Anomalous currents in a hadronic fluid

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Based on: J.L.Mañes, EM, M.Valle, M.A.V.M, JHEP1811 ('18), JHEP1912 ('19).

Other references: K.Landsteiner, EM, F.Pena-Benitez, PRL107('11);

Lect. Notes Phys. 81 ('13); EM, M.Valle, JHEP1411 ('14).

Issues

1. Introduction: Anomalous Transport
   - The Chiral Magnetic Effect
   - Hydrodynamics of Relativistic Fluids
2. Equilibrium Partition Function Formalism to Hydrodymamics
   - Equilibrium Partition Function
   - Derivative Expansion
3. Non-Abelian Anomalies
   - The chiral anomaly
   - Partition function. Currents without Goldstone bosons
   - The Wess-Zumino-Witten partition function

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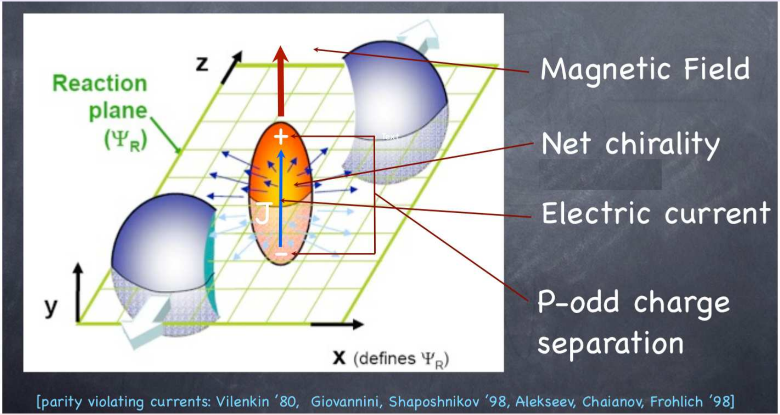
The Chiral Magnetic Effect (CME)

[Kharzeev, McLerran, Warringa '07]



Strong Magnetic field induces a $\mathcal{P}$-odd charge separation $\Longrightarrow$

$\Longrightarrow$ Electric current: $\vec{J} = \sigma^{\mathcal{B}} \vec{B}$.

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Hydrodynamics of Relativistic Fluids

[Son,Surowka], [Eling,Neiman,Oz], [Erdmenger et al.], [Banerjee et al.], [Loganayagam], [Kharzeev, Yee], [Sadovyev et al.], [Landsteiner, EM, Pena-Benitez], . . .

$$\langle T^{\mu\nu}\rangle = \underbrace{(\varepsilon + \mathcal{P})u^\mu u^\nu + \mathcal{P} g^{\mu\nu}}_{\text{Ideal Hydro}} + \underbrace{\langle T^{\mu\nu}\rangle_{\text{diss \& anom}}}_{\text{Dissipative \& Anomalous}} ,$$

$$\langle J^\mu\rangle = \underbrace{n u^\mu}_{\text{Ideal Hydro}} + \underbrace{\langle J^\mu\rangle_{\text{diss \& anom}}}_{\text{Dissipative \& Anomalous}} .$$

- Landau frame: $\langle T^{0i}\rangle \sim u^i$

$$\langle T^{\mu\nu}\rangle_{\text{diss \& anom}} = -\eta P^{\mu\alpha} P^{\nu\beta}\left(D_\alpha u_\beta + D_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta} D^\lambda u_\lambda\right) - \zeta P^{\mu\nu} D^\alpha u_\alpha + \cdots$$

$$\langle J^\mu\rangle_{\text{diss \& anom}} = -\sigma T P^{\mu\nu} D_\nu \left(\frac{\mu}{T}\right) + \sigma E^\mu + \underbrace{\sigma^{\mathcal{B}} B^\mu}_{\text{CME}} + \underbrace{\sigma^{\mathcal{V}} \omega^\mu}_{\text{CVE}} + \cdots$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, and
vorticity: $\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda} u_\nu D_\rho u_\lambda$ ➞ Chiral Vortical Effect (CVE).

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Equilibrium Partition Function

[Banerjee et al '12], [Jensen et al '13], [Bhattacharyya '14], [EM, Valle '14]

- Relativistic Invariant Quantum Field Theory on the manifold

$$ds^2 = -e^{2\sigma(\vec{x})}(dt + a_i(\vec{x})dx^i)^2 + g_{ij}(x)dx^i dx^j$$

  and time independent background $U(1)$ gauge connection:

$$\mathcal{A} = \mathcal{A}_0(\vec{x})dx^0 + \mathcal{A}_i(\vec{x})dx^i \, .$$

- Partition function of the system:

$$Z = \mathrm{Tr}\, e^{-\frac{H-\mu_0 Q}{T_0}}$$

  → Dependence of Z on σ, g_{ij}, a_i and $\mathcal{A}_\mu$?

- Most general partition function consistent with:
  - 3-dim diffeomorphism invariance.
  - Kaluza-Klein (KK) invariance: $t \to t + \phi(\vec{x})$, $\vec{x} \to \vec{x}$.
  - $U(1)$ *time-independent* gauge invariance (up to an anomaly).

Equilibrium Partition Function

- Stress Tensor and $U(1)$ current ➞ under t-indep variations

$$\delta \log Z = \frac{1}{T_0} \int d^3x \sqrt{-g_3} \left(-\frac{1}{2} T_{\mu\nu} \delta g^{\mu\nu} + J^\mu \delta \mathcal{A}_\mu \right)$$

➞ $T_{\mu\nu} = -2 T_0 \frac{\delta \log Z}{\delta g^{\mu\nu}}$, $\qquad J^\mu = T_0 \frac{\delta \log Z}{\delta \mathcal{A}_\mu}$.

- In particular, for $\log Z = \mathcal{W}(e^\sigma, A_0, a_i, A_i, g^{ij}, T_0, \mu_0)$, where

$$A_0 = \mathcal{A}_0 \, , \qquad A_i = \mathcal{A}_i - a_i \mathcal{A}_0 \, ,$$

are KK invariant quantities, one gets the consistent currents

$$\langle J^i \rangle_{\rm cons} = \frac{T_0}{\sqrt{-G}} \frac{\delta \mathcal{W}}{\delta A_i} \, , \qquad \langle J_0 \rangle_{\rm cons} = -\frac{T_0 e^{2\sigma}}{\sqrt{-G}} \frac{\delta \mathcal{W}}{\delta A_0} \, ,$$

$$\langle T_0{}^i \rangle = \frac{T_0}{\sqrt{-G}} \left(\frac{\delta \mathcal{W}}{\delta a_i} - A_0 \frac{\delta \mathcal{W}}{\delta A_i} \right) \, , \qquad \langle T_{00} \rangle = -\frac{T_0 e^{2\sigma}}{\sqrt{-G}} \frac{\delta \mathcal{W}}{\delta \sigma} \, .$$

- ➞ $\mathcal{W}$ is a generating functional for the hydrodynamic constitutive relations.

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Equilibrium Partition Function at first order

- Partition function at 1st order in derivative expansion **[Banerjee et al '12; EM, M.Valle '14]**:

$$\mathcal{W}^{(1)}=\int d^3x\sqrt{g_3}\left[\alpha_1(\sigma,A_0)\epsilon^{ijk}A_iA_{jk}+\alpha_2(\sigma,A_0)\epsilon^{ijk}A_if_{jk}+\alpha_3(\sigma,A_0)\epsilon^{ijk}a_if_{jk}\right]$$

where $A_{ij}=\partial_iA_j-\partial_jA_i\,,\ f_{ij}=\partial_ia_j-\partial_ja_i$.

- Ideal gas of Dirac fermions ⟶ from a computation of $\langle {T_0}^i\rangle$ and $\langle J^i\rangle$, one gets

$$\alpha_1(\sigma,A_0)=-\frac{C}{6T_0}A_0\,,\quad \alpha_2(\sigma,A_0)=-\frac{1}{2}\left(\frac{C}{6T_0}A_0^2-C_2T_0\right)\,,\quad \alpha_3(\sigma,A_0)=0\,.$$

where: $\begin{cases} C=\frac{1}{4\pi^2} & \textit{(chiral anomaly)}\text{: [Son, Surowka '09], [Erdmenger et al '09], ...} \\ C_2=\frac{1}{24} & \textit{(gauge-gravitational anomaly)}\text{: [Landsteiner, EM, Pena-Benitez '11]} \end{cases}$

- Coefficients related to chiral magnetic and vortical conductivities:

$$\sigma^{\mathcal{B}}=C\mu\,,\qquad \sigma^{\mathcal{V}}=\frac{1}{2}C\mu^2+C_2T^2\mu\,,\qquad [\mu=e^{-\sigma}A_0,\ T=e^{-\sigma}T_0]\,.$$

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The chiral anomaly

- Theory of a chiral fermion:

$$\mathcal{L}_{\rm YM} = i\overline{\psi}\gamma^\mu(\partial_\mu - it_a\mathcal{A}^a_\mu)\psi \,.$$

- Under a general shift $\mathcal{A}^a_\mu \to \mathcal{A}^a_\mu + \delta\mathcal{A}^a_\mu$:

$$\delta\Gamma[\mathcal{A}] = \int d^Dx\, \delta\mathcal{A}^a_\mu\, J^\mu_a(x)_{\rm cons} \,. \qquad (1)$$

- The anomaly is given by the *"failure of the effective action to be invariant under axial gauge transformations"*:

$$\mathcal{A}_\mu \longrightarrow U\mathcal{A}_\mu U^{-1} - i\partial_\mu U\, U^{-1} \,, \qquad U = \exp\left(i\Lambda^{\rm Axial}_a t_a\right) \,,$$

$$\delta_{\rm gauge}\Gamma[\mathcal{A}] = -\int d^Dx\, \Lambda^{{\rm Axial},\, a}\, G_a[\mathcal{A}] \,.$$

- Particularizing **(1)** to $\delta\mathcal{A}^a_\mu = (D_\mu\Lambda^{\rm Axial})^a$ ➞
➞ (non)-conservation law for the consistent current:

$$D_\mu J^\mu_a(x)_{\rm cons} = G_a[\mathcal{A}(x)] \,.$$

The non-abelian anomaly

- Bardeen form of the non-abelian anomaly for the symmetry group $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ **[W.A. Bardeen, PR184, '69]**:

$$\mathcal{L}_{\rm YM} = i\overline{\psi}_L \gamma^\mu (\partial_\mu - i t_a \mathcal{A}^a_{L\,\mu})\psi_L + i\overline{\psi}_R \gamma^\mu (\partial_\mu - i t_a \mathcal{A}^a_{R\,\mu})\psi_R \,.$$

$$G_a[\mathcal{V},\mathcal{A}] = \frac{iN_c}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\Big\{ t_a \Big[\mathcal{V}_{\mu\nu}\mathcal{V}_{\rho\sigma} + \tfrac{1}{3}\mathcal{A}_{\mu\nu}\mathcal{A}_{\rho\sigma} - \tfrac{32}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma + \tfrac{8}{3} i(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{V}_{\rho\sigma} + \mathcal{A}_\mu\mathcal{V}_{\rho\sigma}\mathcal{A}_\nu + \mathcal{V}_{\rho\sigma}\mathcal{A}_\mu\mathcal{A}_\nu)\Big]\Big\} \,.$$

where

$$\mathcal{V}_{\mu\nu} = \partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu - i[\mathcal{V}_\mu,\mathcal{V}_\nu] - i[\mathcal{A}_\mu,\mathcal{A}_\nu] \,,$$
$$\mathcal{A}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{V}_\mu,\mathcal{A}_\nu] - i[\mathcal{A}_\mu,\mathcal{V}_\nu] \,.$$

- This includes *triangle*, *square* and *pentagon* one-loop diagrams.
- The anomaly arises from the breaking of gauge invariance under *'axial'* gauge transformations of the effective action $\Gamma_0[\mathcal{V},\mathcal{A}]$:

$$\mathscr{Y}_a(x)\Gamma_0[\mathcal{V},\mathcal{A}] = 0 \,, \qquad \mathscr{X}_a(x)\Gamma_0[\mathcal{V},\mathcal{A}] = G_a[\mathcal{V},\mathcal{A}] \,. \qquad (2)$$

$(\mathscr{Y}_a(x), \mathscr{X}_a(x)) \equiv$ Local generator of (vector, axial) transform.

- Computation of $\Gamma_0[\mathcal{V},\mathcal{A}]$ from a solution of **(2)** by trial and error.

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The anomalous functional

- Solution of the local functional $\Gamma_0[V, A, G]$ **[Mañes,EM,Valle,MAVZ'18]**:

$$\Gamma_0[V,A,G] = -\frac{N_c}{32\pi^2}\int dt\, d^3x\sqrt{g_3}\,\epsilon^{ijk}\,\mathrm{Tr}\Big\{\frac{32}{3} i\, V_0 A_i A_j A_k + \frac{4}{3}(A_0 A_i + A_i A_0)A_{jk} + 4(V_0 A_i + A_i V_0)V_{jk} + \frac{8}{3}(A_0^2 + 3V_0^2)A_i \partial_j a_k\Big\} + C_2 T_0^2 \int dt\, d^3x \sqrt{g_3}\,\epsilon^{ijk}\,\mathrm{Tr} A_i\, \partial_j a_k$$

- V_μ and A_μ are KK inv fields: $A_0 = \mathcal{A}_0$, $A_i = \mathcal{A}_i - a_i\mathcal{A}_0$, etc.
- Γ_0 can be determined also from differential geometry methods: Chern-Simons effective action → dimensional reduction **[Jensen, Laganayagam, Yarom '14; Mañes, EM, Valle, MAVZ '18 '19].**
- C_2 related to the mixed gauge-gravitational anomaly $\sim \mathrm{Tr}\{t_a\}\,\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\kappa} R_{\rho\sigma}{}^{\lambda\kappa}$ → One should take into account the Riemann tensor contribution in the anomaly polynomial $\sim \mathrm{Tr}\,\mathcal{F}_A\, R^\mu{}_\nu R^\nu{}_\mu$ **[Nair,Ray,Roy PRD86 '12].**

Currents in equilibrium

- The covariant currents are defined by adding to the consistent currents the Bardeen-Zumino (BZ) polynomials:

$$J^\mu_{\rm cov} = J^\mu_{\rm cons} + J^\mu_{\rm BZ} \,,$$

$$\text{with} \quad J^\mu_{\rm BZ\,V} = -\frac{N_c}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}{\rm Tr}\Big\{ t_a\big(\mathcal{A}_\nu\mathcal{V}_{\rho\sigma} + \mathcal{V}_{\rho\sigma}\mathcal{A}_\nu + \tfrac{8}{3} i\,\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\big)\Big\},$$

$$J^\mu_{\rm BZ\,A} = -\frac{N_c}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}{\rm Tr}\Big\{ t_a\left(\mathcal{A}_\nu\mathcal{A}_{\rho\sigma} + \mathcal{A}_{\rho\sigma}\mathcal{A}_\nu\right)\Big\}.$$

- From $W_0 = i\Gamma_0$ the covariant currents and stress tensor are:

$$\langle J^i_{{\rm A}a}\rangle_{\rm cov} = \frac{N_c}{8\pi^2} e^{-\sigma}\epsilon^{ijk}{\rm Tr}\Big\{ t_a\big[(A_0 A_{jk} + A_{jk}A_0) + (V_0 V_{jk} + V_{jk}V_0)$$

$$+2(A_0^2 + V_0^2)\partial_j a_k\big]\Big\}, \qquad \langle J_{{\rm A}a\,0}\rangle_{\rm cov} = 0\,, \qquad t_a = \frac{\lambda_a}{2}\,,$$

$$\langle T_0{}^i\rangle = -\frac{N_c}{8\pi^2} e^{-\sigma}\epsilon^{ijk}\,{\rm Tr}\Big\{ (A_0^2 + V_0^2)A_{jk} + (V_0 A_0 + V_0 A_0)V_{jk}$$

$$+\left(\frac{2}{3}A_0^3 + 2A_0 V_0^2\right)\partial_j a_k\Big\}, \qquad \langle T_{00}\rangle = \langle T^{ij}\rangle = 0\,.$$

Constitutive relations

- Maximal number of chemical potentials to be consistently introduced = dimension of the Cartan subalgebra. Let us consider the background ($N_f = 3$):

$$V_\mu(\boldsymbol{x}) = V_{0\,\mu}(\boldsymbol{x}) t_0 + V_{3\,\mu}(\boldsymbol{x}) t_3 + V_{8\,\mu}(\boldsymbol{x}) t_8 \,, \qquad t_0 = \frac{1}{\sqrt{6}} 1_{2\times 2} \,, \; t_a = \frac{1}{2}\lambda_a$$

$$A_0 = A_{5\,0}\, t_0 \,, \qquad A_i = 0 \,, \qquad \mathcal{A}_i = a_i A_{5\,0}\, t_0 \,.$$

- Equilibrium velocity field $u_\mu = -e^\sigma(1, a_i)$ and equilibrium baryonic, isospin, strangeness and axial chemical potentials:

$$\mu_0 = \mathcal{V}_{0\,0}\, e^{-\sigma} \,, \quad \mu_3 = \mathcal{V}_{3\,0}\, e^{-\sigma} \,, \quad \mu_8 = \mathcal{V}_{8\,0}\, e^{-\sigma} \,, \quad \mu_5 = \mathcal{A}_{5\,0}\, e^{-\sigma} \,.$$

μ_5 controls chiral imbalance **[Gatto, Ruggieri '12; Espriu et al, '13].**

$$J^\mu_{\text{electromagnetic}} = e\bar{\Psi}\gamma^\mu Q\Psi = e\bar{\Psi}\gamma^\mu t_3\Psi + \frac{e}{\sqrt{3}}\bar{\Psi}\gamma^\mu t_8\Psi \,,$$

$$J^\mu_{\text{baryonic}} = \bar{\Psi}\gamma^\mu B\Psi = \sqrt{\frac{2}{3}}\bar{\Psi}\gamma^\mu t_0\Psi \,, \quad J^\mu_{\text{isospin}} = e\bar{\Psi}\gamma^\mu I_3\Psi = \bar{\Psi}\gamma^\mu t_3\Psi \,,$$

$$J^\mu_{\text{S}} = \bar{\Psi}\gamma^\mu S\Psi = -\sqrt{\frac{2}{3}}\bar{\Psi}\gamma^\mu t_0\Psi + \frac{2}{\sqrt{3}}\bar{\Psi}\gamma^\mu t_8\Psi \,.$$

Constitutive relations ($N_f = 3$)

- Non-abelian magnetic field and vorticity:

$$\mathcal{B}^\mu_a = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{V}_{\alpha\beta\, a} \,, \qquad \omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta \,.$$

- Constitutive relations ➞ Out-of-equilibrium expressions ➞ Lorentz covariantization of the equilibrium currents:

$$\langle J^\mu_{a\,\mathrm{V}} \rangle_{\mathrm{cov}} = \frac{N_c}{4\sqrt{6}\pi^2}\mu_5 \mathcal{B}^\mu_a \,, \qquad a = 0,3,8 \,,$$

$$\langle J^\mu_{0\,\mathrm{A}} \rangle_{\mathrm{cov}} = \frac{N_c}{4\sqrt{6}\pi^2}\left[\mu_0\mathcal{B}^\mu_0 + \mu_3\mathcal{B}^\mu_3 + \mu_8\mathcal{B}^\mu_8 + (\mu_0^2+\mu_3^2+\mu_8^2-\mu_5^2)\omega^\mu\right]$$

$$\langle J^\mu_{3\,\mathrm{A}} \rangle_{\mathrm{cov}} = \frac{N_c}{4\sqrt{6}\pi^2}\left[\mu_3\mathcal{B}^\mu_0 + \left(\mu_0 + \frac{1}{\sqrt{2}}\mu_8\right)\mathcal{B}^\mu_3 + \frac{1}{\sqrt{2}}\mu_3\mathcal{B}^\mu_8 + 2\mu_3\left(\mu_0 + \frac{1}{\sqrt{2}}\mu_8\right)\omega^\mu\right]$$

$$\langle J^\mu_{8\,\mathrm{A}} \rangle_{\mathrm{cov}} = \frac{N_c}{4\sqrt{6}\pi^2}\left[\mu_8\mathcal{B}^\mu_0 + \frac{1}{\sqrt{2}}\mu_3\mathcal{B}^\mu_3 + \left(\mu_0 - \frac{1}{\sqrt{2}}\mu_8\right)\mathcal{B}^\mu_8 + \left(2\mu_0\mu_8 + \frac{1}{\sqrt{2}}\left(\mu_3^2-\mu_8^2\right)\right)\omega^\mu\right]$$

- Covariant form of the stress tensor $T^{\mu\nu} = u^\mu q^\nu + u^\nu q^\mu$:

$$q^\mu = \frac{N_c}{4\sqrt{6}\pi^2}\mu_5\left[\mu_0\mathcal{B}^\mu_0 + \mu_3\mathcal{B}^\mu_3 + \mu_8\mathcal{B}^\mu_8 + \left(\mu_0^2+\mu_3^2+\mu_8^2-\frac{1}{3}\mu_5^2\right)\omega^\mu\right].$$

- Chiral Magnetic and Chiral Vortical contributions.

Constitutive relations ($N_f = 3$)

- Change of basis $(t_0, t_3, t_8) \to (B, Q, S)$:

$$V_\mu(\boldsymbol{x}) = \sum_{a=0,3,8} V_{a\,\mu}(\boldsymbol{x}) t_a = V_{B\,\mu}(\boldsymbol{x}) B + V_{3\,\mu}(\boldsymbol{x}) Q + V_{S\,\mu}(\boldsymbol{x}) S\,,$$

$$A_0 = A_{5\,0} t_0 = A_B B\,.$$

- The electromagnetic field $\mathbb{V}_\mu$ is the only (physical) propagating gauge field. Assuming also $\mu_B = \mu_S = 0$:

$$\left.\begin{array}{l} V_{3\,\mu} \equiv e\mathbb{V}_\mu \\ V_{B\,\mu} = 0 \\ V_{S\,\mu} = 0 \end{array}\right\} \quad \Longrightarrow \quad \left.\begin{array}{l} V_{0\,\mu} = 0 \\ V_{3\,\mu} = \sqrt{3} V_{8\,\mu} = e\mathbb{V}_\mu \\ \mathcal{B}_0^\mu = 0\,, \quad \mathcal{B}_3^\mu = \sqrt{3}\mathcal{B}_8^\mu = e\mathcal{B}^\mu \end{array}\right\}$$

- Constitutive relation for the electromagnetic current:

$$\langle J^\mu_{\rm em} \rangle_{\rm cov} = \frac{e^2 N_c}{3\sqrt{6}\pi^2} \mu_5 \mathcal{B}^\mu$$

→ $\nexists$ CVE in vector current for $\mathrm{U}(3)_{\mathrm{V}} \times \mathrm{U}(3)_A$.
But $\exists$ CVE for $\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_A$ **[Landsteiner et al. Lect. Notes. Phys. '13]**.

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Wess-Zumino-Witten partition function

[Kaiser '01], [Son, Stephanov '08], [Fukushima, Mameda '12], [Brauner, Kadam '17], . . .

- Effects of the anomaly when symmetry is spontaneously broken.
- The WZW action describes the interactions of the Goldstone bosons among them and with the background fields.
- Application to QCD in the confined phase ⟶ Hadronic fluids ⟶ Fluid of pions, kaons, . . . interacting with external EM fields.
- WZW action:

$$\Gamma^{WZW}[\xi, A] = \Gamma_0[A] - \Gamma_0[A_{-\xi}]$$

  $\Gamma_0 \equiv$ anomalous functional in absence of symmetry breaking (computed above).
  $A_{-\xi} \equiv$ gauge field transformed with gauge parameters $\Lambda_a = -\xi_a$.
  [Wess, Zumino '71; Witten '83; Mañes '85; Chu, Ho, Zumino '96].
- $\xi_a \equiv$ Goldstone bosons

$$U(\xi) = \exp\left(2i\sum_a \xi_a t_a\right) .$$

Goldstone bosons

- Spontaneous breaking of axial symmetry:
$$\mathrm{U}(2)_L \times \mathrm{U}(2)_R \to \mathrm{U}(2)_V .$$
- The matrix of Goldstone bosons (GB)
$$U(\xi) = \exp\left(2i \sum_{a=1}^{3} \xi_a t_a\right) ,$$
includes three GBs from the broken $\mathrm{SU}(2)_A$ symmetry. The fourth GB ξ_0 is absent, as the $\mathrm{U}(1)_A$ sym is violated by non-pert effects.
- In terms of conventionally normalized GB fields $\pi^0, \pi^\pm$:
$$2 \sum_{a=1}^{3} \xi_a t_a = \frac{\sqrt{2}}{f_\pi} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix} ,$$
where $f_\pi \approx 92\,\mathrm{MeV}$ is the pion decay constant.
- Lagrangian to zeroth order
$$\mathcal{L} = -\frac{f_\pi^2}{4} G^{\mu\nu} \mathrm{Tr}\left\{ D_\mu U (D_\nu U)^\dagger \right\} , \qquad D_\mu U = \partial_\mu U - i[\mathcal{V}_{3\,\mu} t_3, U] .$$

Wess-Zumino-Witten partition function

- Spontaneous breaking of axial symmetry:

$$\mathrm{U}(2)_L \times \mathrm{U}(2)_R \to \mathrm{U}(2)_V .$$

  Suitable gauge transformation is an axial transformation with parameters $\Lambda_a^{\mathrm{Axial}}(x) = -\xi_a(x)$.

- WZW partition function:

$$\begin{aligned}
T_0 \mathcal{W}^{WZW} =& \frac{N_c}{8\pi^2} \int d^3x \sqrt{g}\, V_{0\,0}\, \epsilon^{ijk} \Big[-\tfrac{1}{2} \mathrm{Tr}\{ \partial_i (R_j + L_j) t_3 \} V_{3\,k} + \tfrac{1}{6} i\, \mathrm{Tr}\{ L_i L_j L_k \} \Big] \\
&+ \frac{N_c}{16\pi^2} \int d^3x \sqrt{g}\, \epsilon^{ijk} \mathrm{Tr}\{ (R_i + L_i) t_3 \} \\
&\times \left(V_{0\,0}\, \partial_j V_{3\,k} + V_{3\,0}\, \partial_j V_{0\,k} + V_{0\,0} V_{3\,0}\, \partial_j a_k \right) \\
&+ \frac{N_c}{48\pi^2} A_{5\,0} \int d^3x \sqrt{g}\, \epsilon^{ijk} \Big[\mathrm{Tr}\{ (L_i - R_i) t_3 \} + 2 \mathrm{Tr}\{ t_3^2 - U t_3 U^{-1} t_3 \} V_{3\,i} \Big] \\
&\times \left(\partial_j V_{3\,k} + V_{3\,0} \partial_j a_k \right)
\end{aligned}$$

where $L_j = i\partial_j U\, U^{-1}, \quad R_j = iU^{-1}\partial_j U.$

Constitutive relations in hydrododynamics

- Contribution to the constitutive relations:

$$T^{\mu\nu} = T^{\mu\nu}_{\text{perfect}} + \underbrace{\pi^{\mu\nu}}_{\text{Dissipative \& Anomalous}} , \qquad J^{\mu} = J^{\mu}_{\text{perfect}} + \underbrace{\delta J^{\mu}}_{\text{Dissipative \& Anomalous}} .$$

- Consistent expansion ⟶

$$T^{\mu\nu}_{\text{perfect}}(\mu_{a\,0}+\delta\mu_a, T_0+\delta T, \ldots) , \quad J^{\mu}_{\text{perfect}}(\mu_{a\,0}+\delta\mu_a, T_0+\delta T, \ldots).$$

- Choose the frame in which the first derivative corrections to the fluid quantities vanish, i.e. $\delta\mu_a^{(1)} = 0$, $\delta T^{(1)} = 0$ and $\delta u^{\mu\,(1)} = 0$
- In this frame

$$\pi^{\mu\nu} = 0 ,$$
$$\delta J^{\mu} = -\underbrace{(\delta J^{\nu} u_{\nu})}_{\text{Longitudinal}} u^{\mu} + \underbrace{P^{\mu}{}_{\nu}\delta J^{\nu}}_{\text{Transverse}} ,$$

where $P^{\mu\nu} = G^{\mu\nu} + u^{\mu}u^{\nu}$.

Constitutive relations at first order in derivatives

- Written in terms of 5 pseudo-scalars:

$$\mathbb{S}_{1(a)} \equiv \mathcal{I}_\mu \mathcal{B}^\mu_a \,, \qquad \mathbb{S}_2 \equiv \mathcal{I}_\mu \omega^\mu \,,$$
$$\mathbb{S}_3 \equiv \epsilon^{\mu\nu\alpha\beta} u_\mu \left[\mathcal{V}_{3\,\nu} \partial_\alpha \mathcal{I}_\beta - \frac{i}{3} \mathrm{Tr}\{L_\nu L_\alpha L_\beta\} \right] \,,$$
$$\mathbb{S}_{4(a)} \equiv \mathcal{T}_\mu \mathcal{B}^\mu_a \,, \qquad \mathbb{S}_5 = \mathcal{T}_\mu \omega^\mu \,,$$

- and 4 pseudo-vectors:

$$P^\mu_{1(a)} \equiv \frac{1}{T} \epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{I}_\alpha \, \mathcal{E}_{\beta(a)} \,, \qquad P^\mu_2 \equiv \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha \mathcal{I}_\beta \,,$$
$$P^\mu_{3(a)} \equiv \frac{1}{T} \epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{T}_\alpha \mathcal{E}_{\beta(a)} \,, \qquad P^\mu_4 \equiv \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha \mathcal{T}_\beta \,.$$

- We define:

$$\mathcal{E}_{\mu(a)} \equiv \mathcal{V}_{\mu\nu\,a} u^\nu = T \partial_\mu \left(\mu_a / T \right) \qquad \text{(non-abelian electric field)} \,,$$
$$\mathcal{H} \equiv \mathrm{Tr}\{(U^{-1} Q U - Q) Q\} \,, \qquad \mathcal{I}_\mu \equiv \mathrm{Tr}\{(L_\mu + R_\mu) Q\} \,,$$
$$\mathcal{T}_\mu \equiv \mathrm{Tr}\{(R_\mu - L_\mu) Q\} + 2 \mathcal{V}_{3\,\mu} \mathcal{H} \,.$$

Constitutive relations at first order in derivatives

- Baryonic currents [In color the BZ contributions: CME and CVE]:

$$u_\mu \langle \delta J^\mu_{0\,\mathrm{V}} \rangle_{\mathrm{cov}} = -\frac{N_c}{16\pi^2}\left[\mathbb{S}_{1(3)} + \mathbb{S}_3\right] ,$$

$$P^\mu{}_\nu \langle \delta J^\nu_{0\,\mathrm{V}} \rangle_{\mathrm{cov}} = -\frac{N_c}{16\pi^2}\left[T\mathbb{P}^\mu_{1(3)} - \mu_3 \mathbb{P}^\mu_2 - 2\mu_5 \mathcal{B}^\mu_0\right] ,$$

$$u_\mu \langle \delta J^\mu_{0\,\mathrm{A}} \rangle_{\mathrm{cov}} = -\frac{N_c}{48\pi^2}\mathbb{S}_{4(3)} ,$$

$$P^\mu{}_\nu \langle \delta J^\nu_{0\,\mathrm{A}} \rangle_{\mathrm{cov}} = -\frac{N_c}{48\pi^2}\left[T\mathbb{P}^\mu_{3(3)} + 2\mu_3 \mathcal{H}\mathcal{B}^\mu_3 + 4\mu_5^2 \omega^\mu\right] .$$

- Isospin currents:

$$u_\mu \langle \delta J^\mu_{3\,\mathrm{V}} \rangle_{\mathrm{cov}} = -\frac{N_c}{48\pi^2}\left[3\mathbb{S}_{1(0)} + 2\mu_5 \mathbb{S}_5\right] ,$$

$$P^\mu{}_\nu \langle \delta J^\nu_{3\,\mathrm{V}} \rangle_{\mathrm{cov}} = -\frac{N_c}{48\pi^2}\left[3T\mathbb{P}^\mu_{1(0)} + \mu_5 \mathbb{P}^\mu_4 + 4\mu_3\mu_5 \mathcal{H}\omega^\mu - 2\mu_5(\mathcal{H} + 3)\mathcal{B}^\mu_3\right]$$

$$u_\mu \langle \delta J^\mu_{3\,\mathrm{A}} \rangle_{\mathrm{cov}} = -\frac{N_c}{48\pi^2}\left[3\mathbb{S}_{4(0)} - 2\mu_5 \mathbb{S}_2\right] ,$$

$$P^\mu{}_\nu \langle \delta J^\nu_{3\,\mathrm{A}} \rangle_{\mathrm{cov}} = -\frac{N_c}{48\pi^2}\left[3T\mathbb{P}^\mu_{3(0)} + 6\mu_3 \mathcal{H}\mathcal{B}^\mu_0 - \mu_5 \mathbb{P}^\mu_2\right] .$$

Constitutive relations at first order in derivatives

Properties:

- The only dependence in the background is through the gauge fields $\mathcal{V}_{a\,\mu}$, so that the dependence in a_i has disappeared ➞ Constitutive relations should be expressed in terms of quantities of the fluid: fluid velocity, external fields, etc.
- Chiral electric effect (CEE) in vector current **[Neiman,Oz, JHEP09 '11]**

$$\langle \delta J^i_{\rm em} \rangle_{\rm cov} = \frac{e^2 N_c}{12\pi^2}\left[\frac{1}{f_\pi}\epsilon^{ijk}\partial_j\pi^0\mathcal{E}_k + \frac{5}{3}\mu_5\mathcal{B}^i + \frac{2}{f_\pi^2}\mu\mu_5\pi^+\pi^-\omega^i\right] + \cdots ,$$

- Chiral electric effect in axial currents:

$$\langle \delta J^i_{0\,\rm A} \rangle_{\rm cov} = \frac{N_c}{24\pi^2 f_\pi^2}\left[ie\epsilon^{ijk}(\pi^-\partial_j\pi^+ - \pi^+\partial_j\pi^-)\mathcal{E}_k + 2e\mu\pi^+\pi^-\mathcal{B}^i - 2\mu_5^2\omega^i\right] + \cdots$$

- We provide explicit values for the corresponding transport coefficients of the CEE. Derived from an equilibrium partition function ➞ CEE cannot lead to entropy production ➞ CEE is non-dissipative.

Conclusions

- We have studied non-dissipative transport effects up to 1st order in the hydrodynamic expansion in non-abelian theories.
- Effects are induced by *external electromagnetic fields*, *vortices* and curvature in a relativistic fluid $\Longrightarrow$ Anomalous Transport.
- Equilibrium partition function method can only account for non-dissipative effects: *time reversal properties*.
- Dissipative effects (shear viscosity, electric conductivity, ...) ➞ other methods: Kubo formulae, Fluid/gravity correspondence, ...
- Anomalous effects in presence of Spontaneous Breaking of non-abelian gauge symmetries:
  - Interactions of Goldstone bosons among them and with external electromagnetic fields.
  - Application to QCD in confined phase: Fluid of pions, kaons, etc.
- Future directions:
  - Application to other sectors of the SM ➞ ElectroWeak sector.
  - Application to condensed matter systems ➞ Superfluids, Weyl semi-metals **[Basar, Kharzeev, Yee '14], [Landsteiner '14].**

Thank You!