2} \left[\frac{1}{2} m_a \dot{\boldsymbol{x}}^2 \delta^3(\boldsymbol{x} - \boldsymbol{x}_a(t)) + \frac{\lambda}{2} \left(-\frac{\lambda}{2} P_{\mu\nu:\alpha\beta} \int \Delta_{\text{ret}}(x-y) T^{\alpha\beta}_{pp\,b}(y) \, \mathrm{d}^4 y \right) T^{\mu\nu}_{pp\,a}(x) \right] \tag{8}$$

$$= \int \mathrm{d}t \, \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2) - \iint \frac{\mathrm{d}^4 x \, \mathrm{d}^4 y}{c} \frac{\lambda}{2} T^{\mu\nu}_{pp\,1}(x) \Delta_{\text{ret}}(x-y) P_{\mu\nu:\alpha\beta} \frac{\lambda}{2} T^{\alpha\beta}_{pp\,2}(y) \tag{9}$$

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$$S_{pp} \overset{\frac{v}{c}\sim 0}{=} \int \left[\frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2 \right) + \frac{\lambda^2}{8} \frac{m_1 m_2 c^4}{4\pi r} \right] \mathrm{d}t \tag{10}$$

$$= \int \left[\frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2 \right) + \frac{G m_1 m_2}{r} \right] \mathrm{d}t \tag{11}$$

This potential can be expressed graphically as

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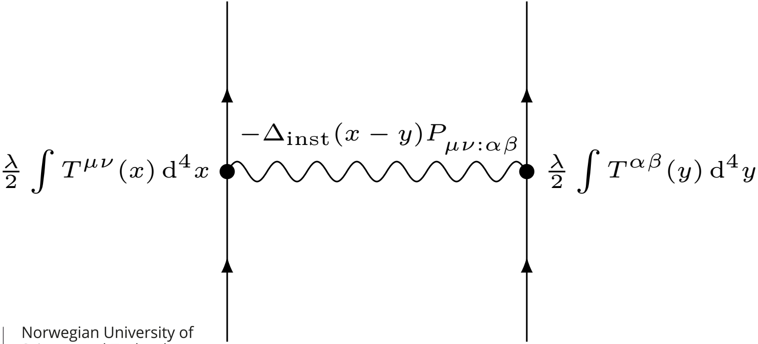
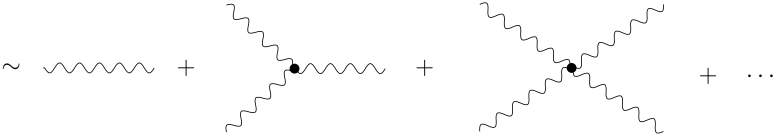


Using Feynman diagrams to compute orbital energy - Expanding the action

Until now, the action has only been expanded to quadratic order in h, but GR is a non-linear theory and thus contain terms of all orders in h.

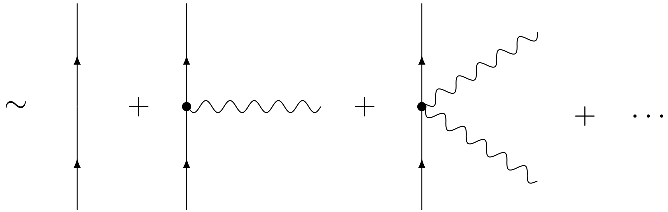
$$S_{\text{EH}} \sim \frac{1}{2} \int \mathrm{d}^4x \left[(\partial h)^2 + \lambda h (\partial h)^2 + \lambda^2 h^2 (\partial h)^2 + \ldots \right] \tag{12}$$



Using Feynman diagrams to compute orbital energy - Expanding the action

Likewise, taking the interaction term to be the geodesic, also the interaction acquires higher order terms.

$$
\begin{aligned}
S_{pp} &= -mc^2 \int \mathrm{d}\tau \sqrt{1 - \lambda h_{\mu\nu} \frac{\dot{x}^\mu}{c} \frac{\dot{x}^\nu}{c}} \qquad (13) \\
&= -mc^2 \int \gamma^{-1} \mathrm{d}t + \frac{\lambda}{2} \int \mathrm{d}t\, h_{\mu\nu} \gamma^{-1} m \dot{x}^\mu \dot{x}^\nu + \frac{m\lambda^2}{8c^2} \int \mathrm{d}t\, \gamma^{-1} \left(h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right)^2 + \dots
\end{aligned}
$$



Using Feynman diagrams to compute orbital energy - Assigning PN order to diagrams

Separating the field into potential modes H (dashed lines) and radiation modes $\mathcal{H}$ (squiggly lines), $h_{\mu\nu} = H_{\mu\nu} + \mathcal{H}_{\mu\nu}$, the potential propagator can be PN expanded *off shell*.

$$\Delta_{\text{inst}}(k) \equiv \frac{1}{-k_\mu k^\mu} = \frac{1}{-\boldsymbol{k}^2} \frac{1}{1 - k_0^2/\boldsymbol{k}^2} = \frac{1}{-\boldsymbol{k}^2} \left(1 + \left(\frac{k_0}{\boldsymbol{k}}\right)^2 + \left(\frac{k_0}{\boldsymbol{k}}\right)^4 + \dots \right). \tag{14}$$

Each factor of $k_0^2/\boldsymbol{k}^2$ results in a factor of v^2/c^2, and are denoted by $\otimes$ in graphs.

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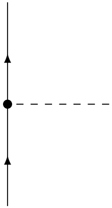
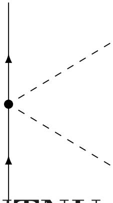
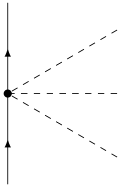
It can be shown that factors of $H \sim \frac{\sqrt{v}}{r}$, and $\lambda \sim \frac{\sqrt{rmv^2}}{m} \equiv \frac{\sqrt{Lv}}{m}$ in the final expression. Then, it is straight forward to predict the PN order of different diagrams.



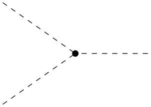
$$\sim \sqrt{L} \qquad (15)$$



$$\sim v^2 \qquad (16)$$



$$\sim v^4/\sqrt{L} \qquad (17)$$



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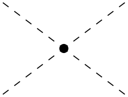
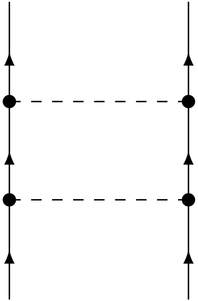
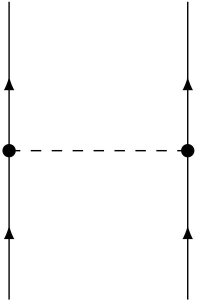


$$\sim v^4/L \qquad (19)$$

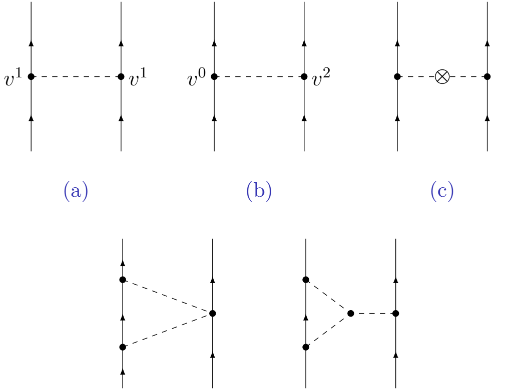
All graviton lines must remain connected if the particle lines are stripped of. This is because particle lines are not propagating, thus disconnected graviton lines belong to separate diagrams.



$$= 2\times$$



Using Feynman diagrams to compute orbital energy - The resulting 1PN diagrams



(d) (e)

$$V_{(a)} = \frac{Gm_1m_2}{r}\frac{4\boldsymbol{v}_1\cdot\boldsymbol{v}_2}{c^2}$$

$$V_{(b)} = -\frac{Gm_1m_2}{r}\frac{3}{2}\frac{\boldsymbol{v}_1^2+\boldsymbol{v}_2^2}{c^2}$$

$$V_{(c)} = -\frac{Gm_1m_2}{r}\frac{\boldsymbol{v}_1\cdot\boldsymbol{v}_2-(\boldsymbol{v}_1\cdot\hat{\boldsymbol{r}})(\boldsymbol{v}_2\cdot\hat{\boldsymbol{r}})}{2c^2}$$

$$V_{(d)} = -\frac{Gm_1m_2}{r}\frac{G(m_1+m_2)}{2rc^2}$$

$$V_{(e)} = \frac{Gm_1m_2}{r}\frac{G(m_1+m_2)}{rc^2}$$

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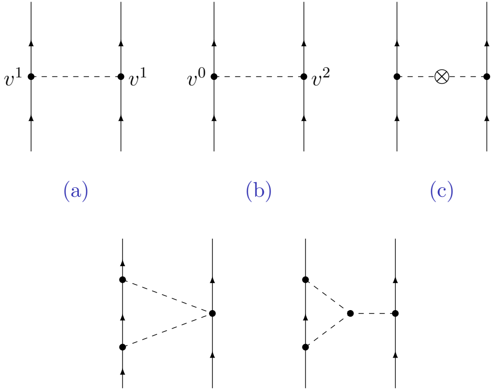

(a) (b) (c) (d) (e)

$$V_{(a)} = \frac{Gm_1m_2}{r}\frac{4\boldsymbol{v}_1\cdot\boldsymbol{v}_2}{c^2}$$

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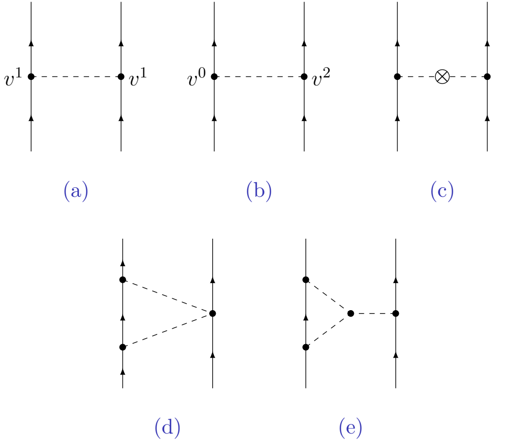
Using Feynman diagrams to compute orbital energy - The resulting 1PN diagrams



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Using Feynman diagrams to compute orbital energy - The resulting Lagrangian

Summing up the 0PN and 1PN diagrams, and adding kinetic energy terms up to 1PN results in the *Einstein-Infeld-Hoffmann Lagrangian*.

$$L_{\mathrm{EIH}} = \sum_a \frac{1}{2} m_a v_a^2 \left(1 + \frac{v_a^2}{4c^2}\right) + \frac{G m_1 m_2}{r} \Bigg\{ 1 + \frac{1}{2c^2} \left[3(\boldsymbol{v}_1^2 + \boldsymbol{v}_2^2) - 7 \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 - (\boldsymbol{v}_1 \cdot \hat{\boldsymbol{r}})(\boldsymbol{v}_2 \cdot \hat{\boldsymbol{r}}) - \frac{GM}{r} \right] \Bigg\}. \tag{20}$$

Imposing circular orbits, EoM's can be found relating r, ω, and v. E.g.

$$\omega^2 = \frac{GM}{r^3} \left\{ 1 - (3 - \eta) \frac{GM}{rc^2} + \mathcal{O}\left(c^{-4}\right) \right\}, \tag{21}$$

which is the 1PN version of Kepler's third law. For the Earth-Moon system, this is a correction of $\delta r \approx 0.5$ cm.

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Using Feynman diagrams to compute orbital energy - The 1PN energy

Provided a Lagrangian, it is a straightforward matter to obtain the energy. Preforming the Legendre transform, the Hamiltonian is obtained

$$H_{\rm EIH} = \frac{1}{2}\mu v^2\left(1+\frac{3}{4}(1-3\eta)\frac{v^2}{c^2}\right) - \frac{GM\mu}{r}\left\{1-\frac{GM}{2rc^2}-\frac{v^2}{2c^2}\left[3+\eta\left(1+\frac{\dot{r}^2}{v^2}\right)\right]\right\}. \quad (22)$$

For circular orbits this is

$$E = -\frac{\mu}{2}(GM\omega)^{2/3}\left\{1-\left(\frac{3}{4}+\frac{1}{12}\eta\right)\frac{(GM\omega)^{2/3}}{c^2}+\mathcal{O}\left(c^{-3}\right)\right\}, \quad (23)$$

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The energy flux - A brief flash of the result

At the large spatial scale, the radiation modes $\mathcal{H}$ couple to the binary masses + potential modes H, which are effectively a point source endowed with multipole structure.

Using the EoM found at short range, this motion can be inserted into the different multipoles and determine the energy flux associated with that type of motion at different PN orders.

The result at 1PN is

$$\mathcal{F} = \frac{32}{5}\frac{\eta^2}{G}\frac{(GM\omega)^{10/3}}{c^5}\left\{1 - \left(\frac{1247}{336} + \frac{35}{12}\eta\right)\frac{(GM\omega)^{2/3}}{c^2} + \mathcal{O}\left(c^{-3}\right)\right\}. \tag{24}$$

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Summary - How to compute GWforms using Feynman diagrams

1. Start by separating the binary evolution into two time scales.
2. In the short time scale, assume conservative motion by ignoring radiation.
   2.1 Make diagrams with desired PN scaling and compute these.
   2.2 Expand lower order diagrams with velocity dependent couplings and relativistic corrections to propagators.
   2.3 Sum up to obtain Lagrangian. Compute the corresponding Hamiltonian to obtain the orbital energy.
3. Using the conservative motion of the short range scale, compute the radiation produced at long range scale. This still in at the short time scale, such that the effect of GW emission can be ignored.
4. Provided with the PN expansion of $E(\omega)$ and $\mathcal{F}(\omega)$, compute the time evolution of $\omega(t)$ from $\mathrm{d}t = -\mathcal{F}^{-1}\frac{\mathrm{d}E}{\mathrm{d}\omega}\mathrm{d}\omega$.

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Thank you for your attention

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