Crystalline phases in dense matter

Stefano Carignano

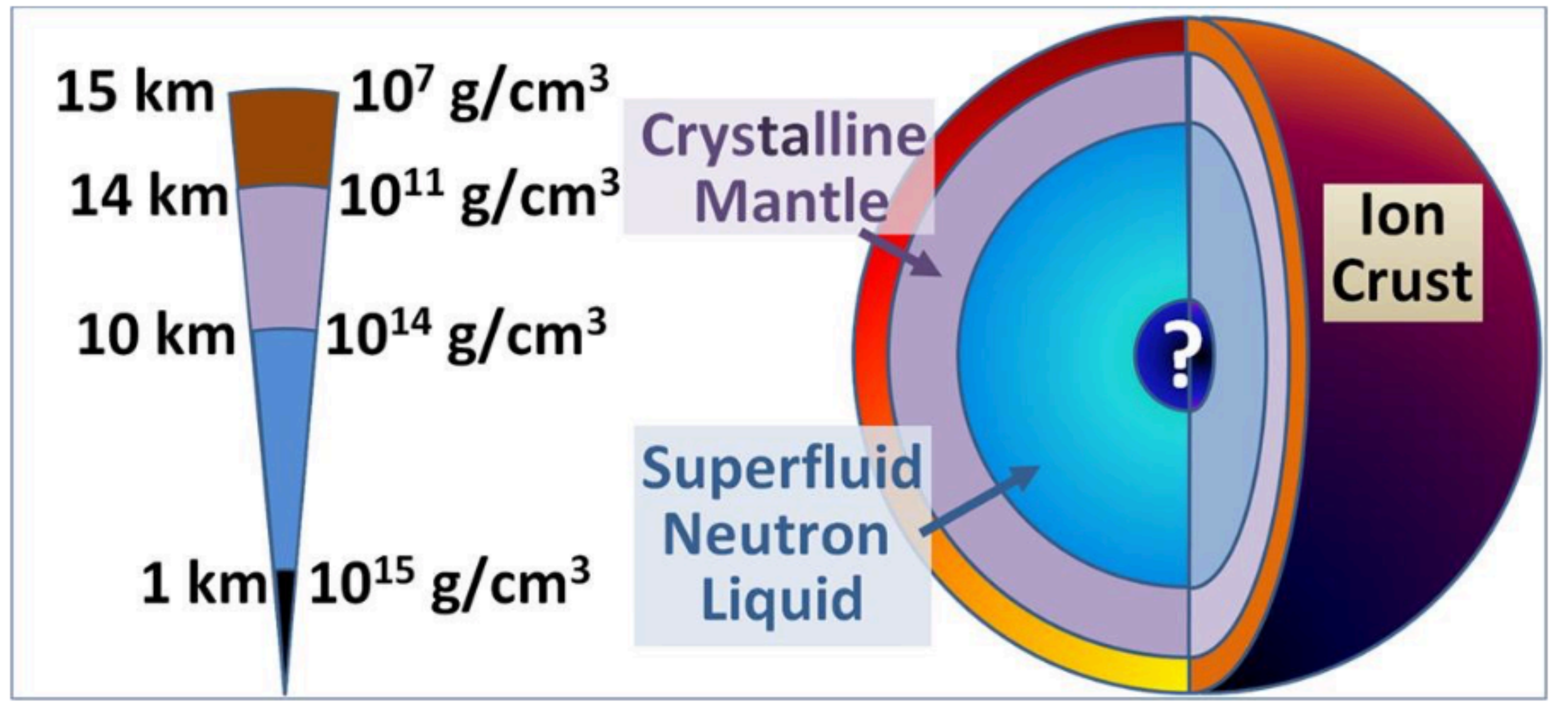
Institute of Space Sciences

CSIC IEEC

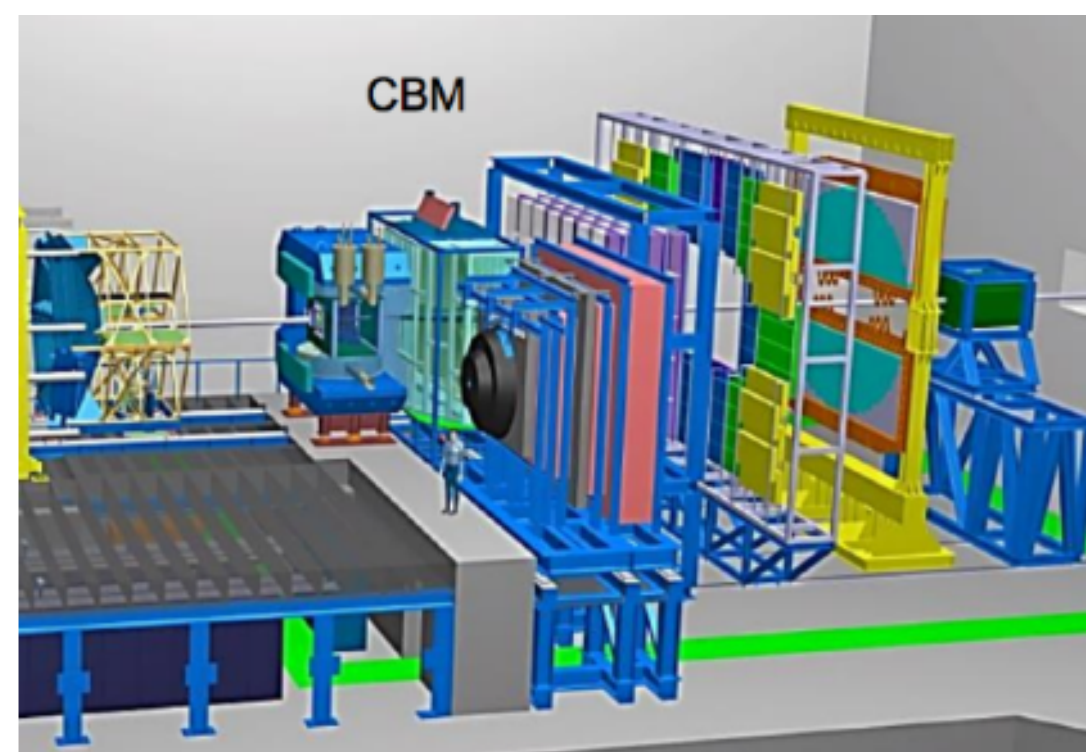
A Virtual Tribute to Quark Confinement and the Hadron Spectrum
August 2021

Motivation: dense QCD

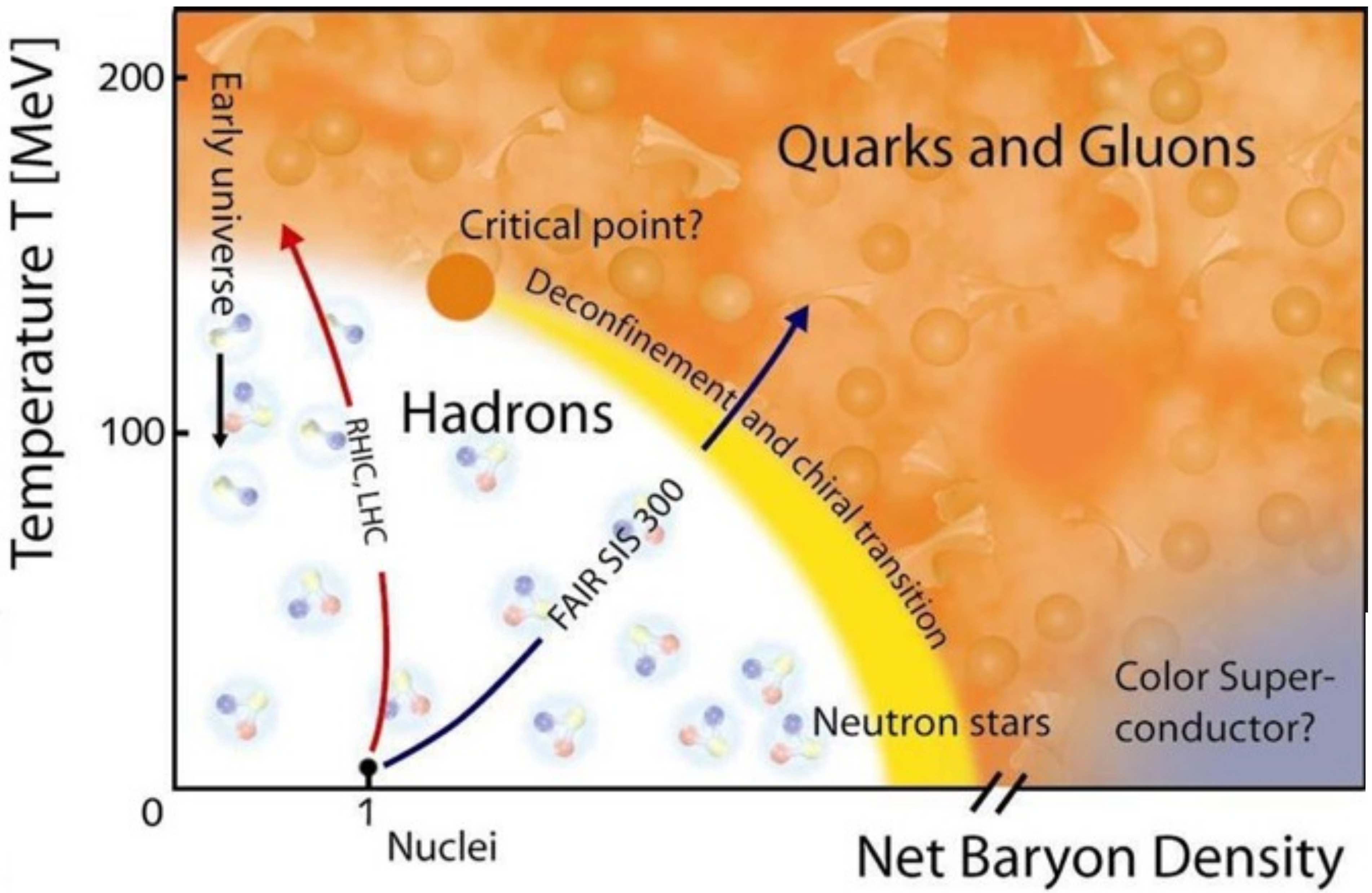
- Compact stars



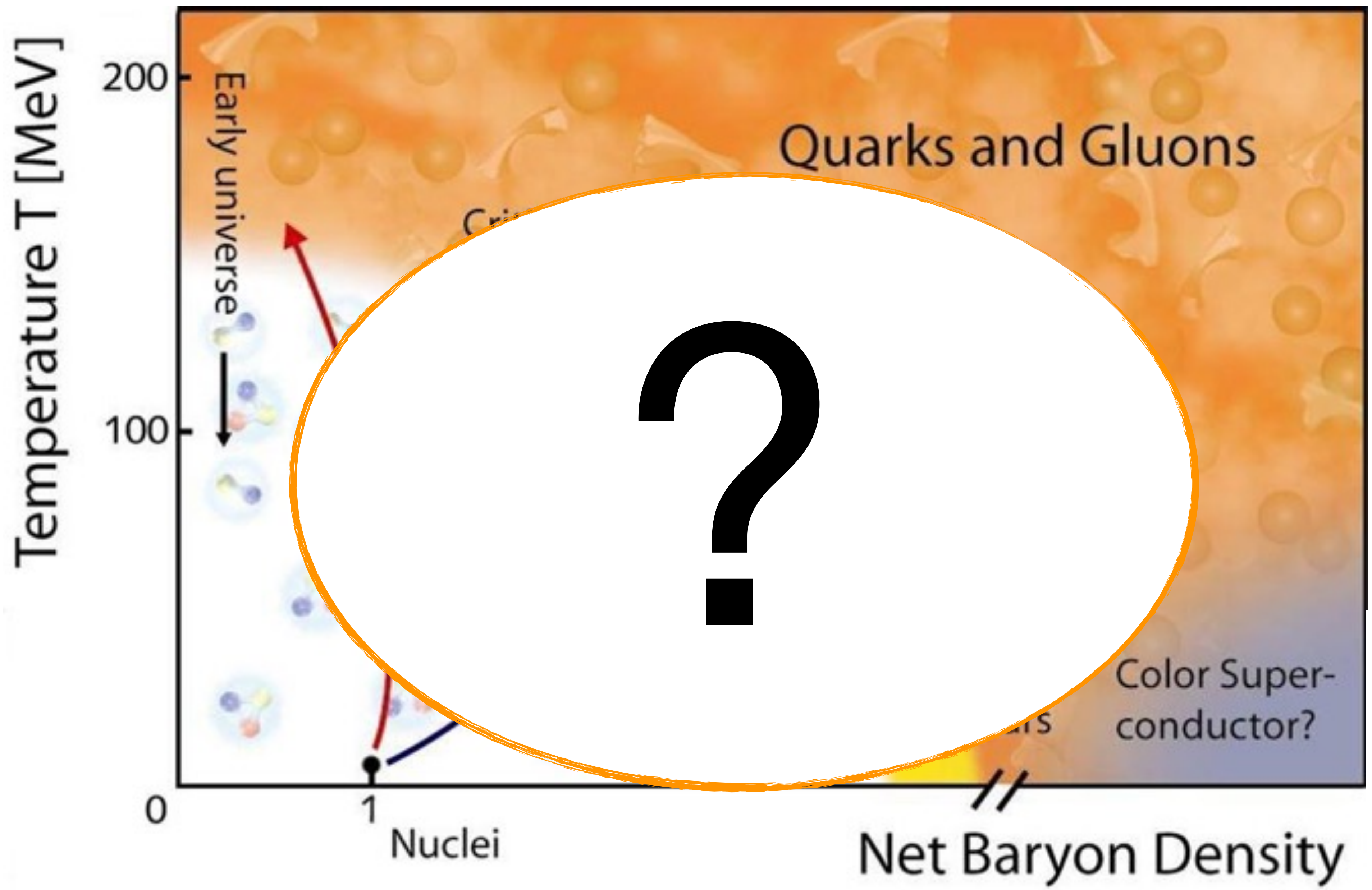
- Heavy ion collisions
  (RHIC BES, FAIR, NICA..)
  and the search for a Critical Point



The QCD phase diagram people have in mind

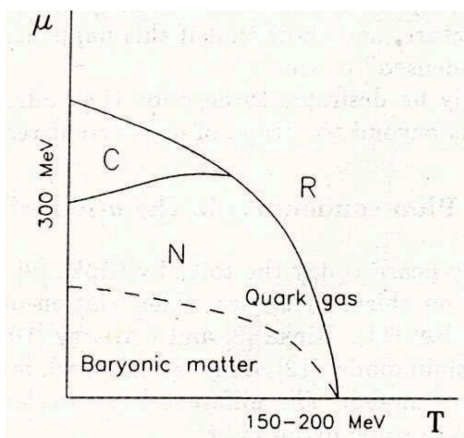


But if you think about it, it should be more like

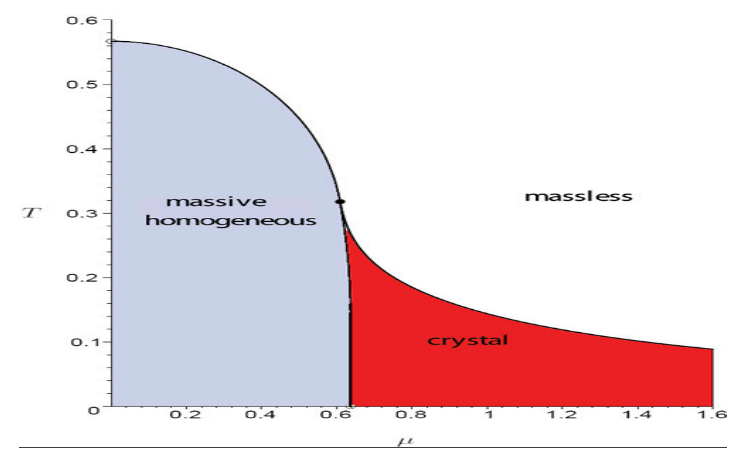


(some) inhomogeneous phases in strong interaction matter

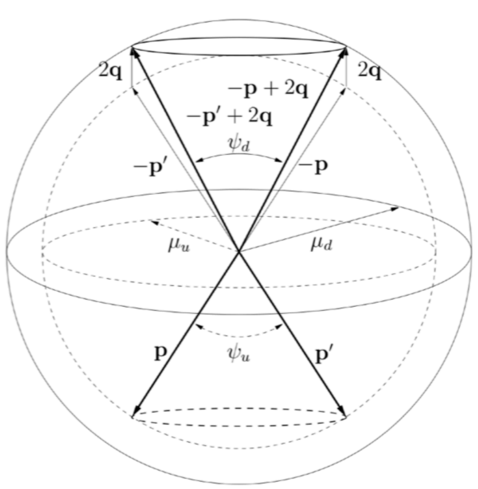
Pion condensation



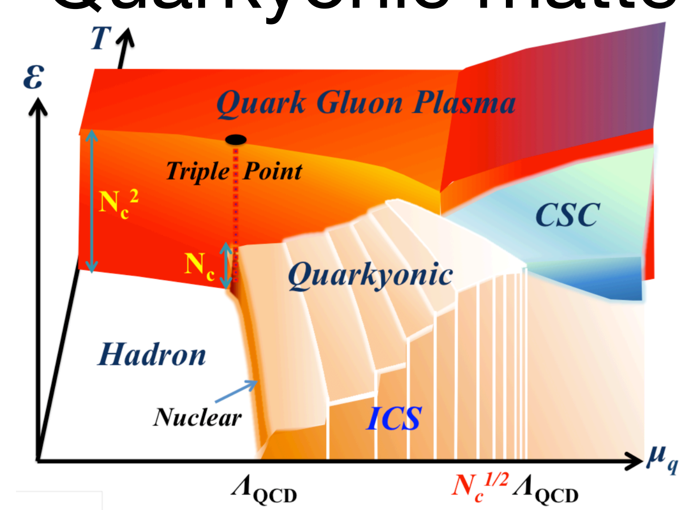
Toy models (Gross-Neveu)…



Color-superconductivity



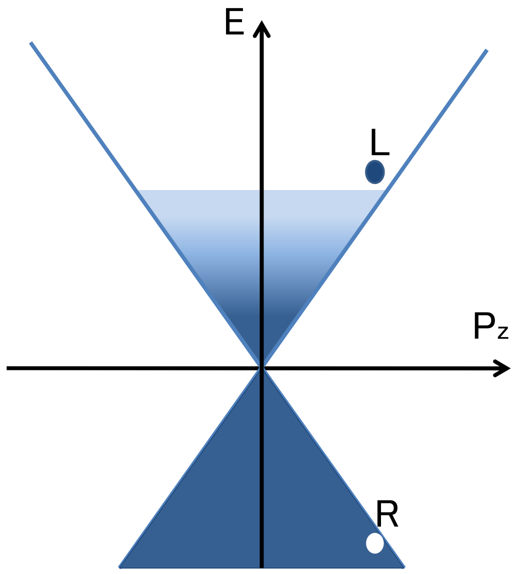
Quarkyonic matter



And more: Skyrmions, chiral soliton lattice, …

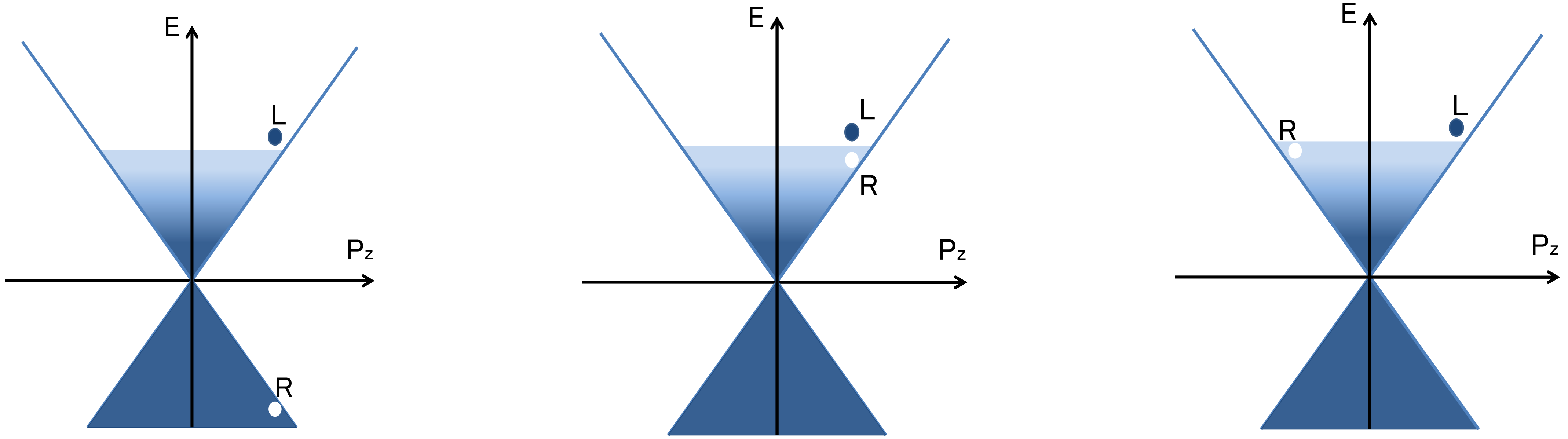
Inhomogeneous chiral condensates

Instead of the standard particle-antiparticle condensate…



Inhomogeneous chiral condensates

...particle-hole pairing at the Fermi surface



- Can occur at finite density: could be relevant at intermediate densities, close to the chiral phase transition

A concrete setup: NJL model

$$\mathcal{L}_{NJL} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma^5 \tau^a \psi)^2]$$

- Simple model (4-fermion interaction)
  with relevant features for a (more or less realistic)
  description of dense QCD
  (symmetries, dynamical mass generation/XSB…)

- Non-renormalizable,
  simplifying assumptions to make things tractable (MFA..)

- A good starting point to investigate qualitative features
  and as input for more refined calculations

Mean-field approximation

- Typical assumption: mean-field approximation $(\bar{\psi}\psi) \approx \langle\bar{\psi}\psi\rangle$
- A constant mean-field chiral condensate acts as constituent quark mass:

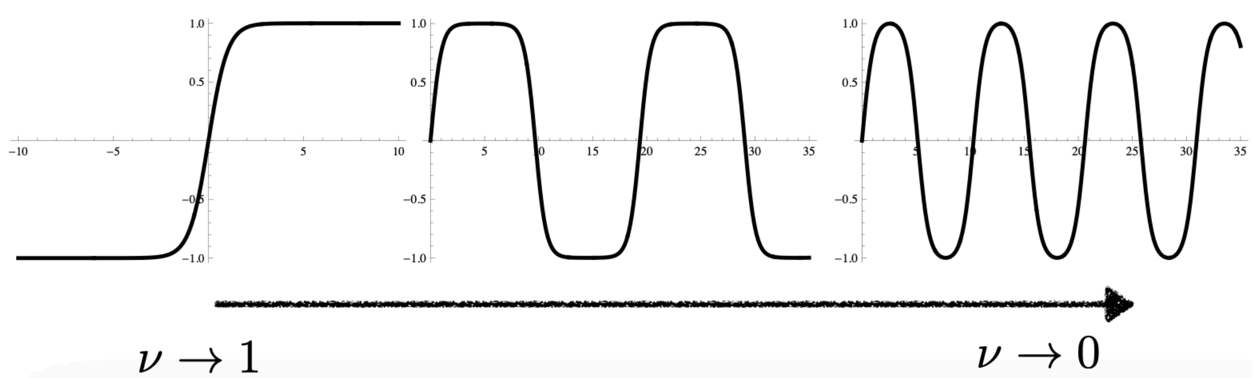
$$M_q = m - 2G\langle\bar{\psi}\psi\rangle$$

- Neglecting fluctuations, it is possible to obtain the free energy of the system as a trace over the inverse quark propagator:

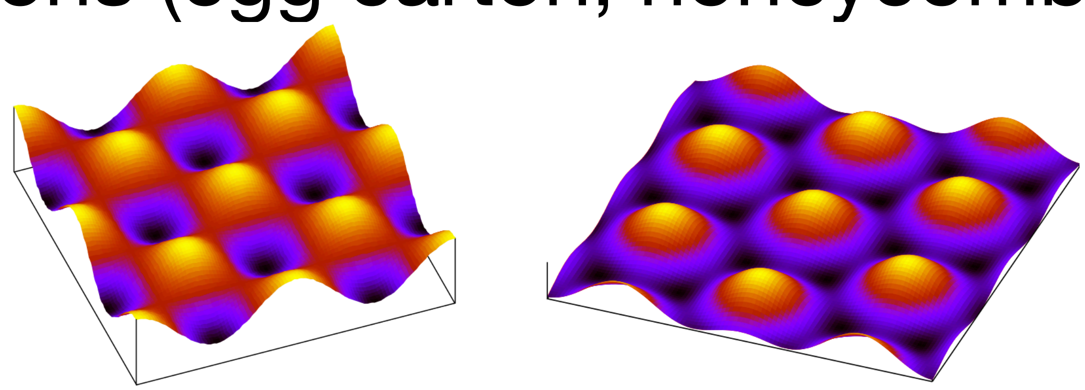
$$\Omega \sim \frac{T}{V}\mathrm{Tr}\log\left(\frac{S^{-1}(M_q)}{T}\right)$$

Favored shape of M(x)?

- Chiral density wave $M(\mathbf{x}) = \Delta e^{iqz}$
- Sinusoidal modulations $M(\mathbf{x}) = \Delta \cos(qz)$
- Real kink crystal $M(\mathbf{x}) = \Delta\nu\,\mathrm{sn}(\Delta z, \nu)$



- 2D modulations (egg carton, honeycomb..)

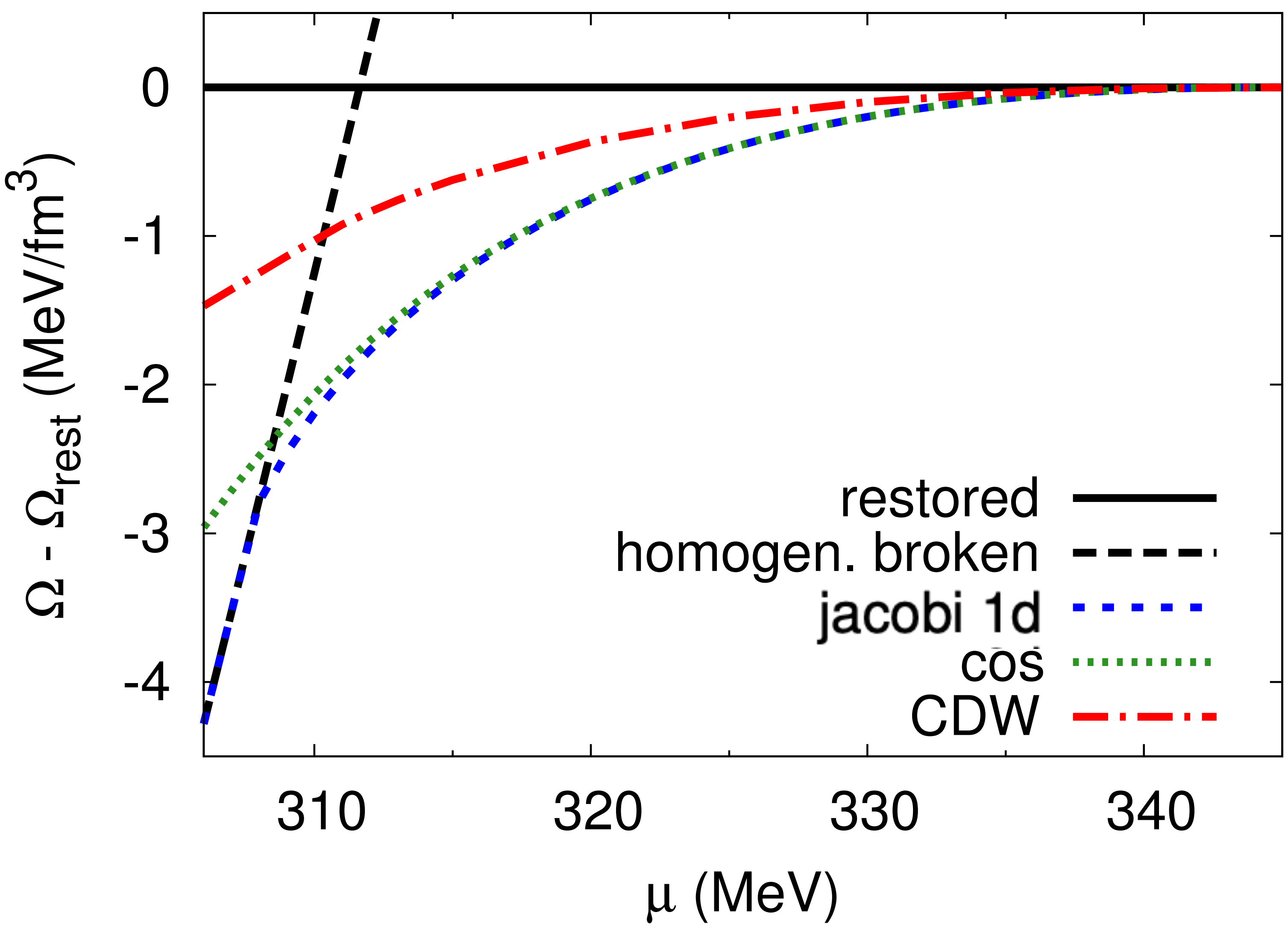


Free energy comparison

- What is the favored phase in the inhomogeneous window? compare free energies for different modulations at T=0

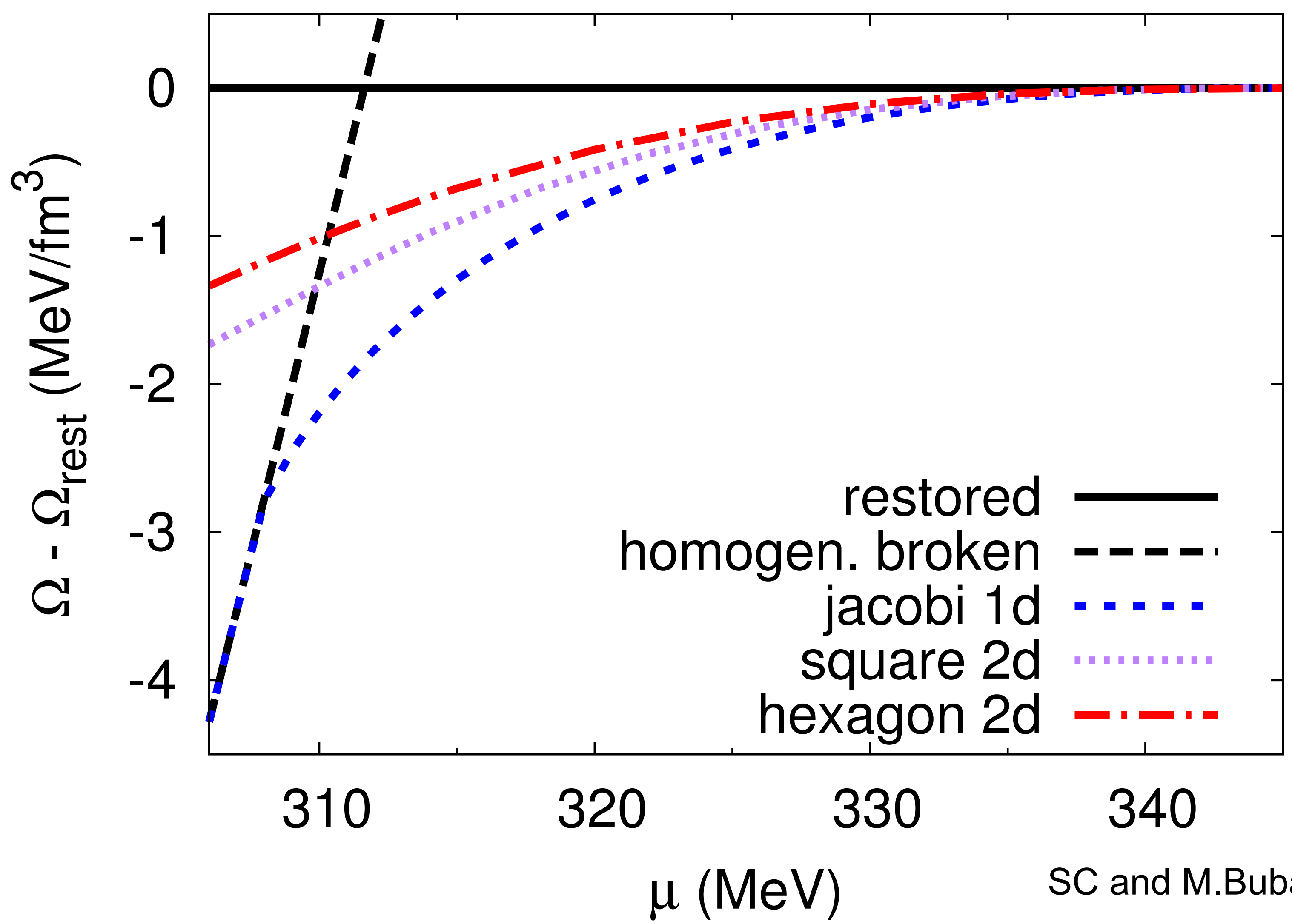
Free energy comparison

- What is the favored phase in the inhomogeneous window? compare free energies for different modulations at T=0. For 1D modulations…



Free energy comparison

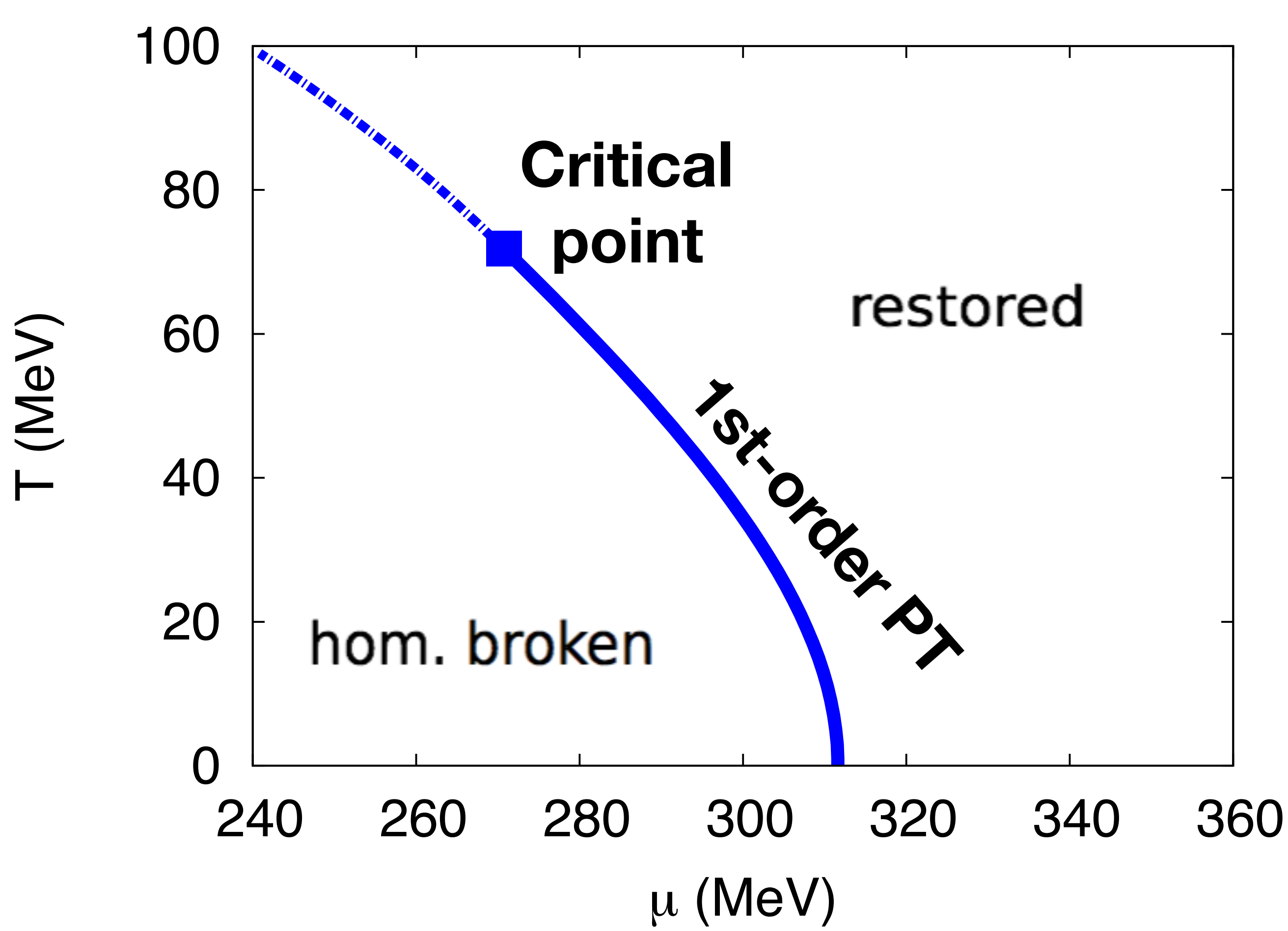
- What is the favored phase in the inhomogeneous window? compare free energies for different modulations at T=0. Including also 2D modulations..



SC and M.Buballa, Phys.Rev. D86 (2012) 074018

NJL phase diagram

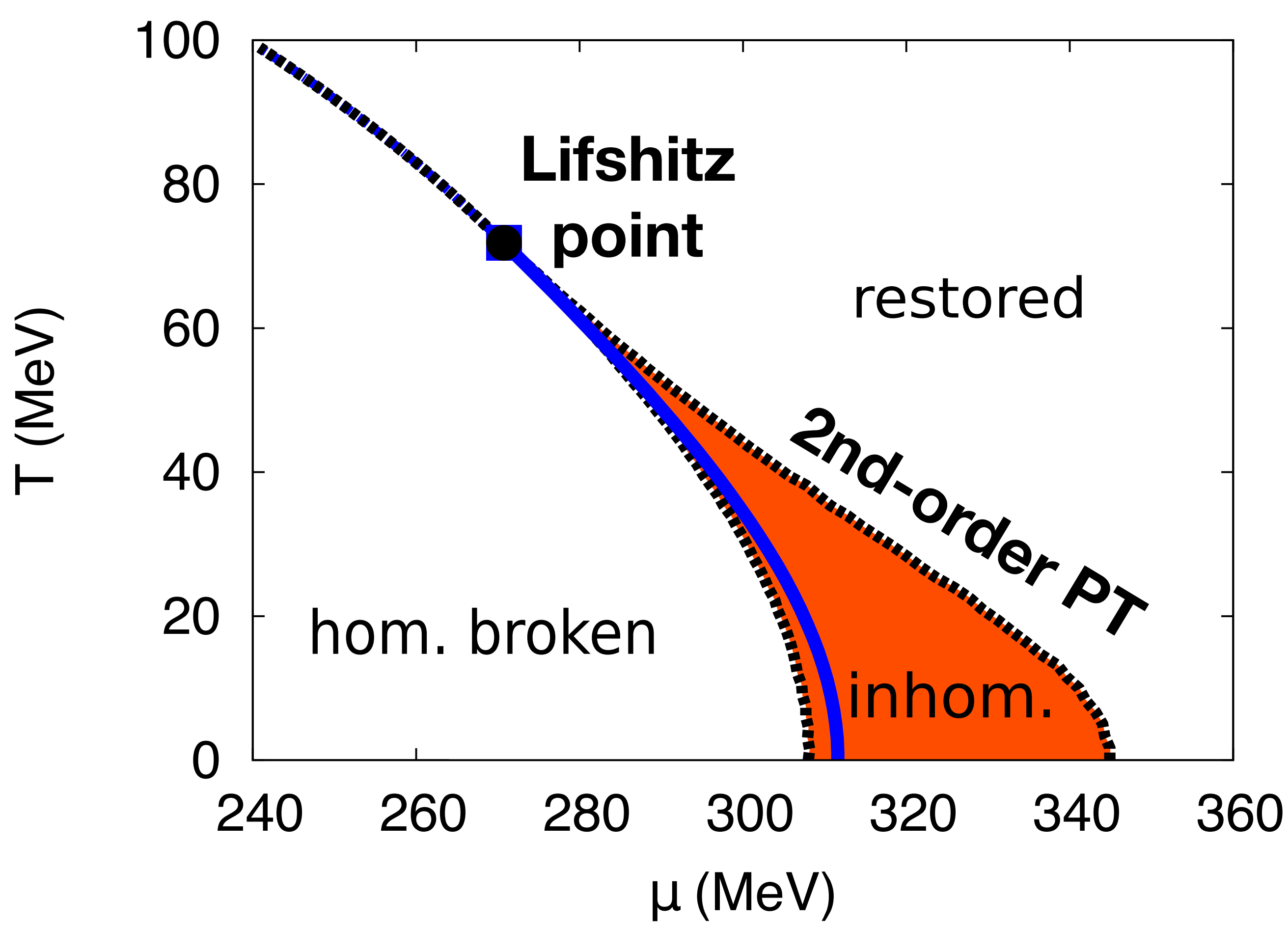
- Allowing for inhomogeneous phases, we go from this…



(chiral limit)

NJL phase diagram

- Allowing for inhomogeneous phases, we go …to this



(chiral limit)

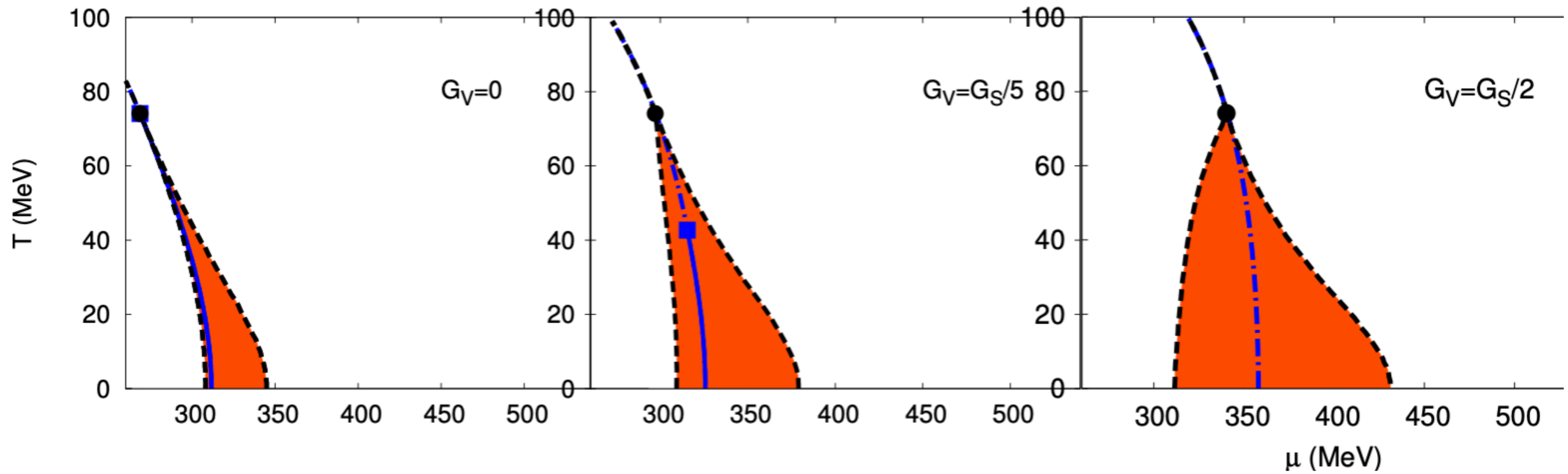
(some) Model extensions and inhomogeneous phases

Vector interactions

Repulsive vector interaction channel:

$$\mathcal{L} = \mathcal{L}_{NJL} - G_V(\bar{\psi}\gamma^\mu\psi)^2$$

Inhomogeneous phase strongly enlarged,
CP falls below the LP and inside the inhomogeneous phase



SC, D.Nickel and M.Buballa, Phys.Rev. D82 (2010) 054009

Density dependent: for n(x) can alter hierarchy of favored phases

SC, M.Schramm and M.Buballa, Phys. Rev. D 98, 014033 (2018)

Going away from the chiral limit

Less straightforward: in the restored phase $M = M_0 \neq 0$

Issues of self-consistency with some solutions (eg. CDW)

-> Work within a modulation-agnostic GL approach:
expand around $M(\mathbf{x}) = M_0 + \delta M(\mathbf{x})$

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_1 \delta M(\mathbf{x}) + \alpha_2 \delta M^2(\mathbf{x}) + \alpha_3 \delta M^3(\mathbf{x}) + \alpha_{4,a}\delta M^4(\mathbf{x}) + \alpha_{4,b}(\nabla \delta M(\mathbf{x}))^2 + \dots\right)$$

M.Buballa and SC, Phys. Lett. B791, 361 (2019)

Going away from the chiral limit

Less straightforward: in the restored phase $M = M_0 \neq 0$

Issues of self-consistency with some solutions (eg. CDW)

-> Work within a modulation-agnostic GL approach:
expand around $M(\mathbf{x}) = M_0 + \delta M(\mathbf{x})$

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_1 \delta M(\mathbf{x}) + \alpha_2 \delta M^2(\mathbf{x}) + \alpha_3 \delta M^3(\mathbf{x}) + \alpha_{4,a}\delta M^4(\mathbf{x}) + \alpha_{4,b}(\nabla \delta M(\mathbf{x}))^2 + \dots\right)$$

$$\text{CEP}: \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$$
$$\text{PLP}: \quad \alpha_1 = \alpha_2 = \alpha_{4,b} = 0$$

→ CP and LP split?

M.Buballa and SC, Phys. Lett. B791, 361 (2019)

Going away from the chiral limit

Less straightforward: in the restored phase $M = M_0 \neq 0$

Issues of self-consistency with some solutions (eg. CDW)

-> Work within a modulation-agnostic GL approach:
expand around $M(\mathbf{x}) = M_0 + \delta M(\mathbf{x})$

$$\Omega[M] = \Omega[M_0] + \frac{1}{V}\int d^3x \left(\alpha_1 \delta M(\mathbf{x}) + \alpha_2 \delta M^2(\mathbf{x}) + \alpha_3 \delta M^3(\mathbf{x}) + \alpha_{4,a}\delta M^4(\mathbf{x}) + \alpha_{4,b}(\nabla \delta M(\mathbf{x}))^2 + \dots\right)$$

$$\mathrm{CEP}: \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$$\mathrm{PLP}: \quad \alpha_1 = \alpha_2 = \alpha_{4,b} = 0$$

CP and LP split? No!

$$\alpha_3 = 4M_0\,\alpha_{4,b}$$

M.Buballa and SC, Phys. Lett. B791, 361 (2019)

Three-flavor quark matter

Add strange quarks with KMT interaction

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - \hat{m}\right)\psi + \mathcal{L}_4 + \mathcal{L}_6$$

$$\mathcal{L}_4 = G\sum_{a=0}^{8}\left[(\bar{\psi}\tau_a\psi)^2 + (\bar{\psi}i\gamma_5\tau_a\psi)^2\right]$$

$$\mathcal{L}_6 = -K\left[\det{}_f\bar{\psi}(1+\gamma_5)\psi + \det{}_f\bar{\psi}(1-\gamma_5)\psi\right]$$

Again modulation-agnostic GL expansion:

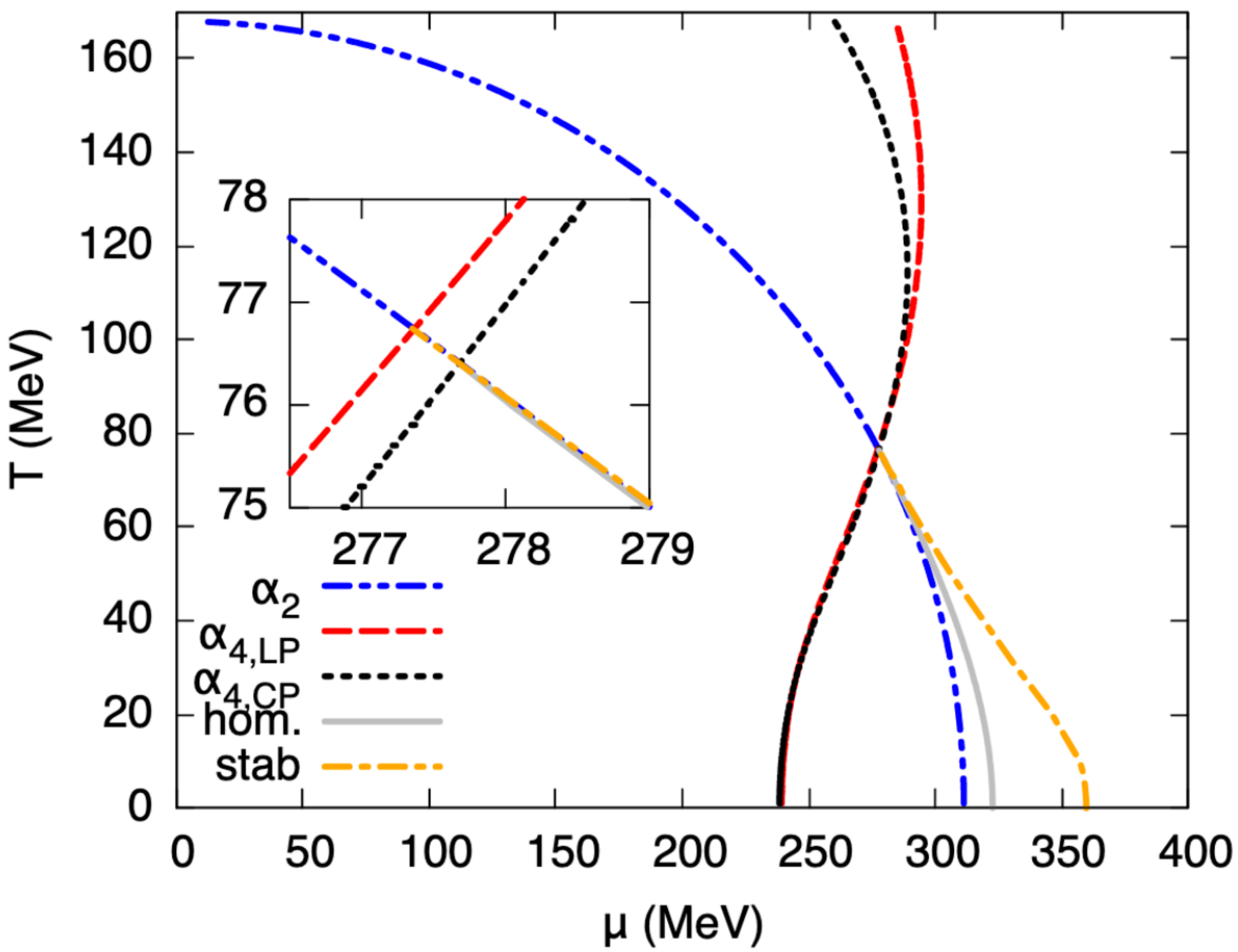
$$\begin{aligned}\omega_{GL}(\Delta_\ell, \Delta_s) = \quad & \alpha_2|\Delta_\ell|^2 + \alpha_{4,a}|\Delta_\ell|^4 + \alpha_{4,b}|\nabla\Delta_\ell|^2 + \dots \\ & +\beta_1\Delta_s + \beta_2\Delta_s^2 + \beta_3\Delta_s^3 + \beta_{4,a}\Delta_s^4 + \beta_{4,b}(\nabla\Delta_s)^2 + \dots \\ & +\gamma_3|\Delta_\ell|^2\Delta_s + \gamma_4|\Delta_\ell|^2\Delta_s^2 + \dots ,\end{aligned}$$

SC and M.Buballa, Phys. Rev. D 101, 014026 (2020)

Three-flavor quark matter

CP and LP split!

For a reasonable parameter set, LP **above** CP



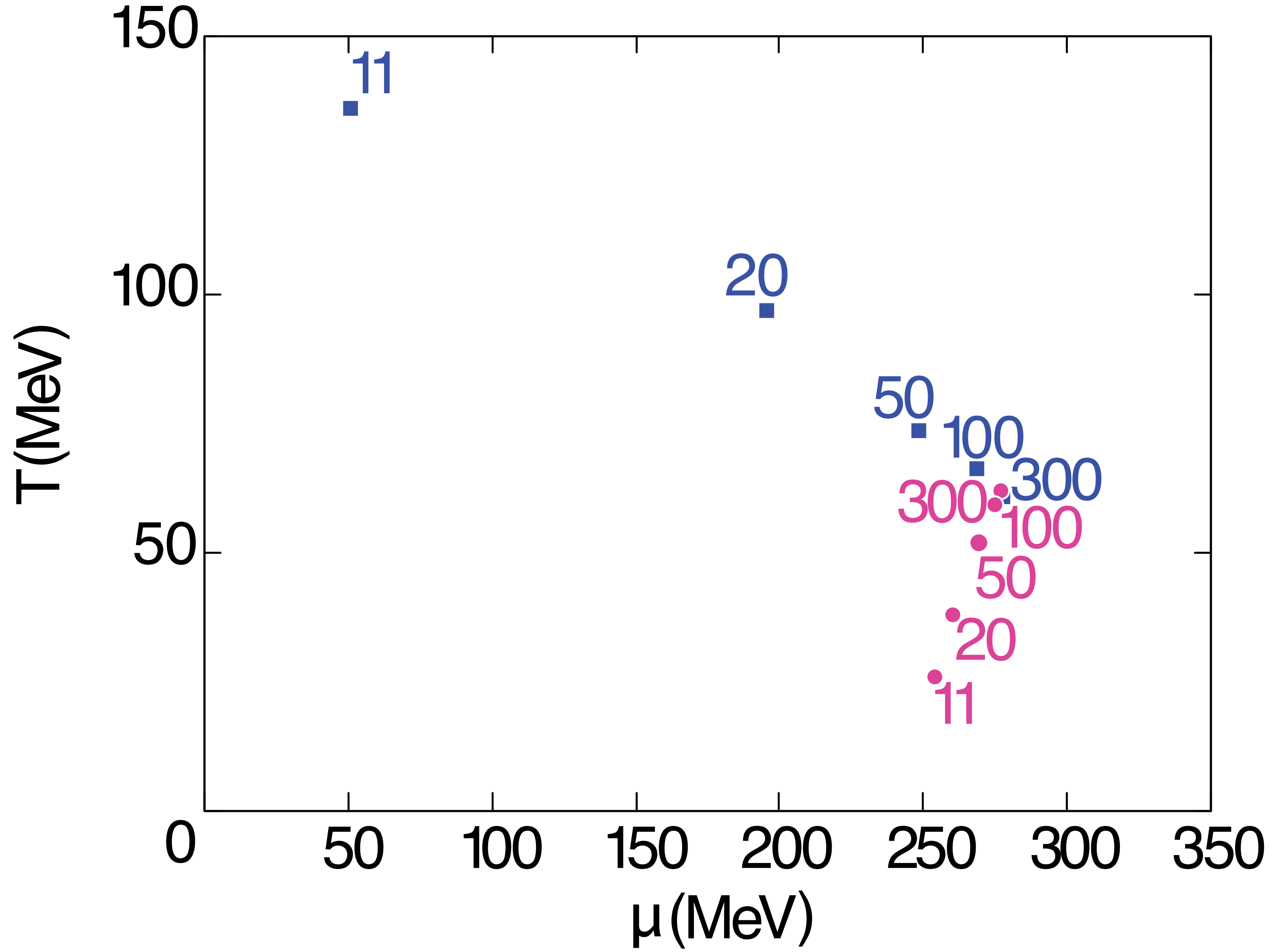
SC and M.Buballa, Phys. Rev. D 101, 014026 (2020)

Varying the current strange quark mass

Lowering ms:

CP moves to the T axis

LP doesn't follow!



SC and M.Buballa, Phys. Rev. D 101, 014026 (2020)

Fluctuations?
Quark-meson model

$$\mathcal{L}_{\mathrm{QM}} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - g(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\right)\psi + \mathcal{L}_{\mathrm{M}}^{\mathrm{kin}} - U(\sigma,\vec{\pi})$$

$$\mathcal{L}_{\mathrm{M}}^{\mathrm{kin}} = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec{\pi}\partial^\mu\vec{\pi}\right)$$

$$U(\sigma,\vec{\pi}) = \frac{\lambda}{4}\left(\sigma^2 + \vec{\pi}^2 - v^2\right)^2$$

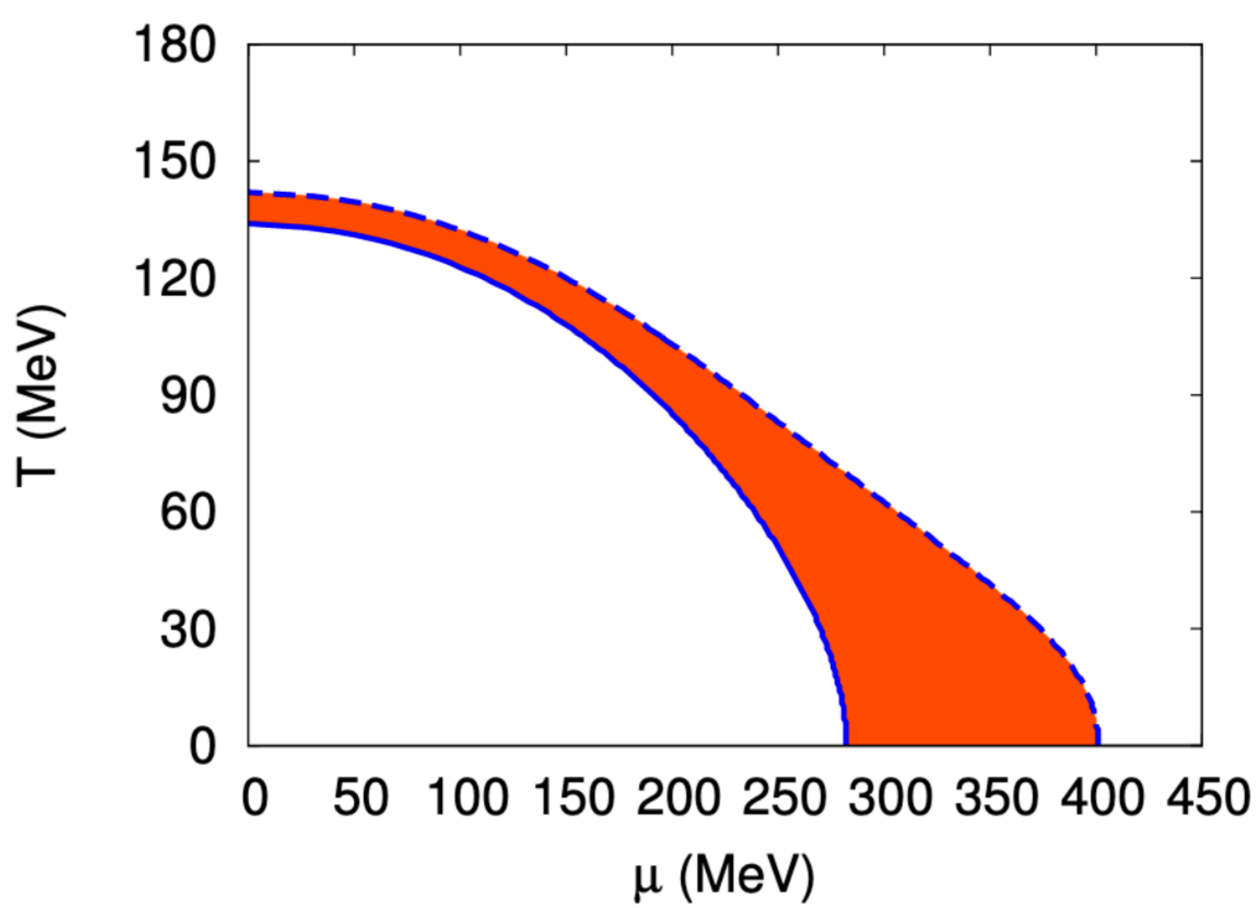
Formally quite similar to the NJL model in MFA
However, renormalizable!

A better starting point to study the role of fluctuations

The role of quark vacuum fluctuations

- Until few years ago typically discarded
- As a consequence, unrealistic phase structure already for homogeneous phases (no CP)

For inhomogeneous phases:



Inhomogeneous at $\mu = 0$?!
Not very likely…

What happens if we include vacuum quark fluctuations?

The role of quark vacuum fluctuations

Including quark fluctuations requires special care in defining vacuum parameters to fit to physical quantities

Inconsistent definitions lead to inconsistent phenomenology, especially (but not only) when dealing with inhomogeneous phases!

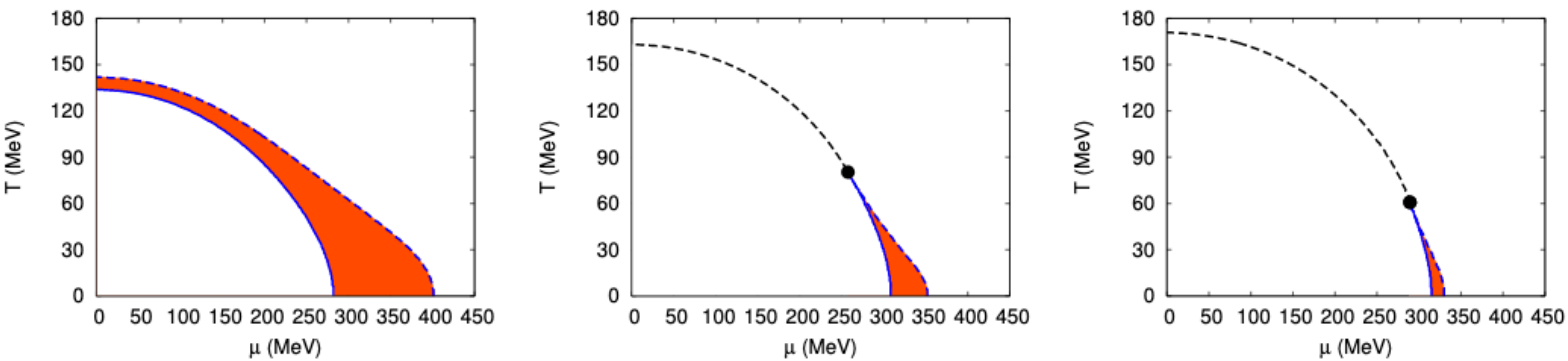
Fit to **pole** masses and **renormalized** decay constants

(instead of the commonly used curvature masses - bare decay constant)

SC, M. Buballa and B-J. Schaefer, Phys.Rev. D90 (2014) 014033
SC, M. Buballa and W. Elkamhawy, Phys.Rev. D94 (2016) 034023

The role of quark vacuum fluctuations

Including fluctuations (“extended” mean-field approximation)

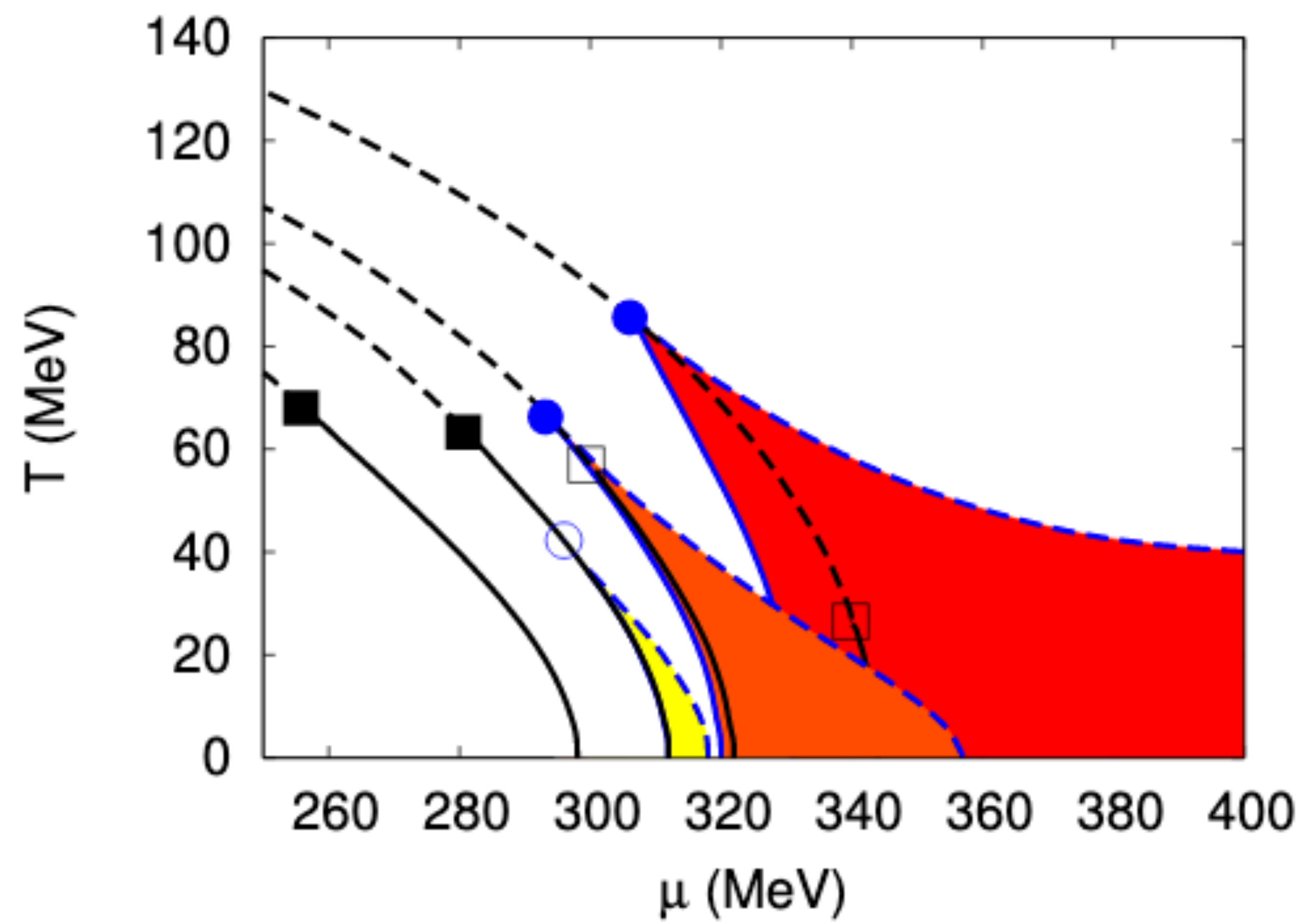


$\Lambda = 0$ $\qquad$ $\Lambda = 600\mathrm{MeV}$ $\qquad$ $\Lambda \to \infty$

CP=LP if $m_\sigma = 2M_{vac}$

SC, M. Buballa and B-J. Schaefer, Phys.Rev. D90 (2014) 014033

Sigma mass influence



Renormalized limit, $m_\sigma =$ 550,590,610,650 MeV

(M_vac = 300 MeV)

SC, M. Buballa and B-J. Schaefer, Phys.Rev. D90 (2014) 014033

QM away from chiral limit

Include explicit chiral symmetry breaking piece:

$$U(\sigma, \boldsymbol{\pi}) = \frac{\lambda}{4}\left(\sigma^2 + \boldsymbol{\pi}^2 - v^2\right)^2 - c\sigma$$

Does the inhomogeneous phase survive?

CDW disfavored at physical pion mass!

J. Andersen and P. Kneschke, Phys. Rev. D 97, 076005 (2018)

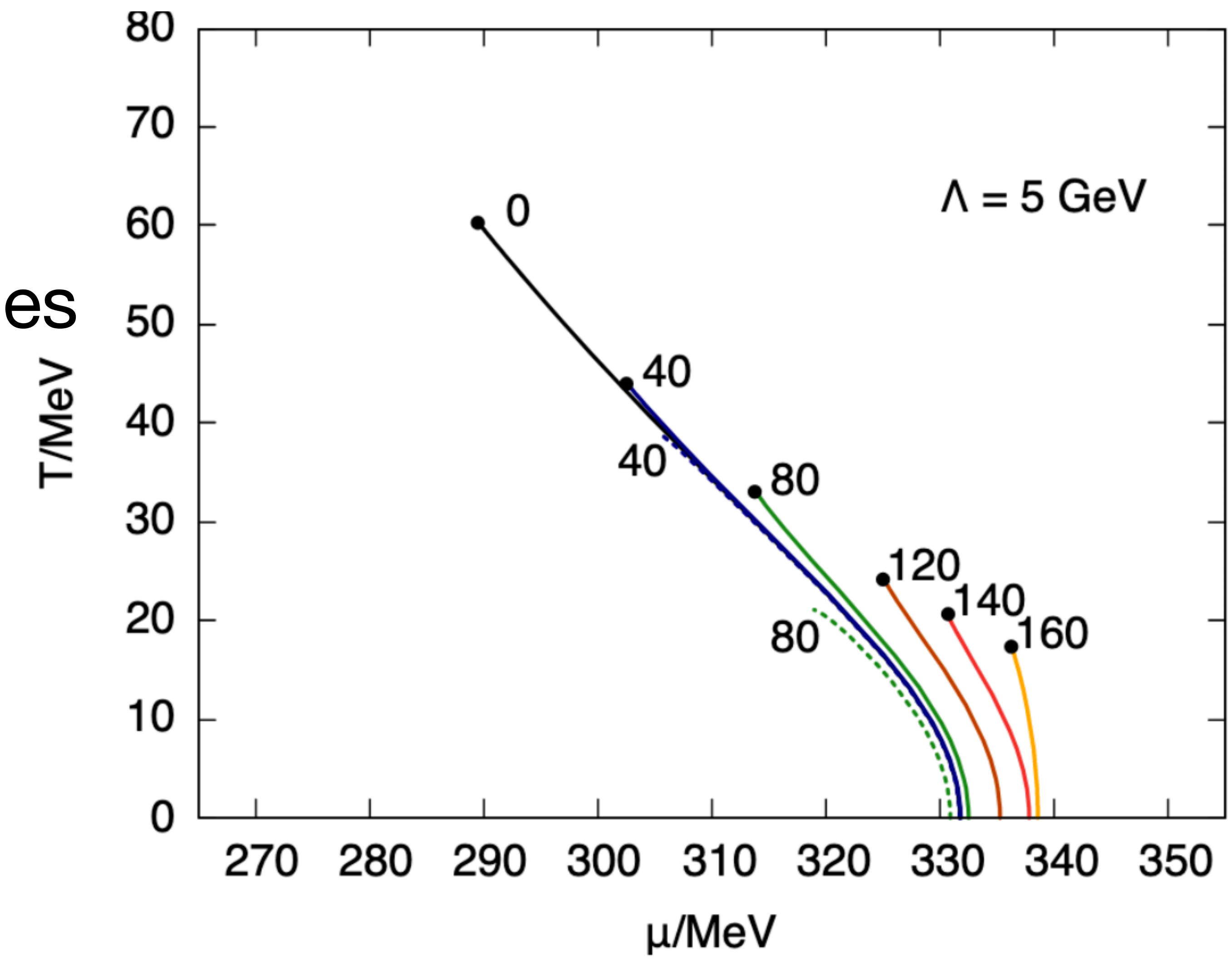
QM away from chiral limit

Include explicit chiral symmetry breaking piece:

$$U(\sigma, \boldsymbol{\pi}) = \frac{\lambda}{4}\left(\sigma^2 + \boldsymbol{\pi}^2 - v^2\right)^2 - c\sigma$$

Stability analysis:
splitting of
scalar-pseudoscalar instability lines

Instability in the scalar channel
survives at physical pion mass!



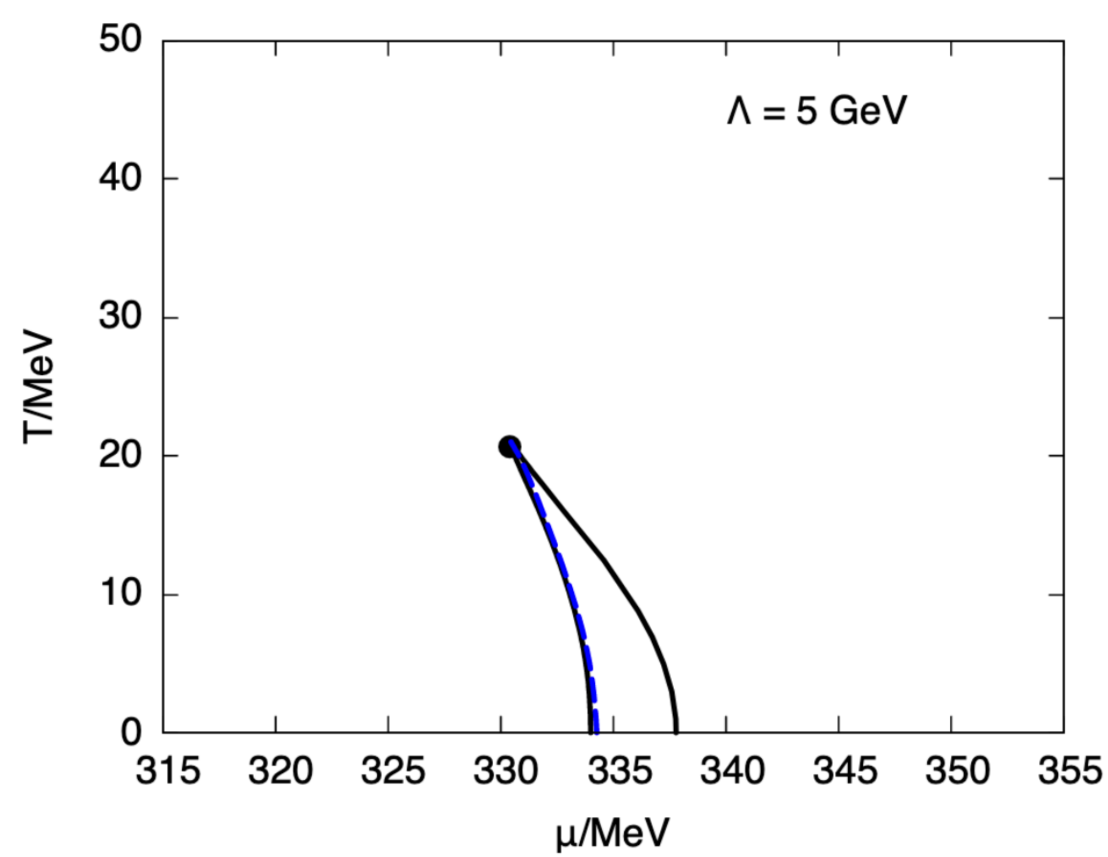
M. Buballa, SC and L. Kurth, arXiv:2006.02133 , EPJ ST 229 (2020)

QM away from chiral limit

Explicit calculation with RKC ansatz:

$$g\sigma(z) \equiv M(z) = \Delta\nu\left[\mathrm{sn}(\Delta z,\nu)\mathrm{sn}(\Delta z + b,\nu)\mathrm{sn}(b,\nu) + \frac{\mathrm{cn}(b,\nu)\mathrm{dn}(b,\nu)}{\mathrm{sn}(b,\nu)}\right]$$

Inhomogeneous phase persists
in the renormalized limit
at physical pion mass

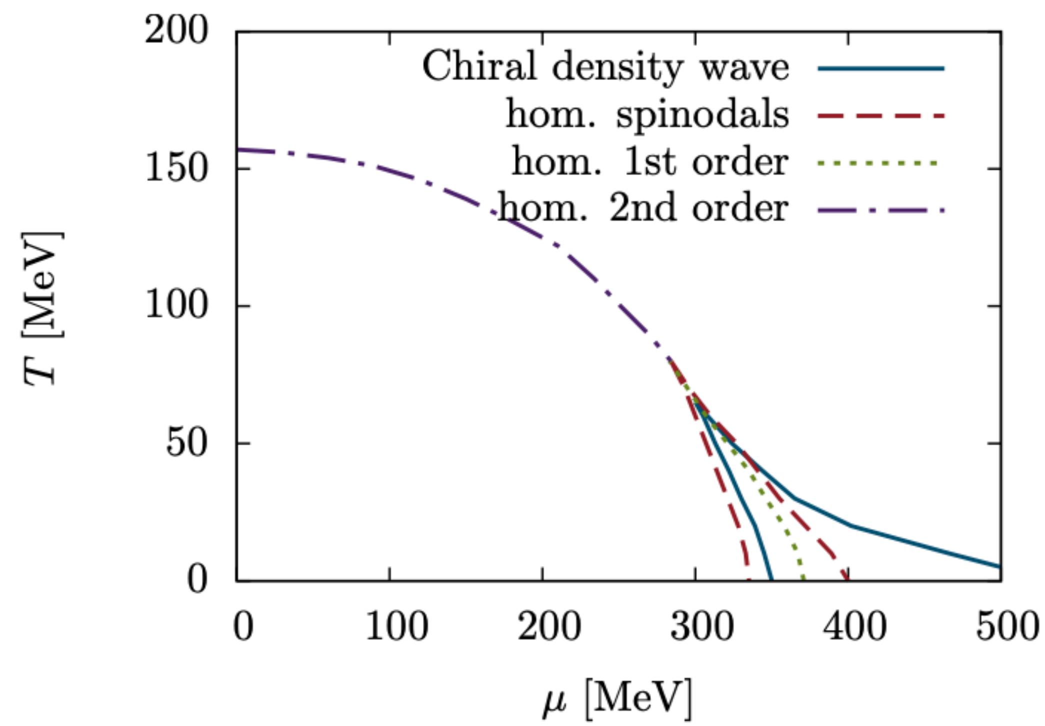


M. Buballa, SC and L. Kurth, arXiv:2006.02133 , EPJ ST 229 (2020)

One might wonder…

Could inhomogeneous phases be a model “artifact”
appearing in simplified quark models?

They appear also
in Dyson-Schwinger studies



D.Müller, M.Buballa and J.Wambach,
Phys.Lett. B727 (2013) 240

FRG approaches also seem to hint at their existence!

W.Fu, J.Pawlowski, F.Rennecke, *Phys.Rev.D* 101 (2020) 5, 054032

Some outlooks

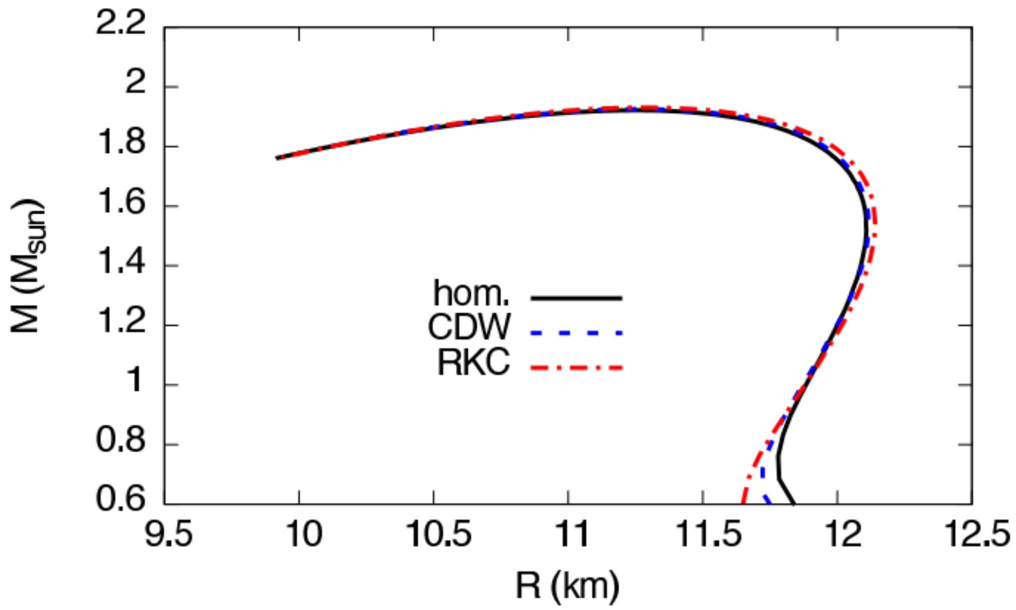
From theory side:
Investigate role of mesonic fluctuations

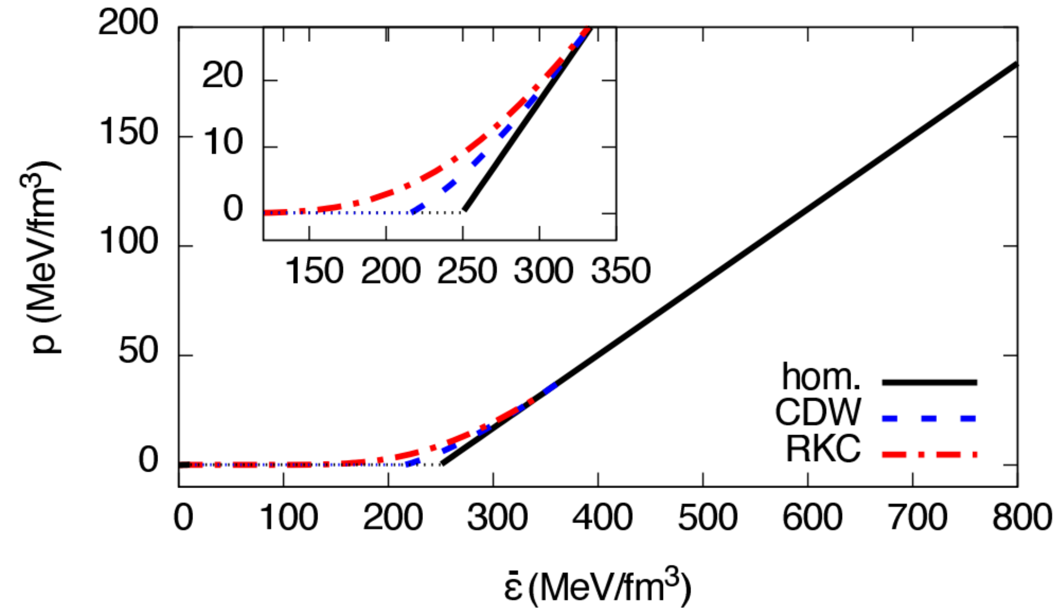
From experiment/observation side:
Investigate possible signatures

Compact star phenomenology?

Compact star phenomenology?

Equation of state/Mass-radius?

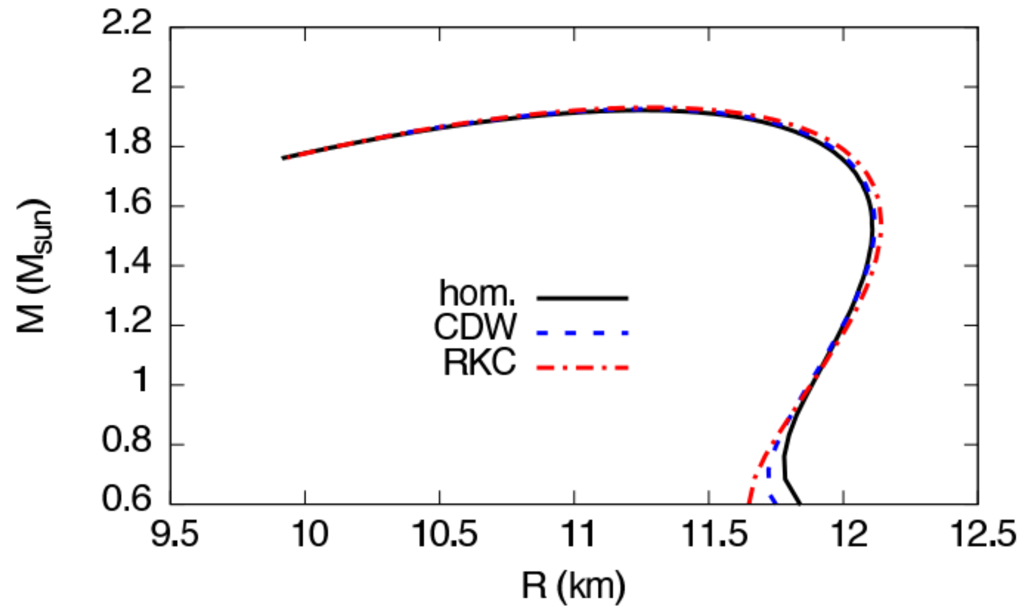


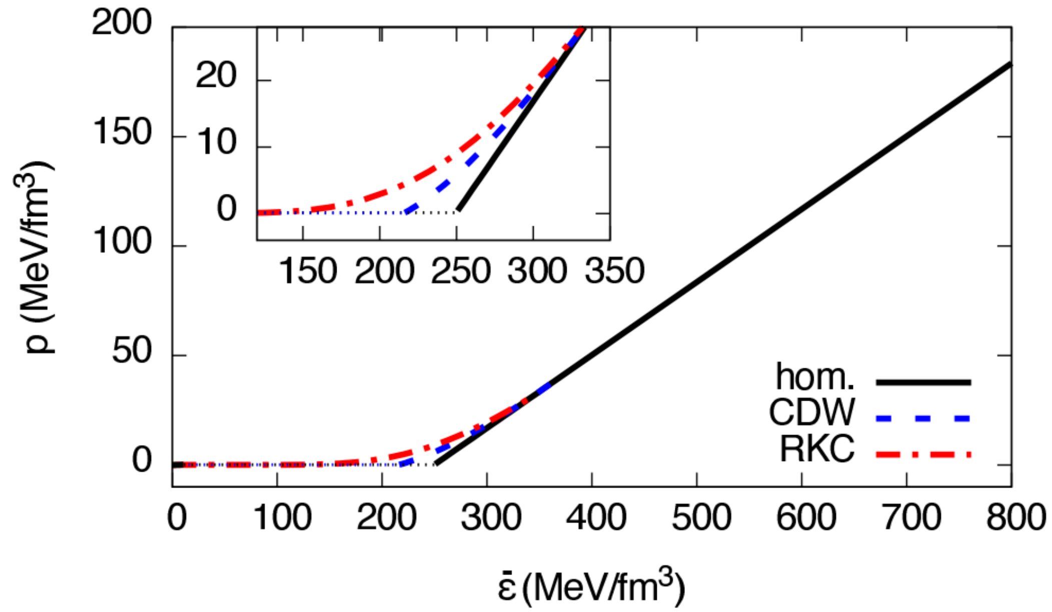


M. Buballa and SC, EPJ A52 (2016) 57

Compact star phenomenology?

Equation of state/Mass-radius?



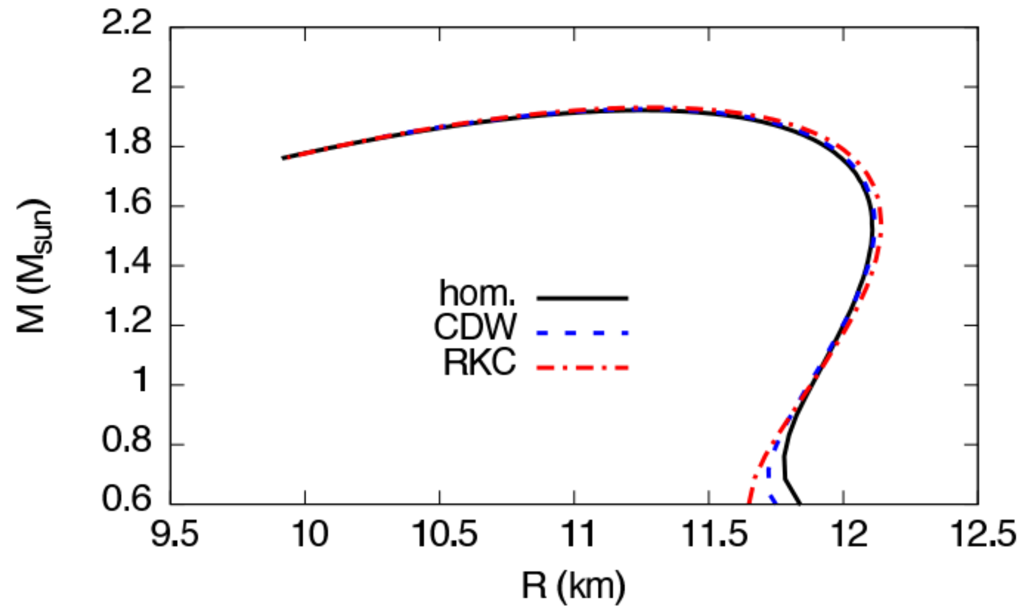


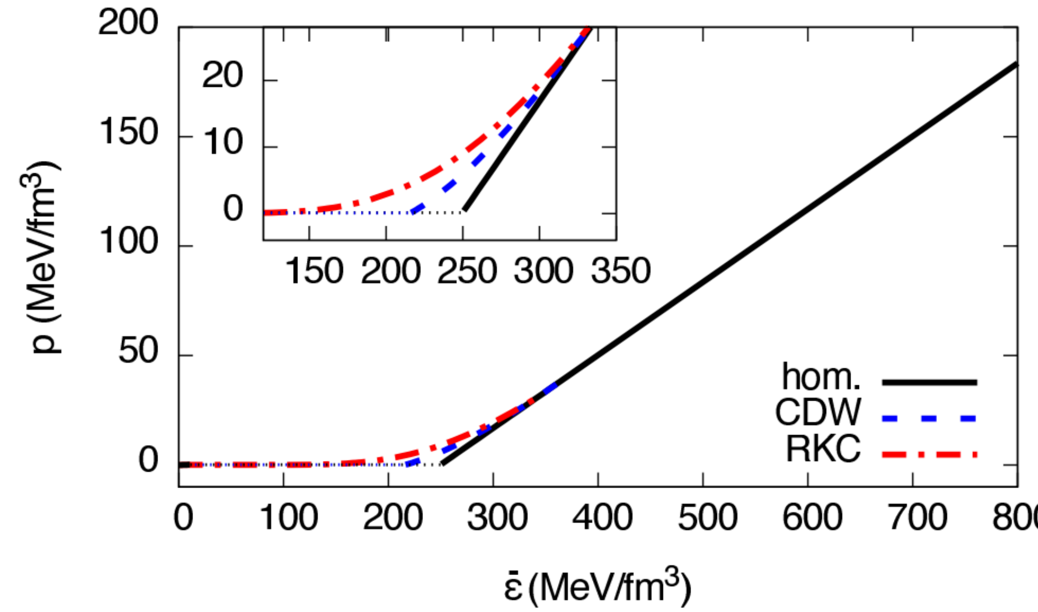
M. Buballa and SC, EPJ A52 (2016) 57

Neutrino emissivity ?

Compact star phenomenology?

Equation of state/Mass-radius?



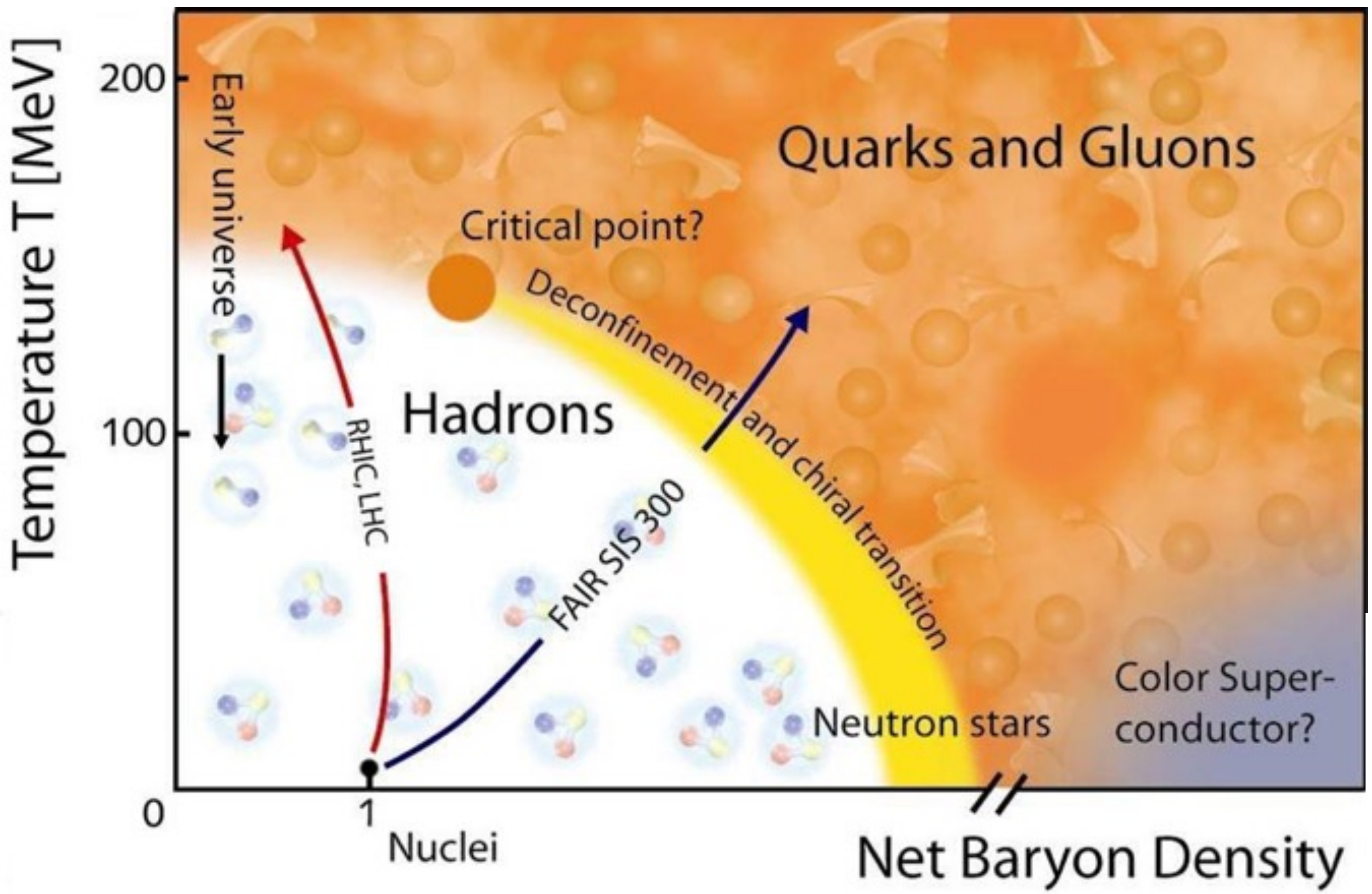


M. Buballa and SC, EPJ A52 (2016) 57

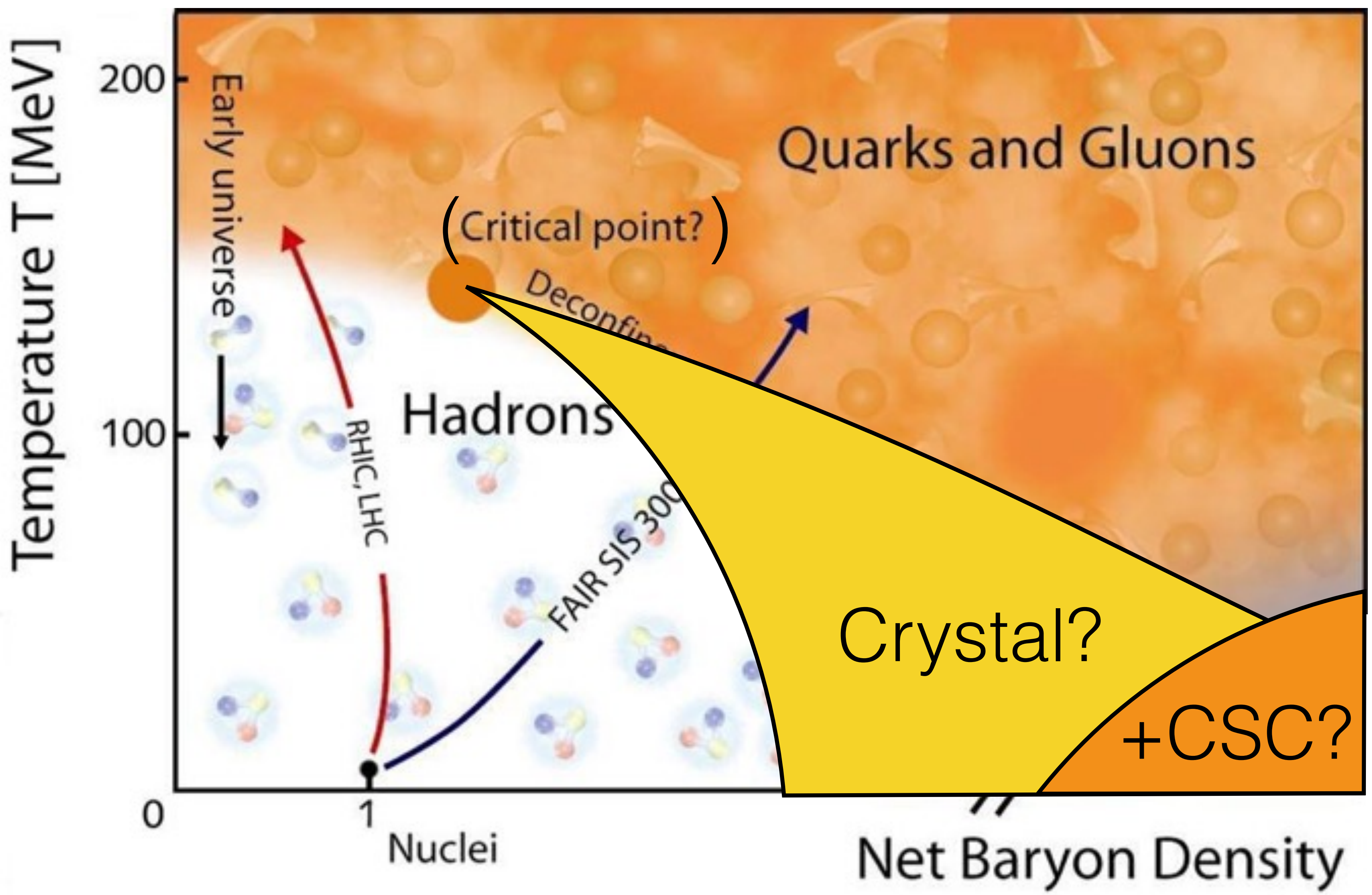
Neutrino emissivity ?

Rigidity - Gravitational waves !?

The QCD phase diagram people have in mind



The QCD phase diagram people *should* have in mind



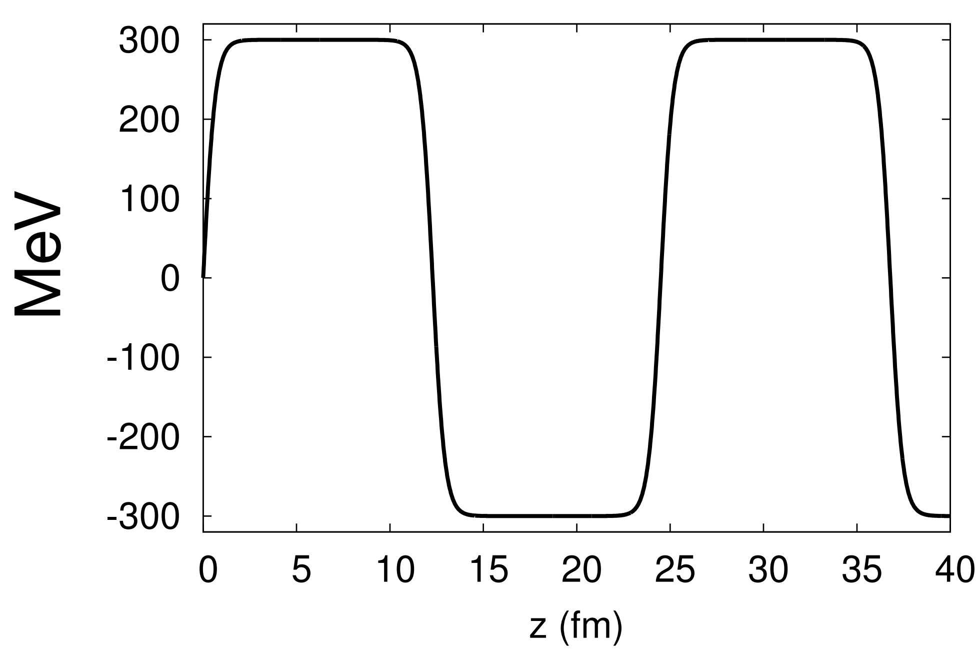
Reviews: M.Buballa and SC,
Prog.Part.Nucl.Phys. 81 (2015) 39 - arXiv:1406.1367
Eur.Phys.J. A52 (2016) 57 - arXiv:1508.04361

Condensate and density

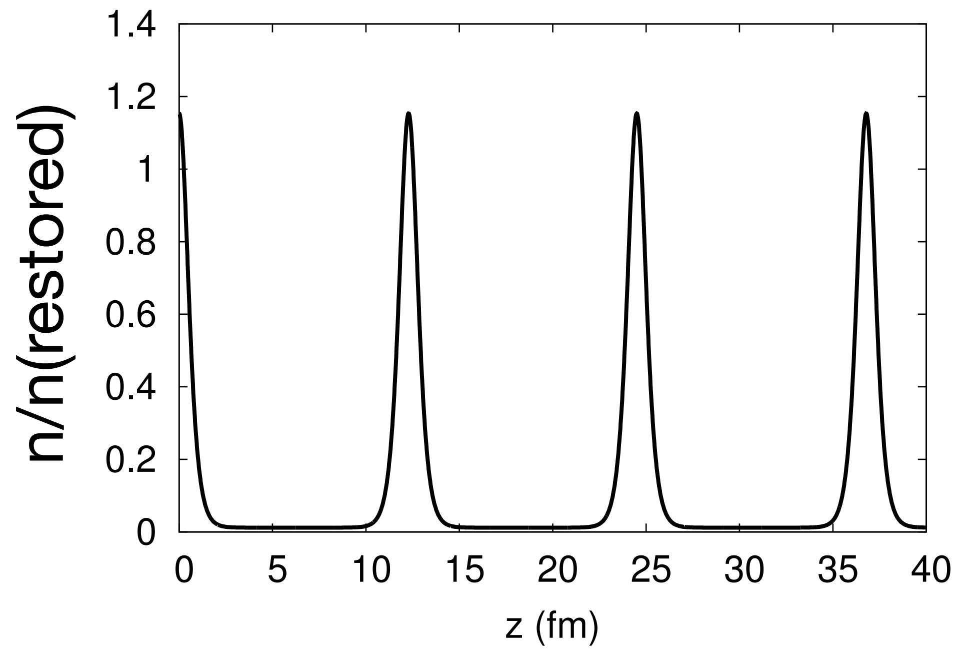
- If the chiral condensate is spatially modulated, the density of the system becomes inhomogeneous as well

- For the real kink crystal

$$M(z) \sim \langle \bar{\psi}\psi \rangle$$



$$n(z) \sim \langle \psi^\dagger \psi \rangle$$



$$\mu \sim 308 \mathrm{M}eV$$

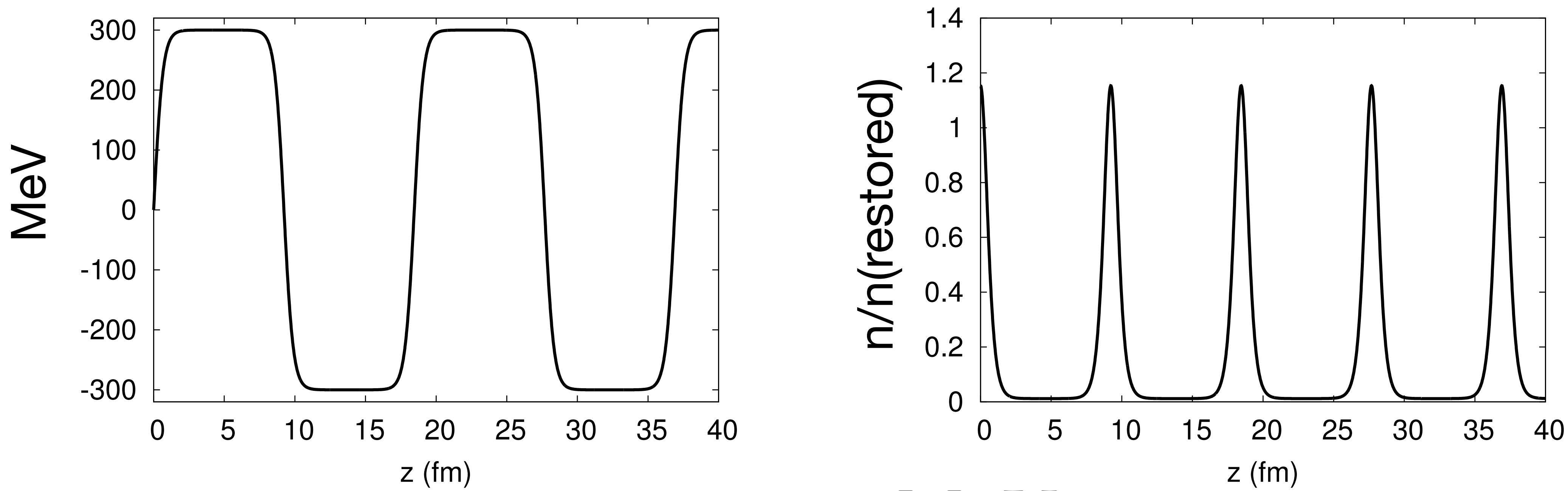
SC, D,Nickel and M.Buballa, Phys.Rev. D82 (2010) 054009
M.Buballa and SC, Phys.Rev. D87 (2013) no.5, 054004

Condensate and density

- If the chiral condensate is spatially modulated, the density of the system becomes inhomogeneous as well

- For the real kink crystal

$$M(z) \sim \langle \bar{\psi}\psi \rangle \qquad n(z) \sim \langle \psi^\dagger \psi \rangle$$



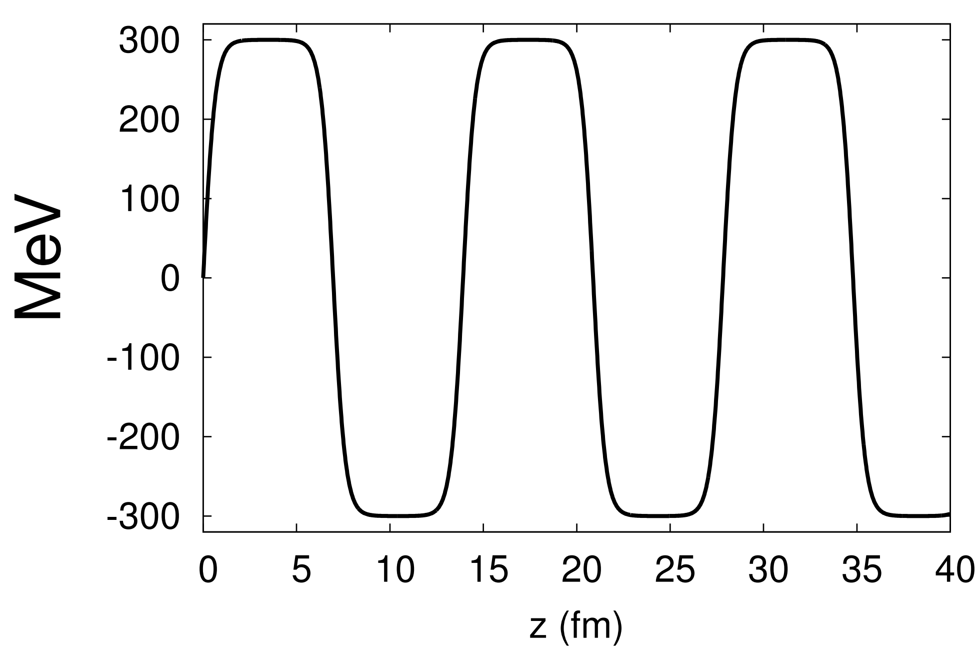
$$\mu \sim 309 MeV$$

Condensate and density

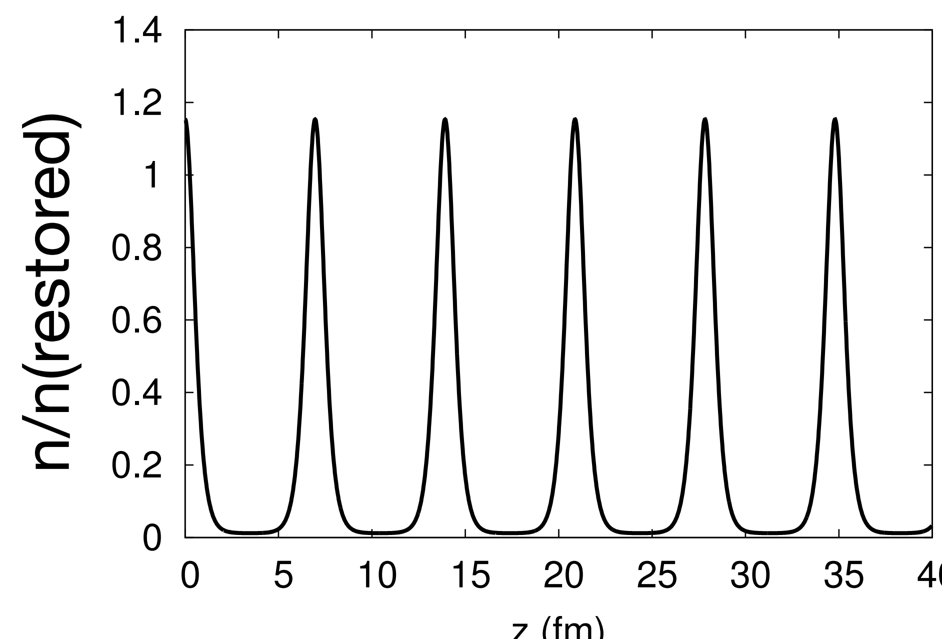
- If the chiral condensate is spatially modulated, the density of the system becomes inhomogeneous as well

- For the real kink crystal

$$M(z) \sim \langle \bar{\psi}\psi \rangle$$



$$n(z) \sim \langle \psi^\dagger \psi \rangle$$



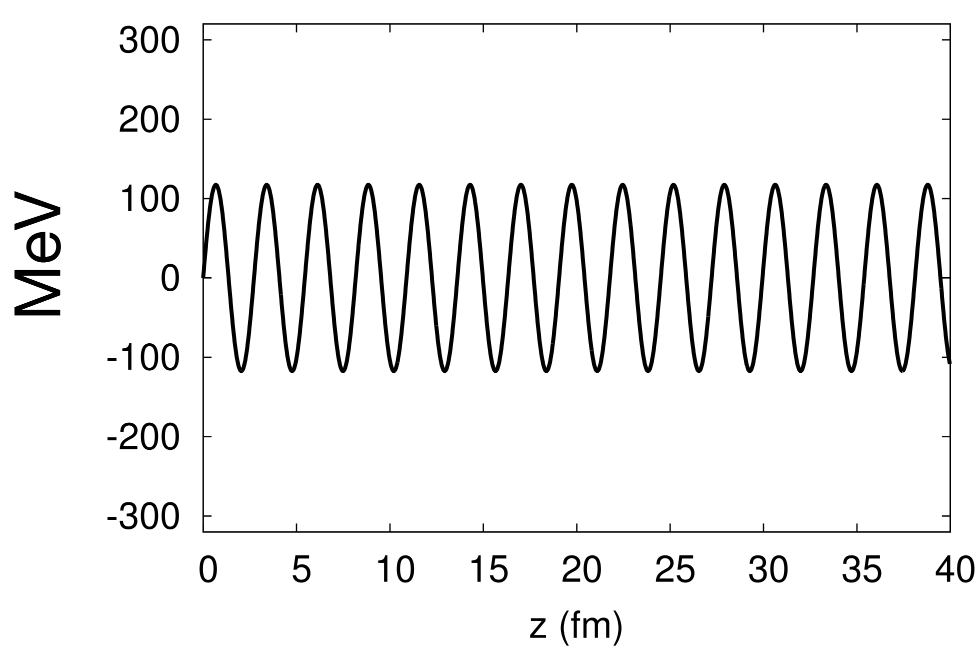
$$\mu \sim 310MeV$$

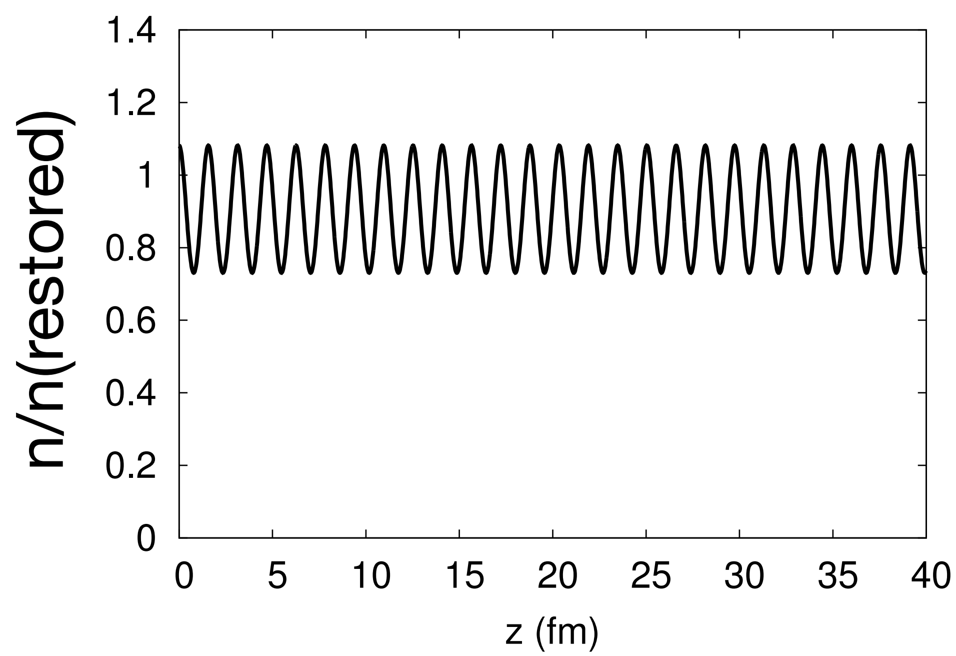
Condensate and density

- If the chiral condensate is spatially modulated, the density of the system becomes inhomogeneous as well

- For the real kink crystal

$$M(z) \sim \langle \bar{\psi}\psi \rangle \qquad n(z) \sim \langle \psi^\dagger \psi \rangle$$





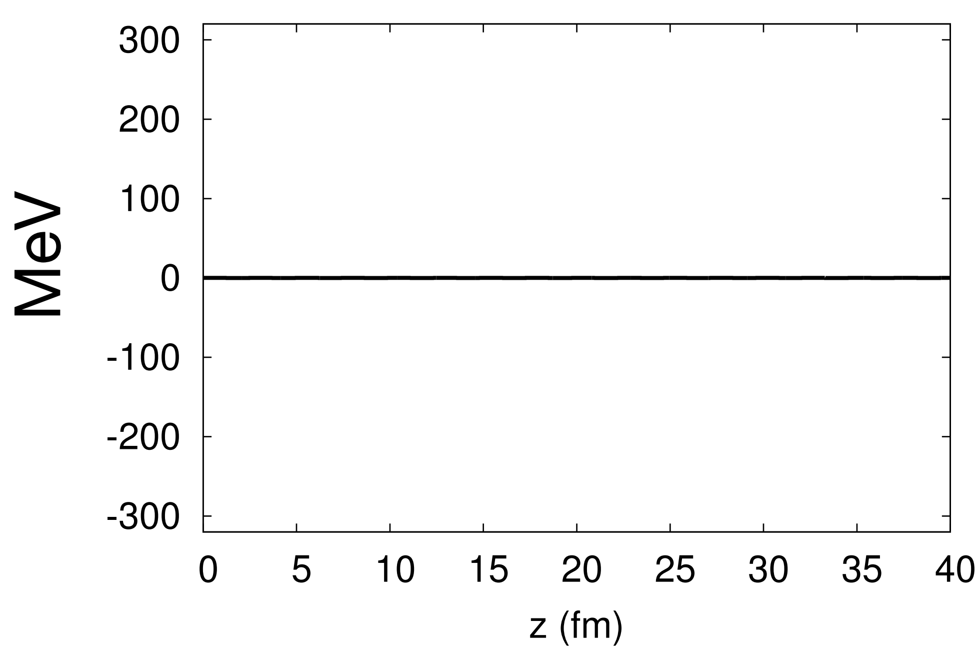
$$\mu \sim 320\mathrm{M}eV$$

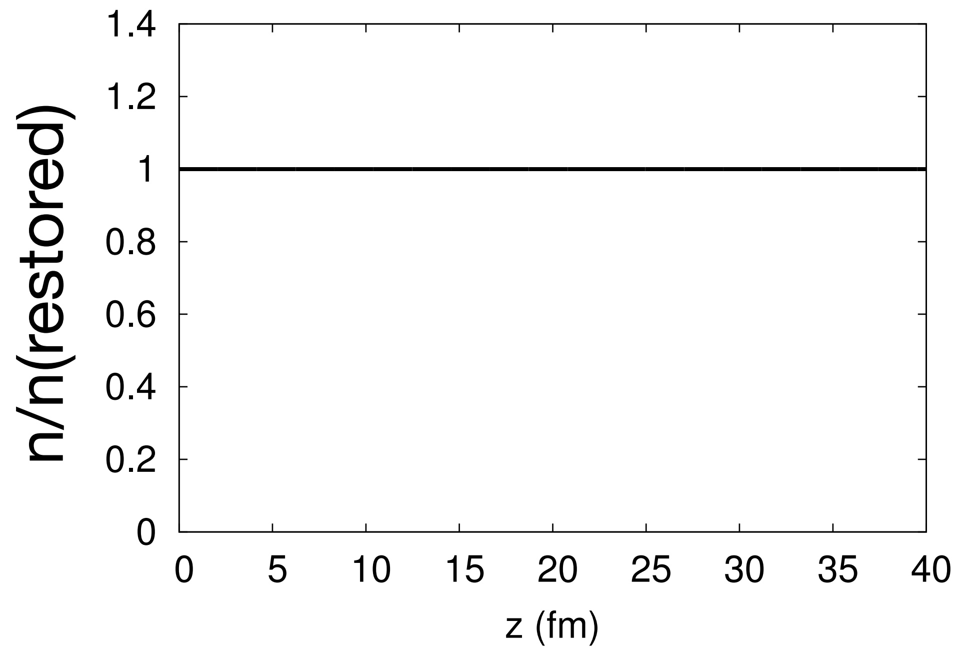
Condensate and density

- If the chiral condensate is spatially modulated, the density of the system becomes inhomogeneous as well

- For the real kink crystal

$$M(z) \sim \langle \bar{\psi}\psi \rangle \qquad n(z) \sim \langle \psi^\dagger \psi \rangle$$





$$\mu \sim 350\mathrm{M}eV$$

Inhomogeneous chiral condensates in NJL

- Allow for a spatially modulated chiral condensate

$$\langle \bar{\psi}\psi \rangle = S(\mathbf{x}) \qquad \langle \bar{\psi} i\gamma^5 \tau_a \psi \rangle = P_a(\mathbf{x})$$

(we can also build $M(\mathbf{x}) = -2G(S(\mathbf{x}) + iP_3(\mathbf{x}))$ (Chiral limit)

- Diagonalize the mean-field quark Hamiltonian in momentum space

$$\mathcal{H}_{\vec{p}_m,\vec{p}_n} = \begin{pmatrix} -\vec{\sigma}\cdot\vec{p}_m\,\delta_{\vec{p}_m,\vec{p}_n} & \sum_{\vec{q}_k} M_{\vec{q}_k}\,\delta_{\vec{p}_m,\vec{p}_n+\vec{q}_k} \\ \sum_{\vec{q}_k} M^*_{\vec{q}_k}\,\delta_{\vec{p}_m,\vec{p}_n-\vec{q}_k} & \vec{\sigma}\cdot\vec{p}_m\,\delta_{\vec{p}_m,\vec{p}_n} \end{pmatrix}$$

Inhomogeneous chiral condensates in NJL

- Then, minimize the thermodynamic potential

$$\Omega(T,\mu;M(\vec{x})) = -\frac{T}{V}\mathrm{Log}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\,\exp\left(\int_{x\in[0,\frac{1}{T}]\times V}(\mathcal{L}_{MF}+\mu\bar{\psi}\gamma^0\psi)\right)$$

$$= -\frac{TN_c}{V}\sum_n \mathrm{Tr}_{D,f,V}\,\mathrm{Log}\left(\frac{1}{T}(i\omega_n+\mathcal{H}_{MF}-\mu)\right)+\frac{1}{V}\int_V \frac{|M(\vec{x})-m|^2}{4G_s}$$

with respect to the mass function M(x)

- Not so easy for an arbitrary M(x) !

- To make the problem tractable, assume specific ansatz for the functional form of M

- So far: results tied to specific Ansätze for M(x)
- Many of them requiring brute-force numerical diagonalizations in momentum space
- Is this the only way?

Ginzburg-Landau analysis

- Systematic expansion of the free energy
  in terms of the order parameter and its gradients

- Reliable if amplitudes and gradients are small

-> close to the Critical/Lifshitz point

$$\Omega_{\rm GL} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\left[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)\right.$$
$$\left. + \alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \dots\right]$$

D.Nickel, Phys.Rev.Lett.103:072301,2009
H.Abuki, D.Ishibashi, K.Suzuki, Phys.Rev.D85:074002,2012

Ginzburg-Landau analysis

$$\Omega_{\rm GL} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\left[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)\right.$$
$$\left. + \alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \ldots\right]$$

D.Nickel, Phys.Rev.Lett.103:072301,2009

H.Abuki, D.Ishibashi, K.Suzuki, Phys.Rev.D85:074002,2012

Ginzburg-Landau analysis

$$\Omega_{\mathrm{GL}} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\left[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)\right.$$
$$\left. + \alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \dots\right]$$

Restored +

Ginzburg-Landau analysis

$$\Omega_{\mathrm{GL}} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\Big[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)$$
$$+\alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \dots\Big]$$

Restored + "homogeneous" +

Ginzburg-Landau analysis

$$\Omega_{\mathrm{GL}} = \Omega[0] + \frac{1}{V}\int d\mathbf{x} \Big[\alpha_2 M^2 + \alpha_4 \left(M^4 + (\nabla M)^2\right) + \alpha_6 \left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)$$
$$+ \alpha_8 \left(M^8 + 14 M^4 (\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5} M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5} M^2 (\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \dots \Big]$$

Restored + **“homogeneous”** + **gradient terms**

- In principle straightforward: for each order add all possible independent terms (considering gradients are of the same order as M)

Ginzburg-Landau analysis

$$\Omega_{\mathrm{GL}} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\Big[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)$$
$$+\alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \ldots\Big]$$

- GL coefficients $\alpha_n(T,\mu)$
  are independent from the shape of the modulation

  -> can be computed relatively easily
  in a chirally restored background!

Ginzburg-Landau analysis

$$\Omega_{\mathrm{GL}} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\left[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)\right.$$
$$\left. + \alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \dots\right]$$

- GL coefficients $\alpha_n(T,\mu)$ are independent from the shape of the modulation

  -> can be computed relatively easily in a chirally restored background!

- But: calculating the relative prefactors between terms of the same order is an extremely tedious task..

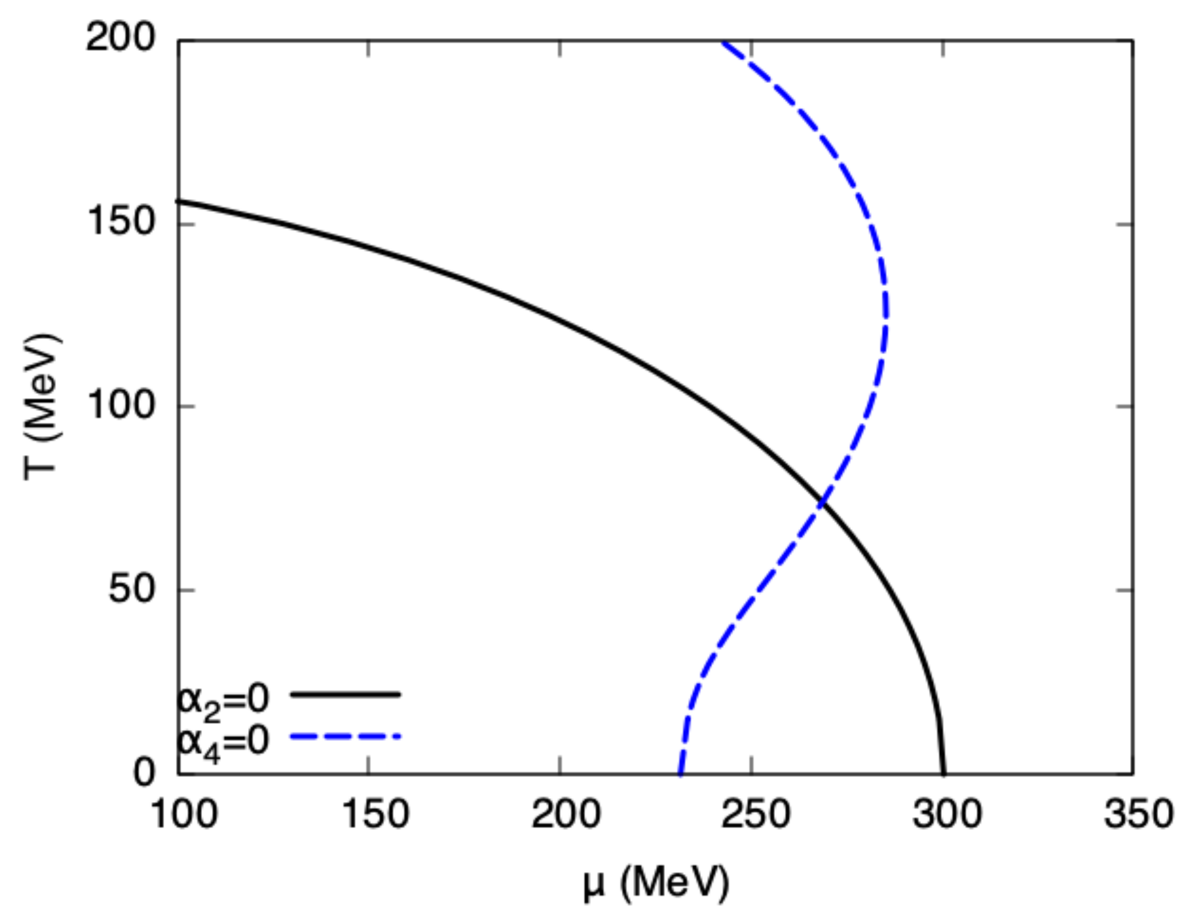
Ginzburg-Landau analysis

Already non-trivial result at lowest order:

$$\Omega_{\mathrm{GL}} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\left[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \dots\right]$$

Critical/Lifshitz point
coincide
and are located at
the point where

$$\alpha_2(T,\mu) = \alpha_4(T,\mu) = 0$$



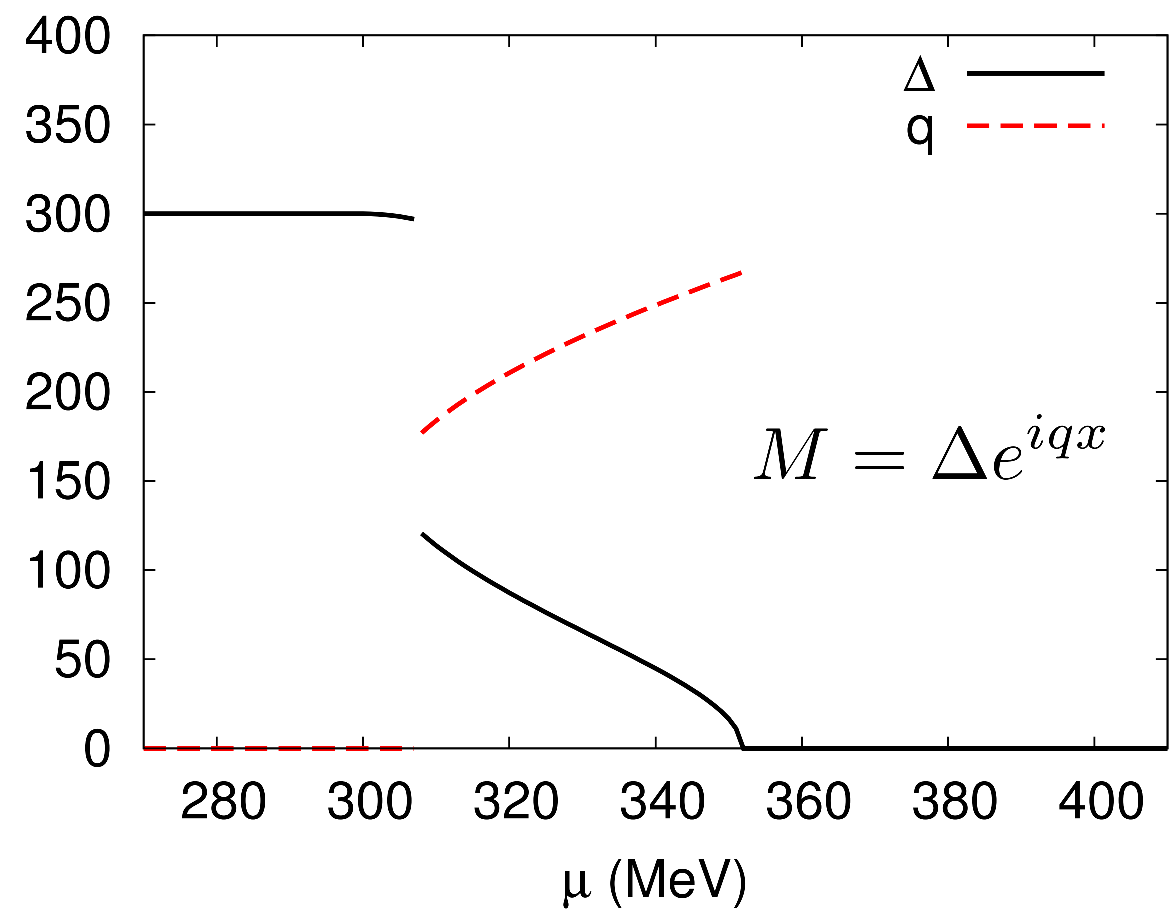
D.Nickel, Phys.Rev.Lett.103:072301,2009

Improved Ginzburg-Landau

- Can we do better ?
  Recall the typical behavior of the order parameters
  (eg. CDW, cosine..)

$$M \sim \Delta$$

$$\nabla M \sim q\Delta$$

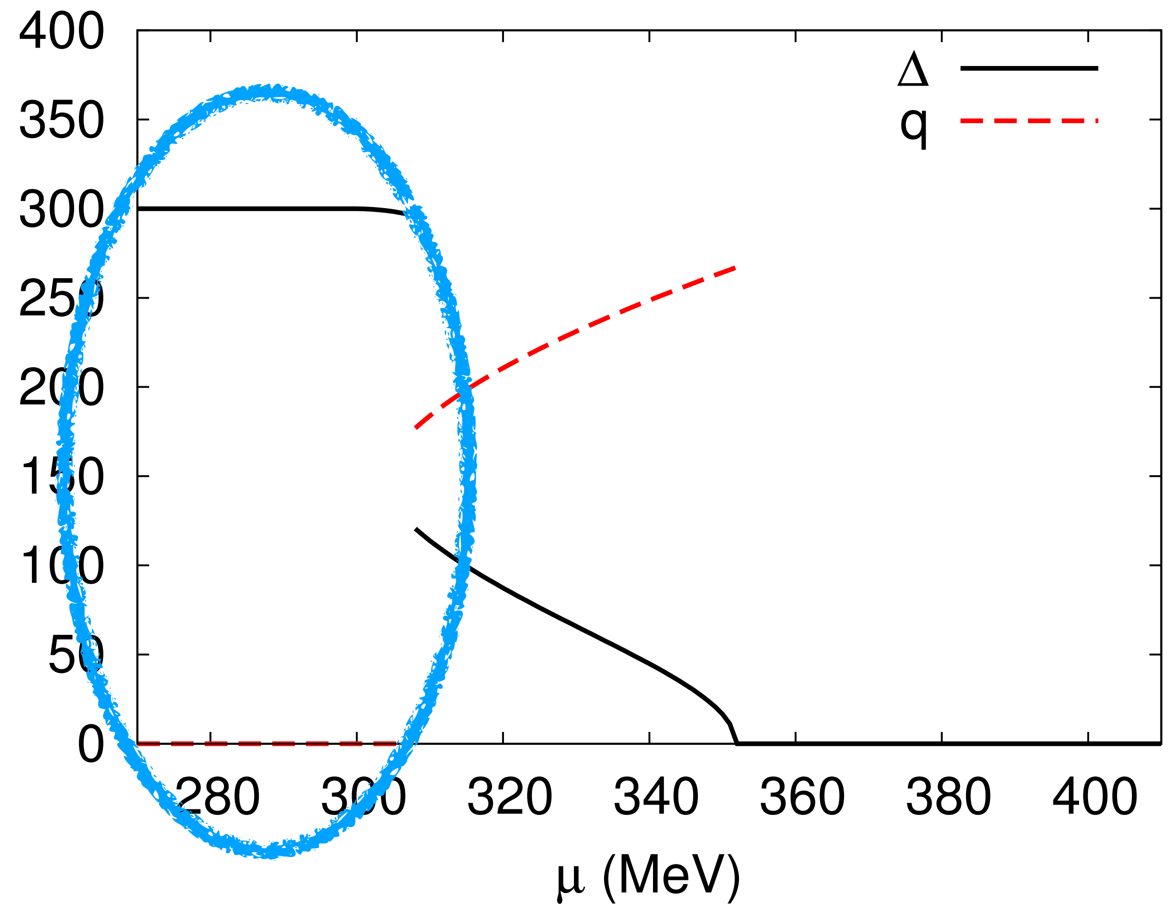


$$\Omega_{\rm GL} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\left[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)\right.$$
$$\left. + \alpha_8\left(M^8 + 14 M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \dots\right]$$

Improved Ginzburg-Landau

- Can we do better ?
  Recall the typical behavior of the order parameters

$$M \sim \Delta$$
$$\nabla M \sim q\Delta$$



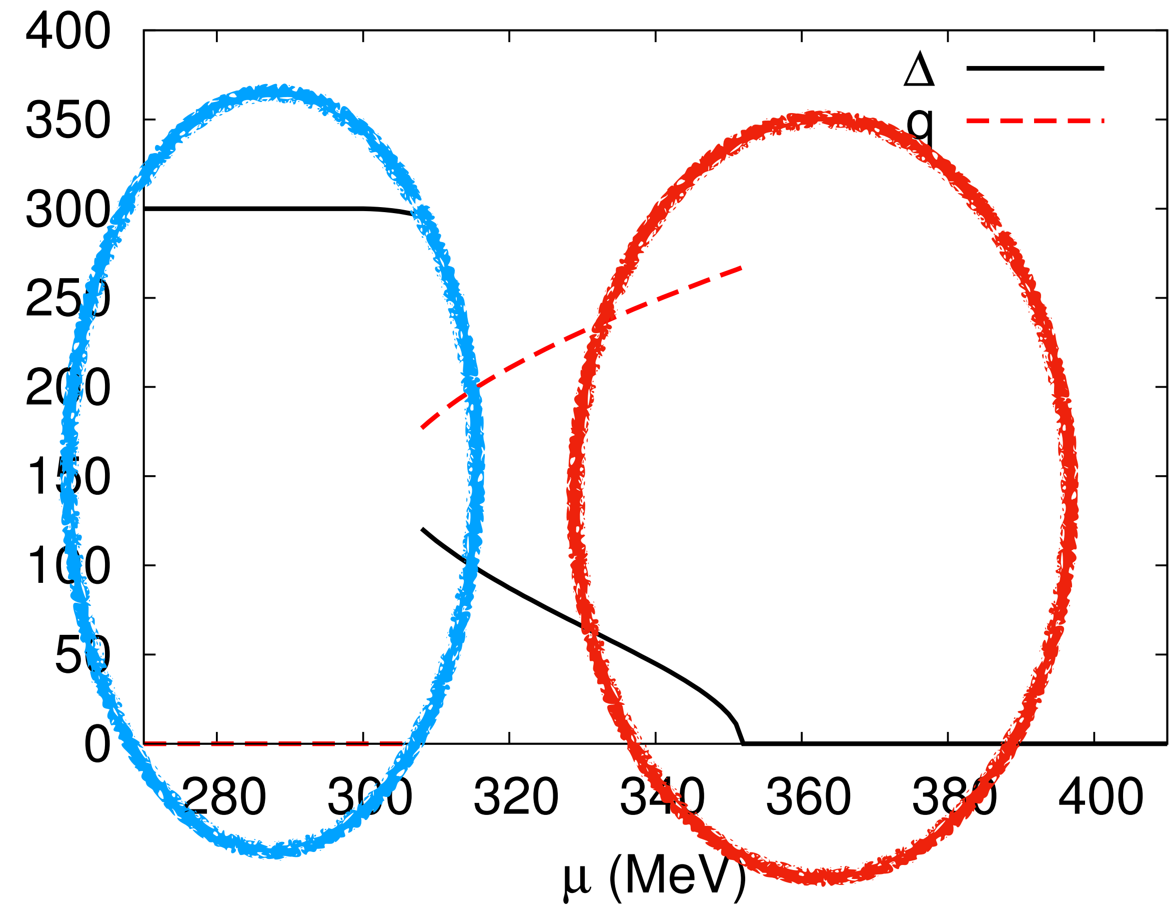
$$\Omega_{\rm GL} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\Big[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)$$
$$+\alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \dots\Big]$$

Improved Ginzburg-Landau

- Can we do better ?
  Recall the typical behavior
  of the order parameters

$$M \sim \Delta$$

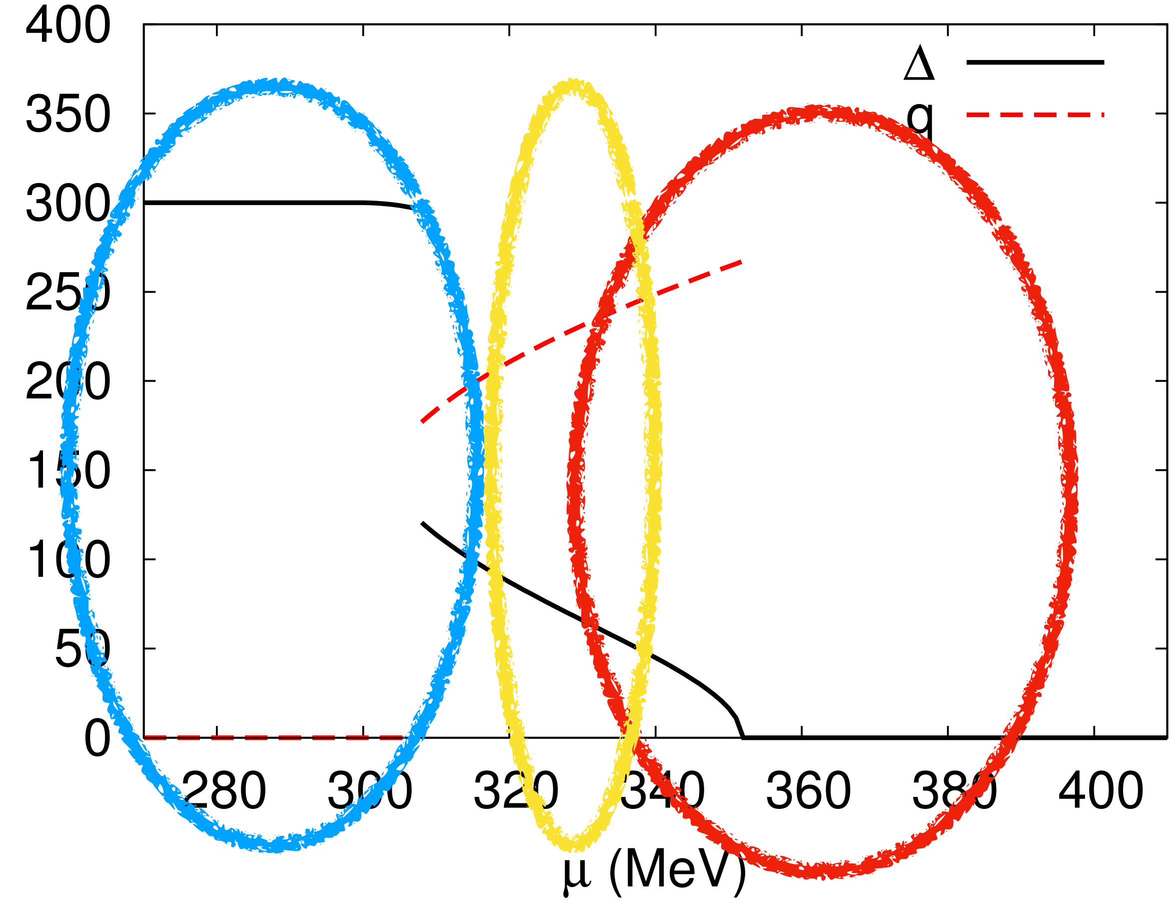
$$\nabla M \sim q\Delta$$



$$\Omega_{\rm GL} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\Big[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)$$
$$+\alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \dots\Big]$$

Improved Ginzburg-Landau

- Can we do better ?
  Recall the typical behavior
  of the order parameters



$$M \sim \Delta$$

$$\nabla M \sim q\Delta$$

$$\Omega_{\rm GL} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\Big[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)$$
$$+\alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \dots\Big]$$

Improved Ginzburg-Landau

$$\Omega_{\rm GL} = \Omega[0] + \frac{1}{V}\int d\mathbf{x} \Big[\alpha_2 M^2 + \alpha_4 \left(M^4 + (\nabla M)^2 \right) + \alpha_6 \left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2 \right)$$
$$+ \alpha_8 \left(M^8 + 14 M^4 (\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5} M (\nabla^2 M)(\nabla M)^2 + \frac{14}{5} M^2 (\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2 \right) + \dots \Big]$$

Improved Ginzburg-Landau

$$\Omega_{\rm GL} = \Omega[0] + \frac{1}{V}\int d\mathbf{x}\Big[\alpha_2 M^2 + \alpha_4\left(M^4 + (\nabla M)^2\right) + \alpha_6\left(M^6 + 3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2 + \frac{1}{2}(\nabla^2 M)^2\right)$$
$$+\alpha_8\left(M^8 + 14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2 + \frac{1}{5}(\nabla^3 M)^2\right) + \dots\Big]$$

straightforward to compute

$$\Omega_{\rm IGL} = \frac{1}{V}\int d\mathbf{x}\Big[\Omega_{\rm hom}(\overline{M^2}) + \alpha_6\left(3(\nabla M)^2 M^2 + \frac{1}{2}(\nabla M^2)^2\right)$$
$$+\alpha_8\left(14M^4(\nabla M)^2 - \frac{1}{5}(\nabla M)^4 + \frac{18}{5}M(\nabla^2 M)(\nabla M)^2 + \frac{14}{5}M^2(\nabla^2 M)^2\right) + \sum_{n\geq 1}\tilde{\alpha}_{2n+2}(\nabla^n M)^2\Big]$$

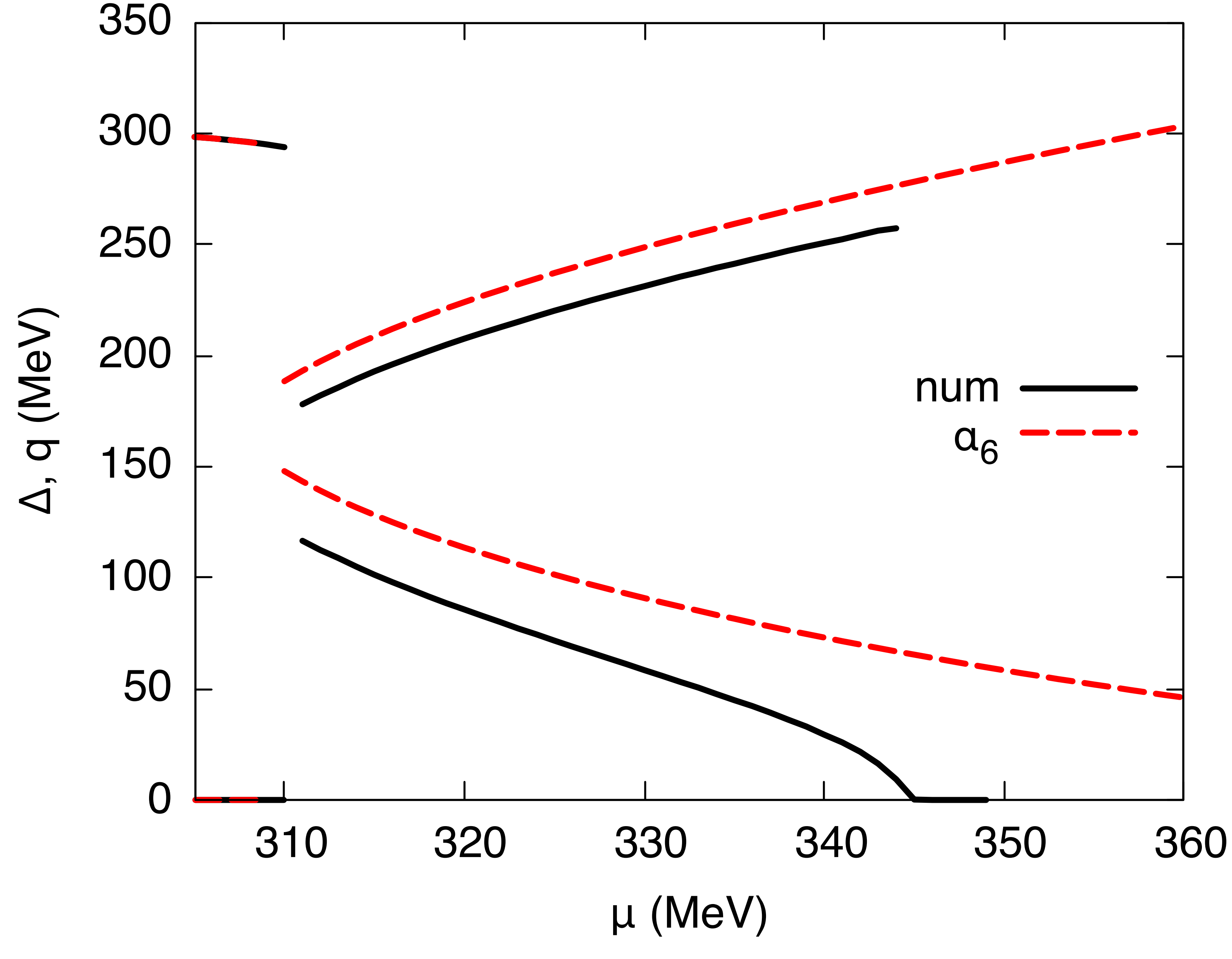
terms $\sim q^{2n}\Delta^2$

easy to compute from the CDW free energy, which is known analytically

SC, F. Anzuini, M. Mannarelli and O. Benhar, Phys. Rev. D 97, 036009 (2018)

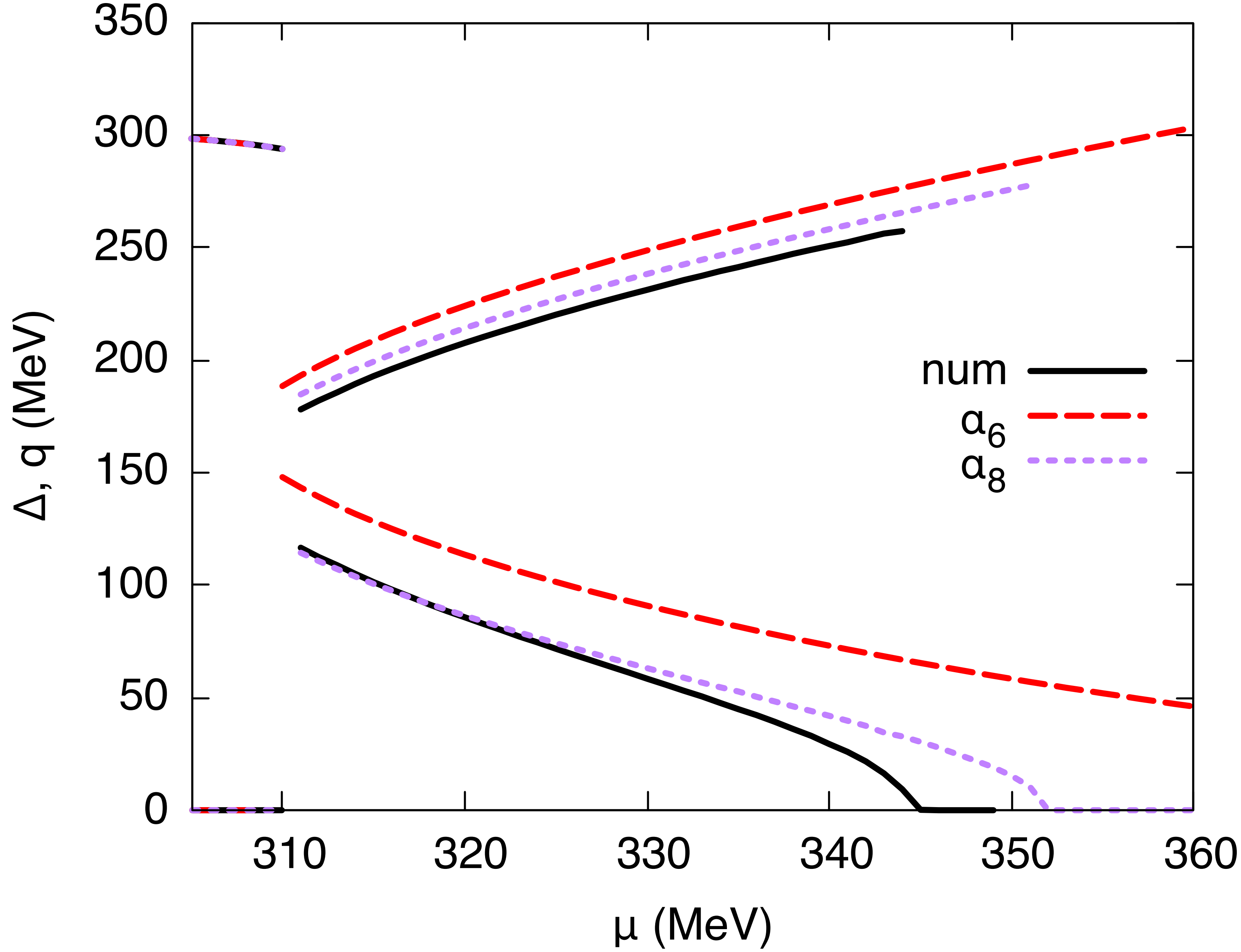
Does it work ?

Minimizing the IGL potential at T=0…



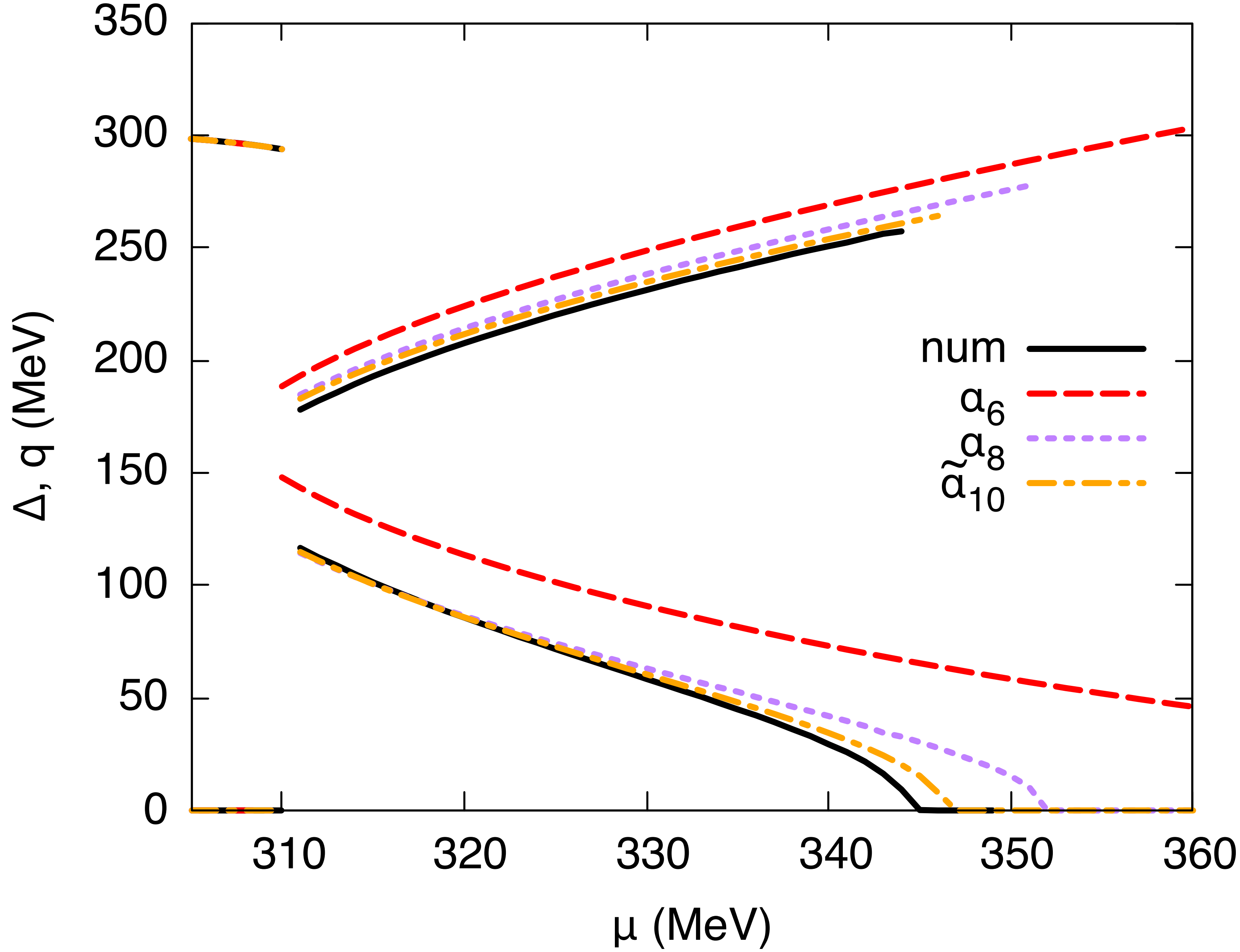
T=0

Does it work ?



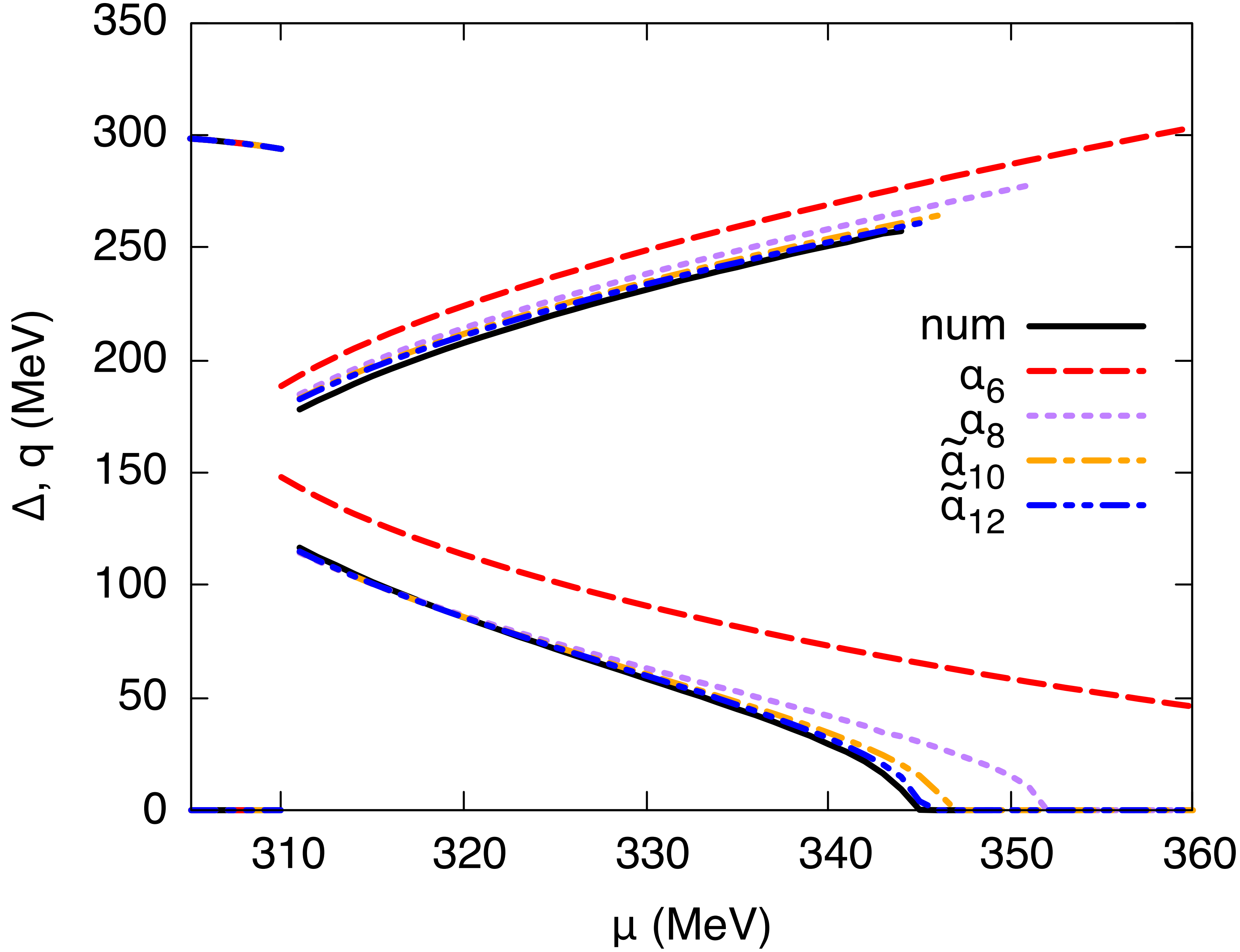
T=0

Does it work ?



T=0

Does it work ?

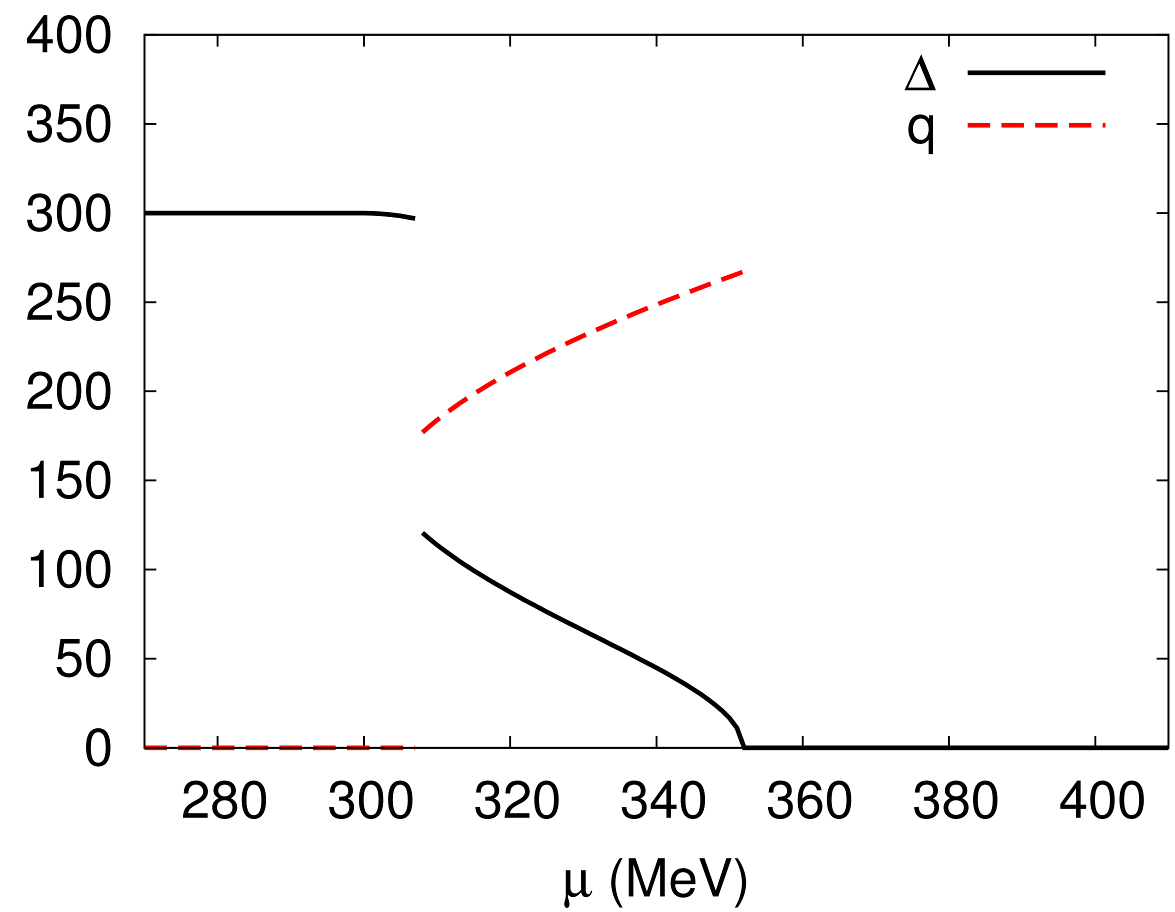


T=0 !!!

Chiral density wave

- Simplest ansatz: 1D plane wave (FF-type): $M(\mathbf{x}) = \Delta e^{iqz}$
- Analytical expression known for the eigenvalue spectrum
- Order parameters: amplitude Δ , wave number q
- Minimizing free energy varying chemical potential (T=0)

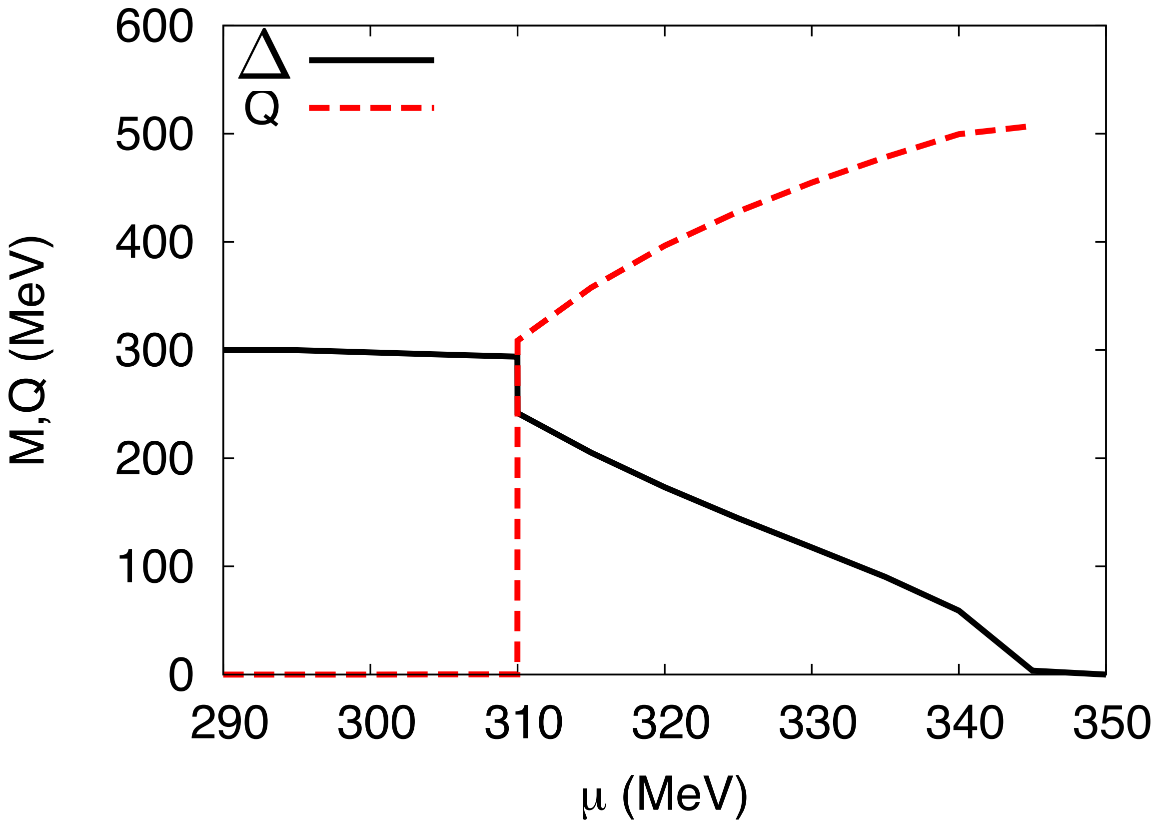
Special feature: constant density!



E. Nakano and T.Tatsumi, Phys.Rev.D71:114006,2005

LOFF-type modulations

- Second-simplest one: 1D cosine ("LOFF") $M(\mathbf{x}) = \Delta \cos(qz)$
- Numerical diagonalization in momentum space required: Computationally intensive, but still doable on my laptop
- Qualitatively similar behavior to the CDW for the order parameters

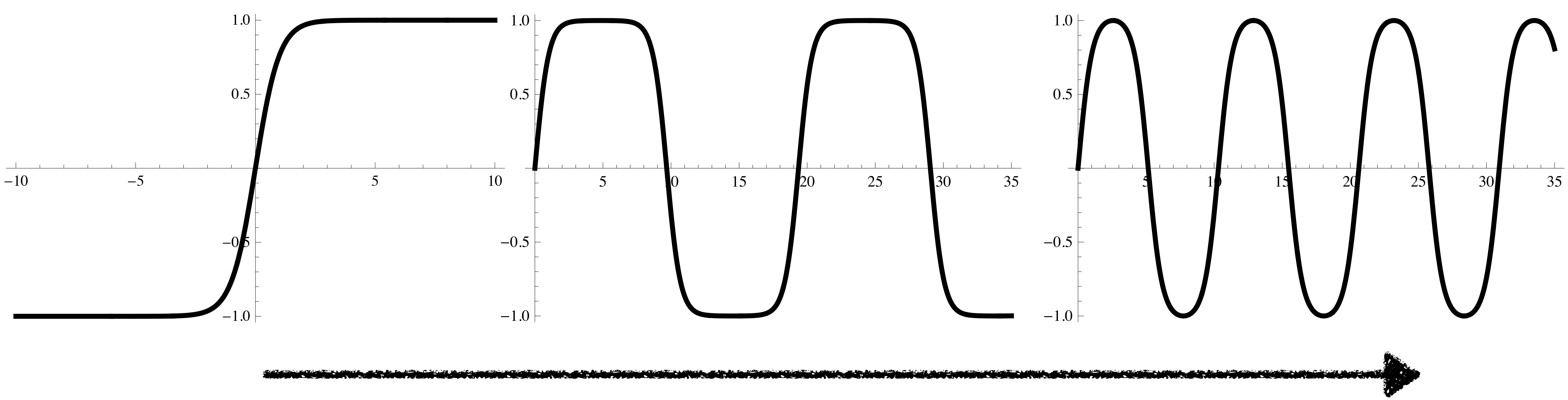


Real-kink crystal

- A more generic one-dimensional structure expressed in terms of Jacobi elliptic functions:

$$M(\mathbf{x}) = \Delta\nu\,\mathrm{sn}(\Delta z, \nu)$$

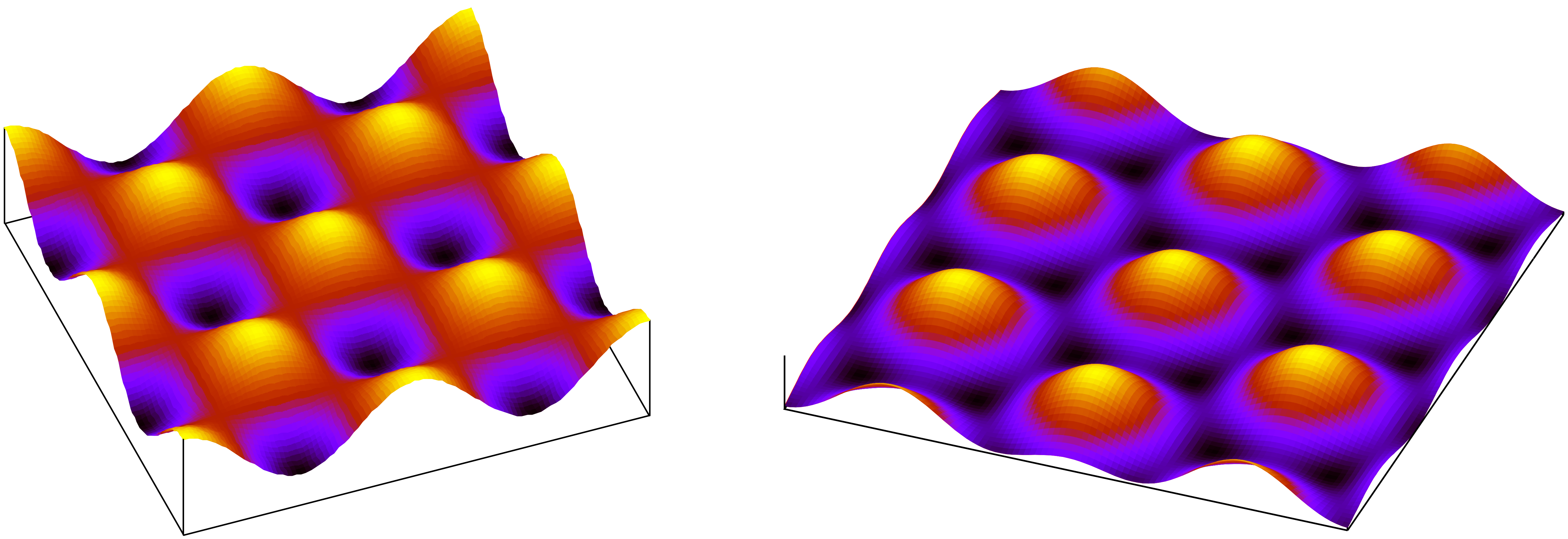
- Parameters: $\Delta\,,\nu$



$\nu \to 1$ $\nu \to 0$

Two-dimensional modulations

- Different lattice structures



- Still numerically doable (on a cluster)

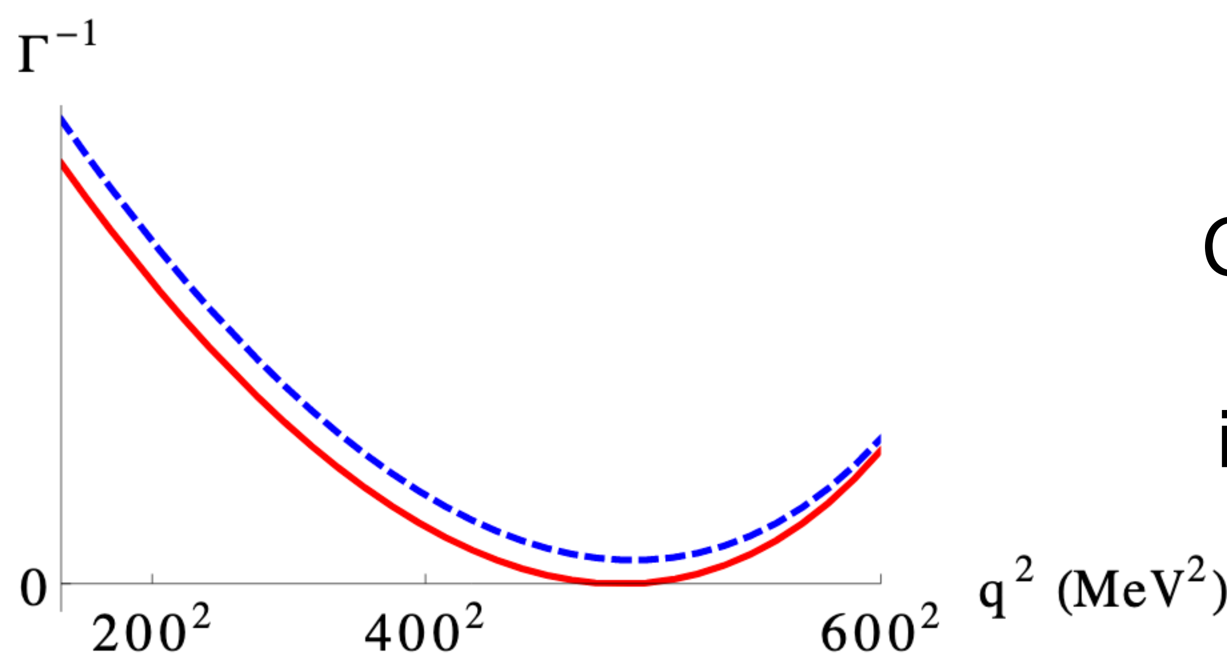
- Qualitatively similar results to 1D mods for order parameters

Stability analysis

Similar spirit to the (I)GL analysis: expand the free energy and look at the second-order piece

$$\Omega^{(2)} = 2G^2 \sum_{\mathbf{q}_k} \left\{ \left|\delta\phi_{S,\mathbf{q}_k}\right|^2 \Gamma_S^{-1}(\mathbf{q}_k^2) + \left|\delta\phi_{P,\mathbf{q}_k}\right|^2 \Gamma_P^{-1}(\mathbf{q}_k^2) \right\}$$

Look for where the correlation functions in either condensation channel changes sign



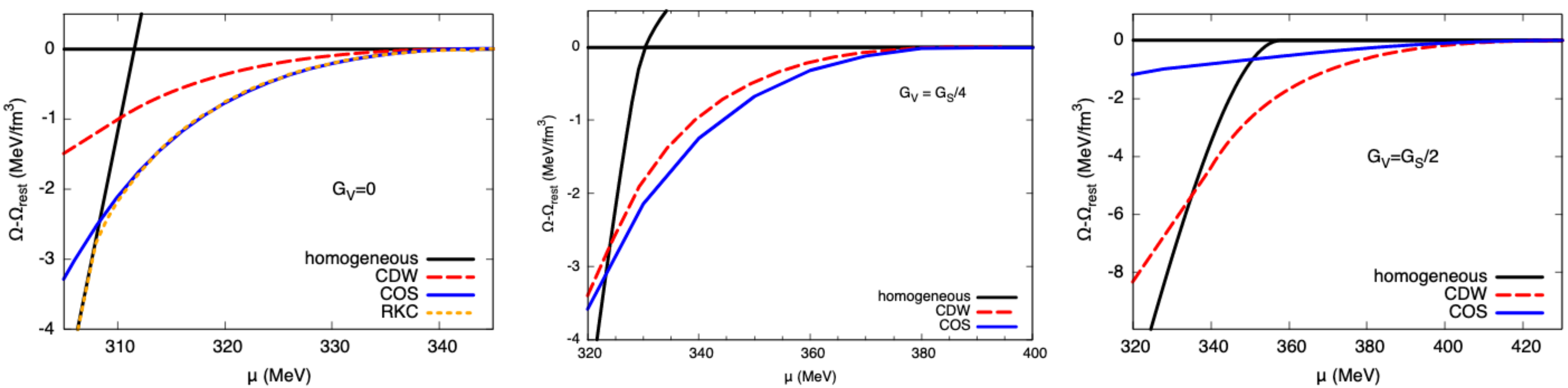
Can be used to determine the phase boundary inhomogeneous-restored

Vector interactions

Going beyond constant density approximation:
vector interactions could alter hierarchy of favored spatial modulations according to their density profile

-> is the RKC still favored over a CDW?

…or numerically at T=0



SC, M.Schramm and M.Buballa, Phys. Rev. D 98, 014033 (2018)

Vector interactions

Repulsive vector interaction channel:

$$\mathcal{L} = \mathcal{L}_{NJL} - G_V(\bar{\psi}\gamma^\mu\psi)^2$$

Mean-field: density-dependent shift of the chemical potential

For inhomogeneous phases: spatially dependent!

$$\tilde{\mu}(\mathbf{x}) = \mu - 2G_V n(\mathbf{x})$$

Technically challenging!

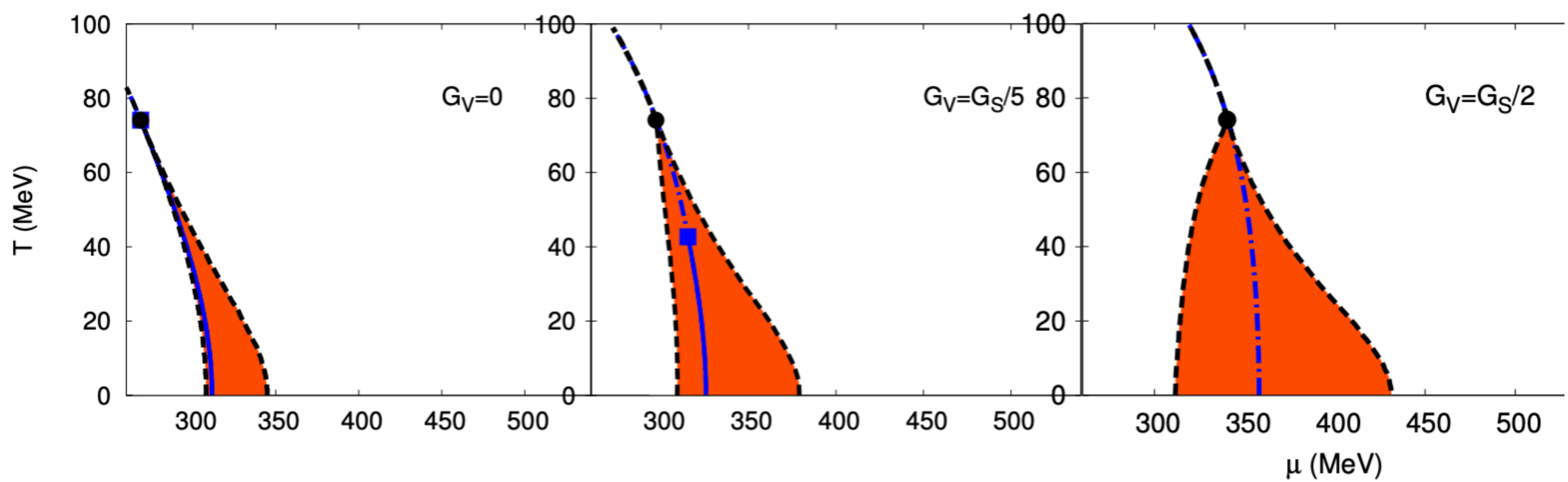
As first approximation consider constant density:

$$n(\mathbf{x}) \to \bar{n} = \langle n(\mathbf{x}) \rangle_{\mathbf{x}}$$

Vector interactions

Constant density approximation:
the inhomogeneous phase enlarges dramatically!

CP falls below the LP and **disappears**
inside the inhomogeneous phase



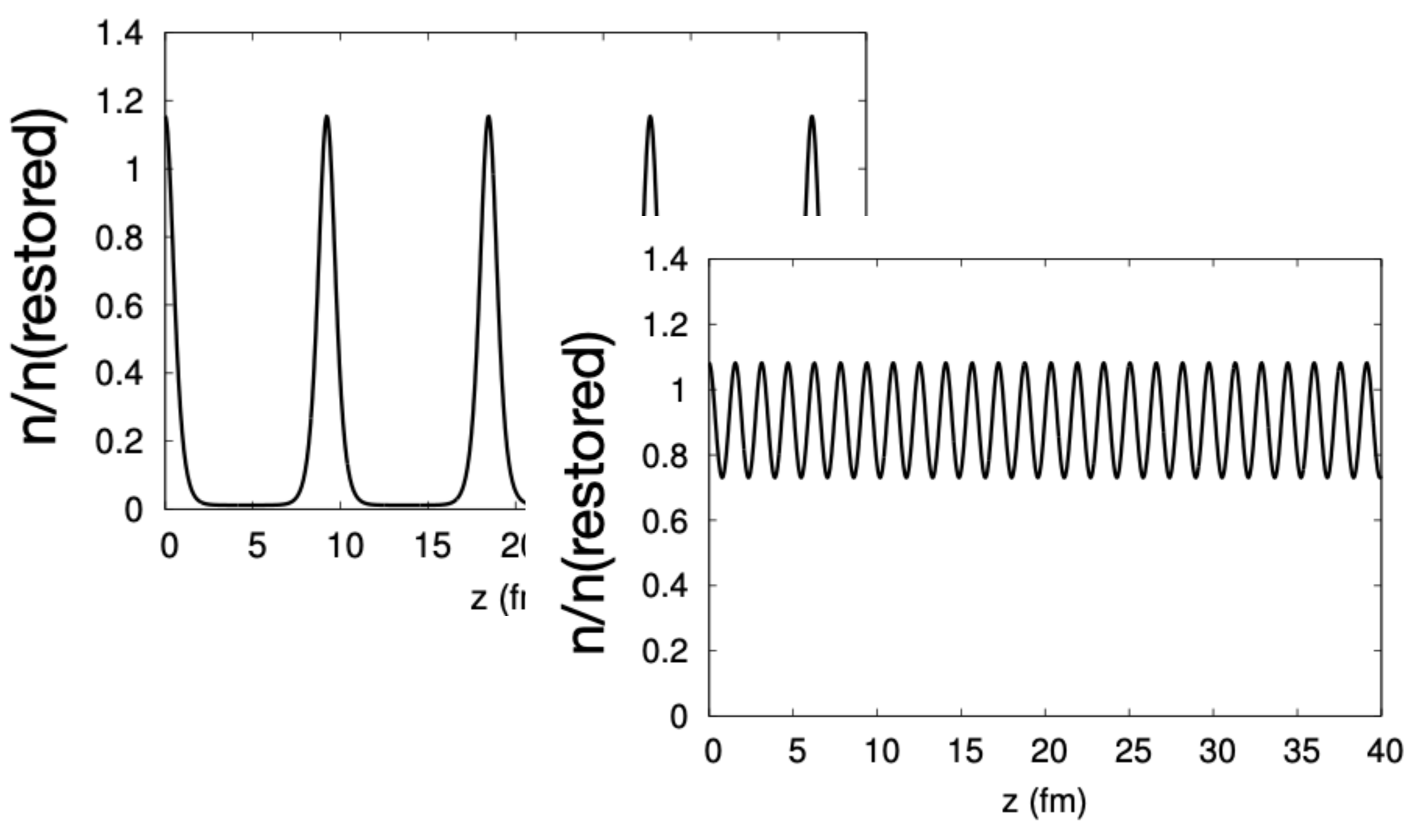
SC, D.Nickel and M.Buballa, Phys.Rev. D82 (2010) 054009

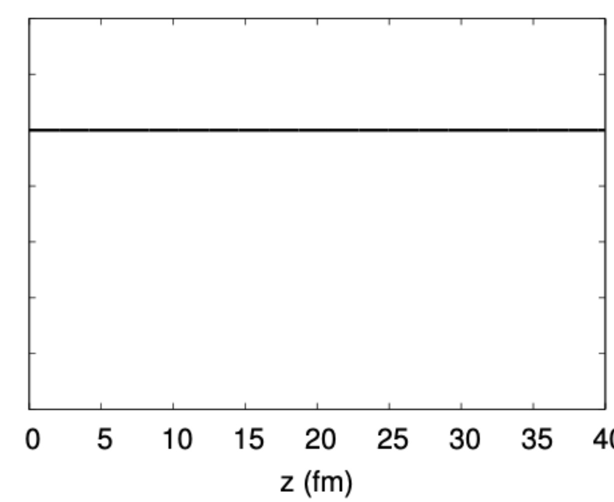
Vector interactions

Going beyond constant density approximation:
vector interactions could alter hierarchy of favored spatial modulations according to their density profile

-> is the RKC still favored over a CDW?

$$n(z) \sim \langle \psi^\dagger \psi \rangle$$



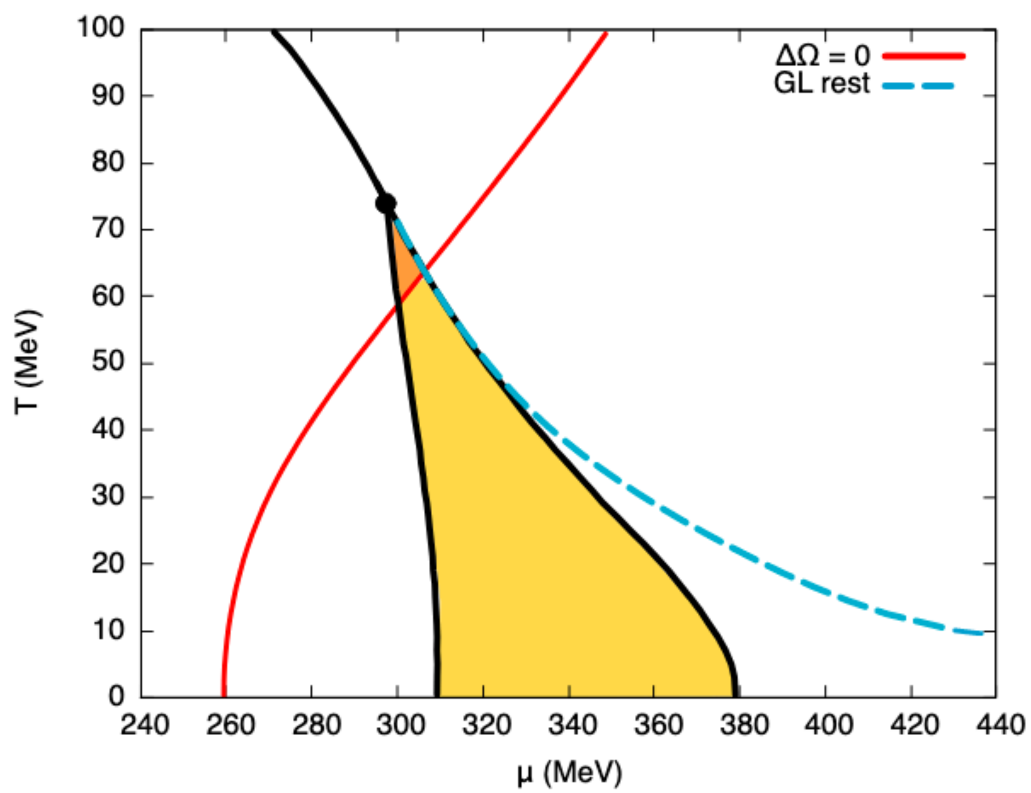


Vector interactions

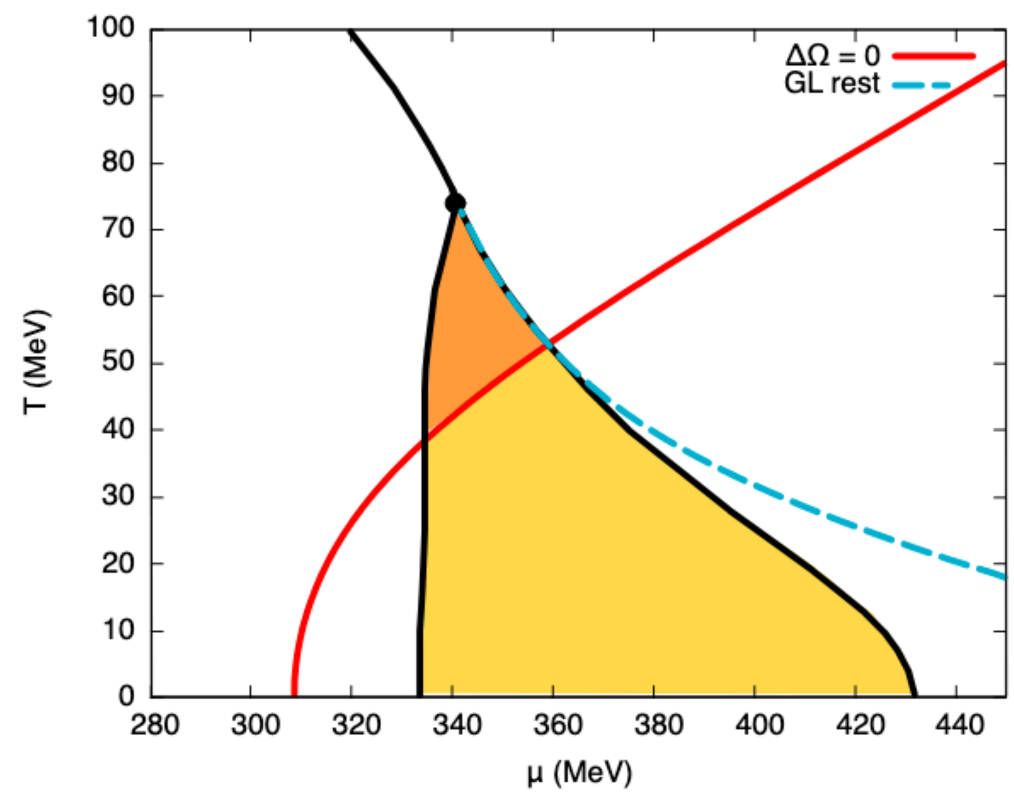
Going beyond constant density approximation:
vector interactions could alter hierarchy of favored spatial modulations according to their density profile

-> is the RKC still favored over a CDW?

Close to LP: GL analysis…



Gv=Gs/5



Gv=Gs/2

SC, M.Schramm and M.Buballa, Phys. Rev. D 98, 014033 (2018)