Higher order QCD corrections to hadronic tau decays and Higgs decays to $b\bar{b}$ using Padé approximants

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In collaboration with Cristiane London, Pere Masjuan, Fabio Oliani

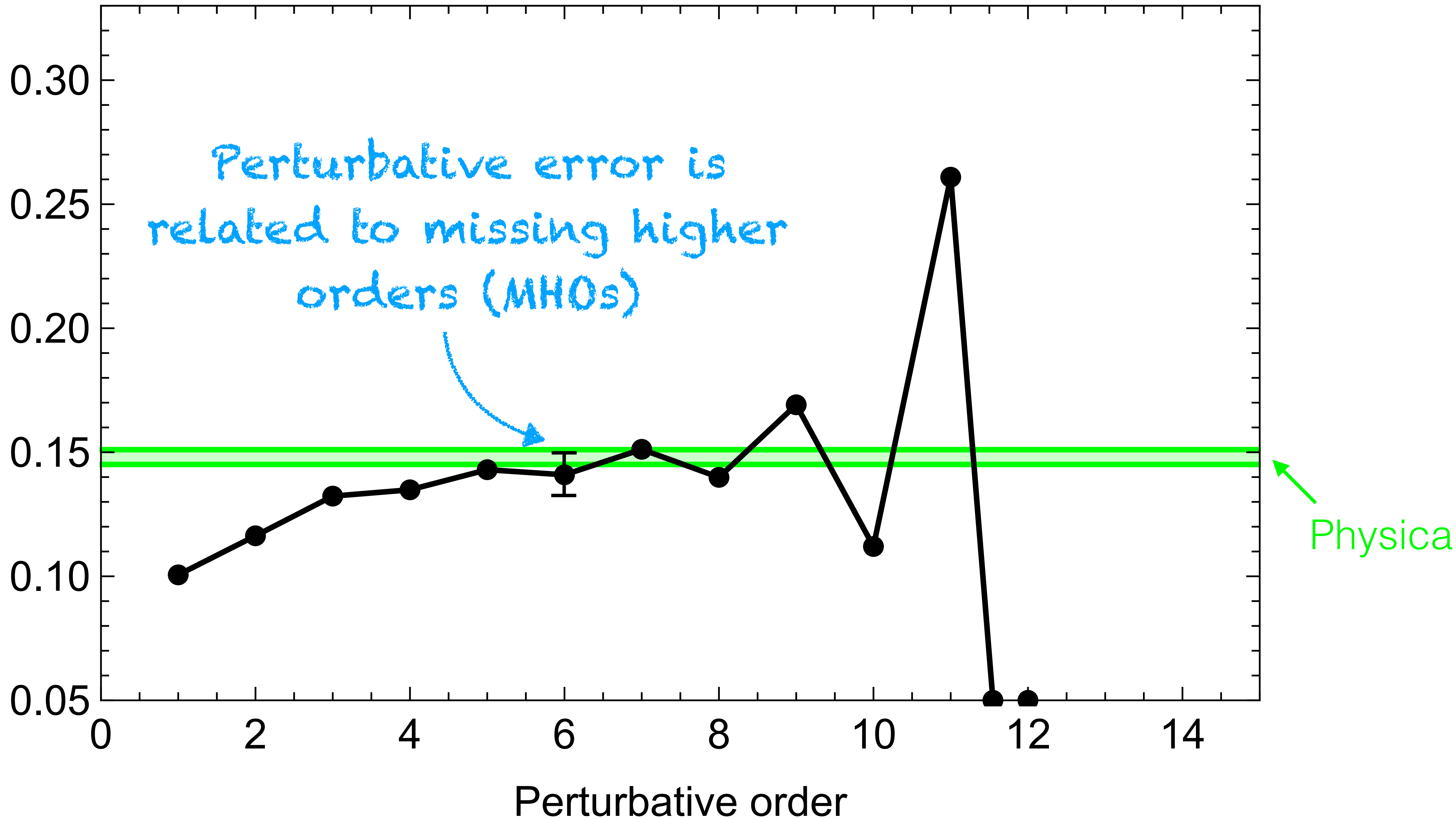
DB, P. Masjuan, F Oliani, JHEP 08 (2018) 075; [arXiv:1807.01567]
DB, C London, P. Masjuan, in preparation

August 2021

Perturbative series, Padé approximants, Borel transform

asymptotic series

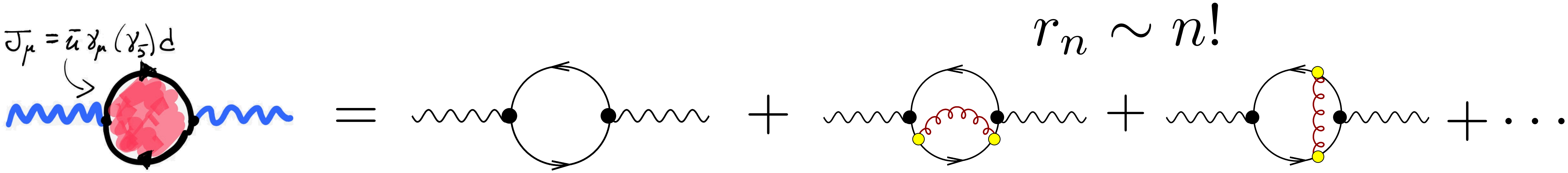
Perturbative expansions in QFTs are (at best) asymptotic



QCD is the limiting factor in several SM observables at present (MHOs and uncertainty in parameters)

Results beyond 5 loops unlikely to appear anytime soon....

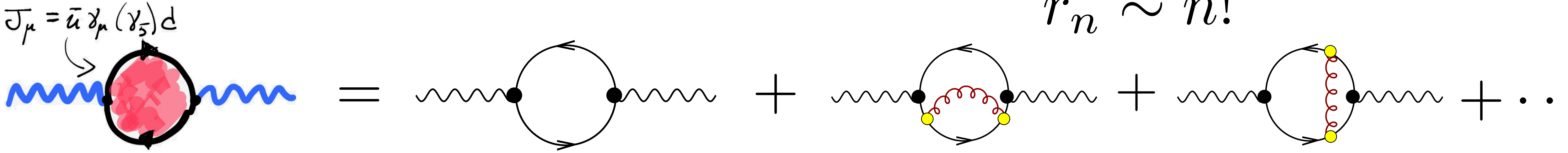
divergences and renormalons



$$R \sim \sum_{n=0}^{n^*} r_n \alpha_s^{n+1} + e^{-p/\alpha_s}$$

divergences and renormalons

$$r_n \sim n!$$

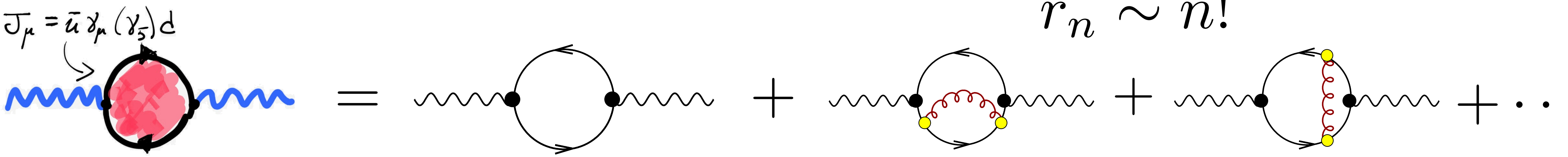


$$R \sim \sum_{n=0}^{n^*} r_n \alpha_s^{n+1} + e^{-p/\alpha_s} \quad \xrightarrow{\alpha_s(Q^2)} \quad R \sim \sum_{n=0}^{n^*} r_n \alpha_s^{n+1} + \left(\frac{\Lambda^2}{Q^2}\right)^p$$

OPE

divergences and renormalons

$r_n \sim n!$



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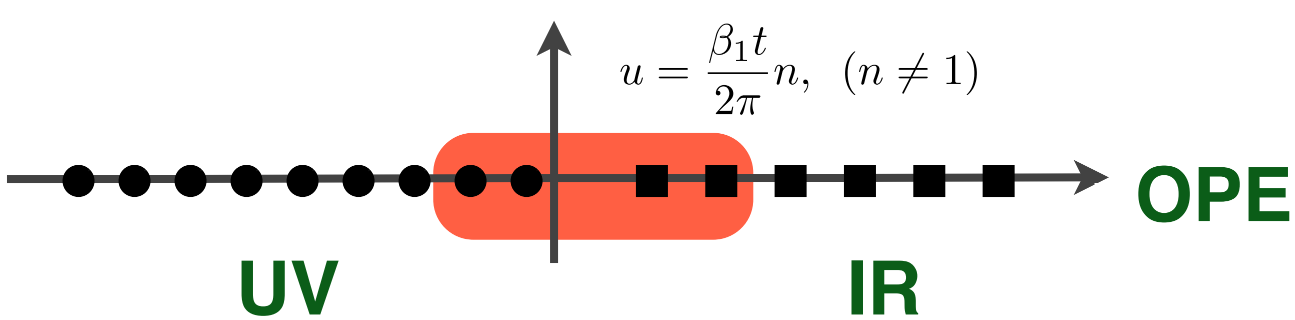
Borel transform method

$$B[R](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}$$ which can be "summed" $\Longrightarrow$ $$\tilde{R} \equiv \int_0^\infty dt\, \mathrm{e}^{-t/\alpha} B[R](t)$$

Singularities in the Borel plane: renormalons

Beneke '99



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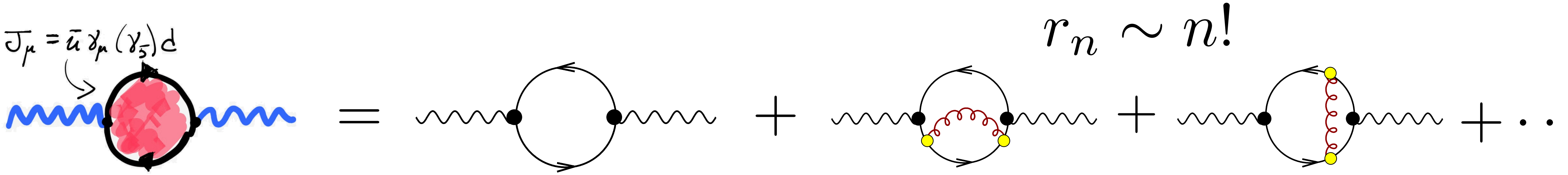

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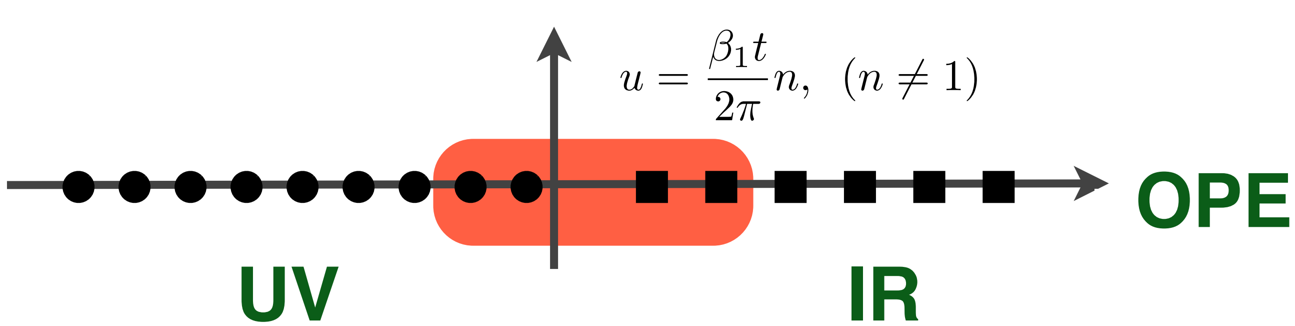

$$B[R](t) = \frac{1}{(u-2)} \mapsto \left(\frac{\Lambda^2}{Q^2}\right)^2$$

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estimating missing higher orders

Use of **Padé approximants** (model independent)

$$P_N^M(x) = \frac{Q_M(x)}{R_N(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_M x^M}{1 + b_1 x + b_2 x^2 + \cdots + b_N x^N}$$

example: $f(x) = \frac{\sqrt{1+\frac{1}{2}x}}{(1+2x)^2}$

Matching the Taylor series

$$f(x) \approx 1 - 3.75x + 10.969x^2 - 28.867x^3 + 71.591x^4 + \cdots$$

$$P_1^1(x) = \frac{a_0 + a_1 x}{1 + b_1 x} \approx a_0 + (a_1 - a_0 b_1)x + (a_0 b_1^2 - a_1 b_1)x^2 + \cdots$$

Estimates of higher orders

$$f(x) \approx 1 - 3.75x + 10.969x^2 - 28.867x^3 + 71.591x^4 + \cdots$$

$$P_1^1(x) \approx 1 - 3.75x + 10.969x^2 - 32.084x^3 + 93.845x^4 + \cdots$$

estimating missing higher orders

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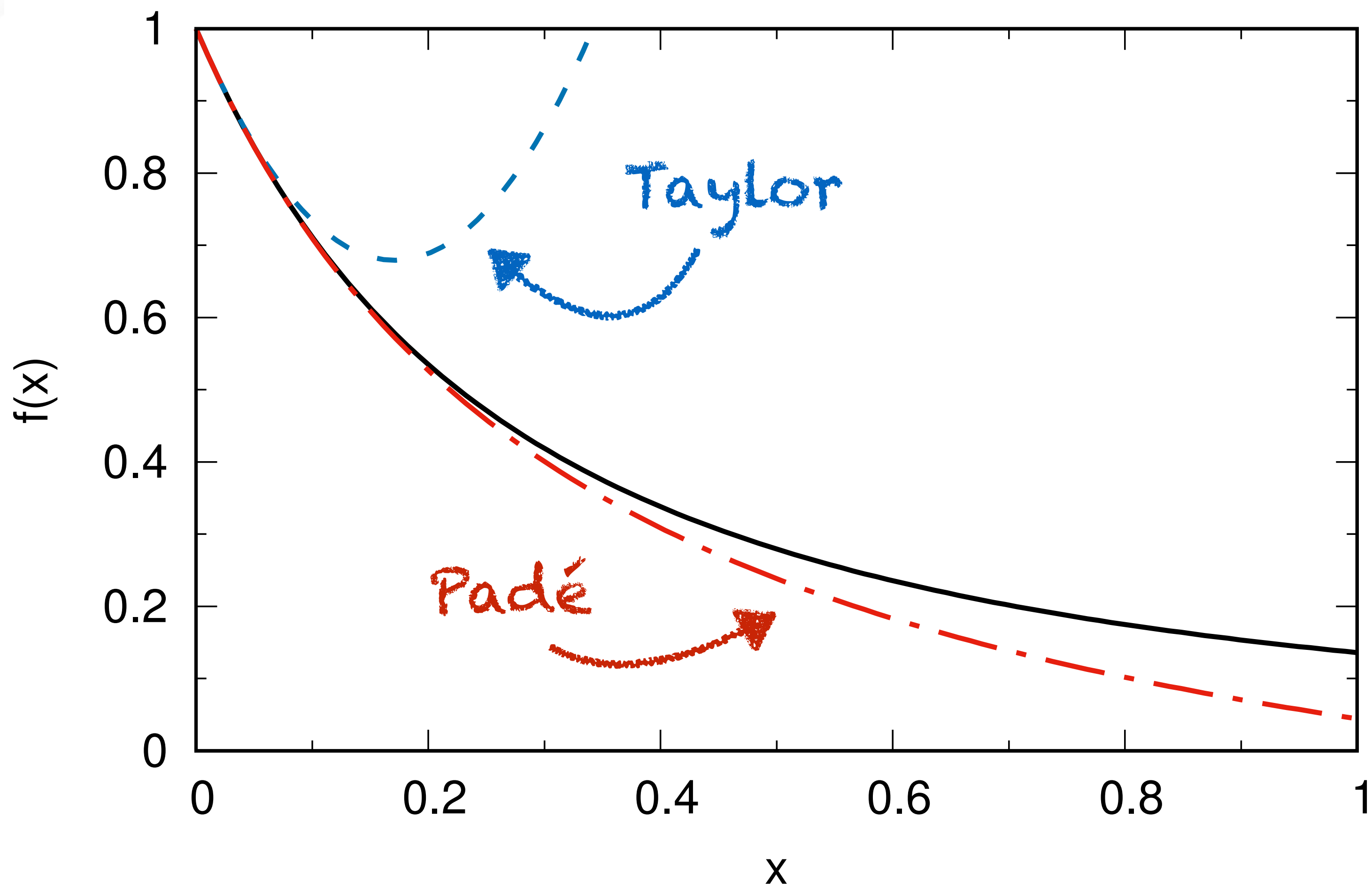
$$f(x) = \frac{\sqrt{1+\frac{1}{2}x}}{(1+2x)^2}, \quad P_1^1(x)$$



- optimised use of the information (a few coefficients)
- very effective
- can reproduce poles
- model independent
- convergence theorems (in some cases) Baker
- provides an error estimate

estimating missing higher orders

D-log Padé approximants
Baker

- Useful for functions with cuts or branch points

$$f(z) = A(z)\,\frac{1}{(\mu - z)^{\gamma}} + B(z) \qquad\qquad F(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln f(z) \approx \frac{\gamma}{(\mu - z)}$$

→ **meromorphic**

- D-log Padé to $f(z)$ is defined as

$$\mathrm{Dlog}^{M}_{N}(z) = f(0)\ \exp\left\{\int \mathrm{d}z\ \ P^{M}_{N}(z)\right\}$$ **PA applied to $F(z)$**

- The D-log Padé is not a rational approximant, however $F(z)$ is meromorphic

- It is possible to determine the position of the cut and its multiplicity

- The D-log Padé recreates the first $M+N+1$ coefficients of the function $f(z)$

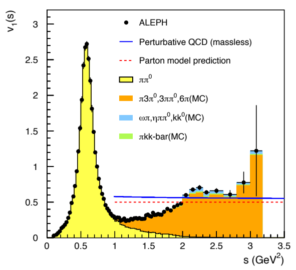
Application to hadronic tau decays

QCD in tau decays

Davier et al '14

$$R_\tau = \frac{\Gamma[\tau \to \mathrm{hadrons}\, \nu_\tau]}{\Gamma[\tau \to e^- \bar{\nu}_e\, \nu_\tau]} = 3.6280 \pm 0.0094$$

$$R_\tau = \int_0^{s_0} ds\, w(s)\, \frac{1}{\pi} \mathrm{Im}\tilde{\Pi}(s)$$



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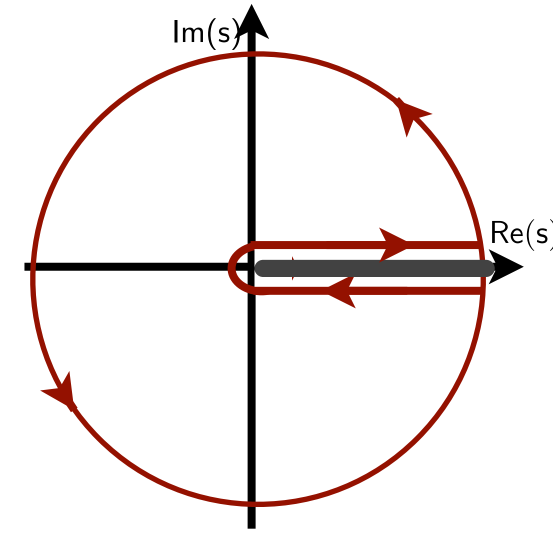

Sum rule (using Cauchy's theorem)

Braaten, Narison, Pich '92

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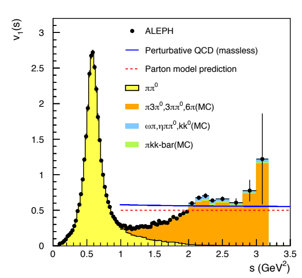

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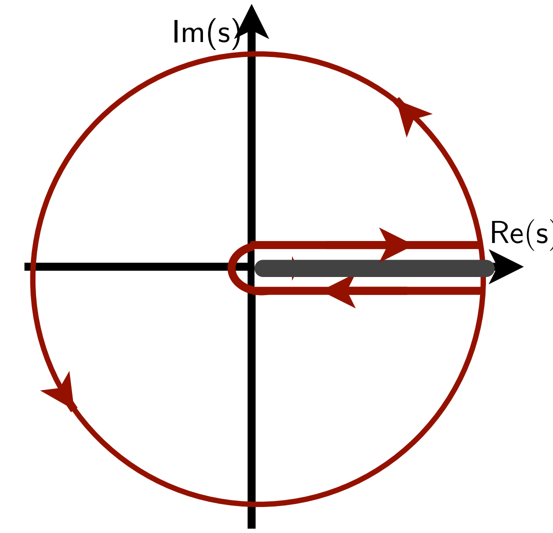

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experiment

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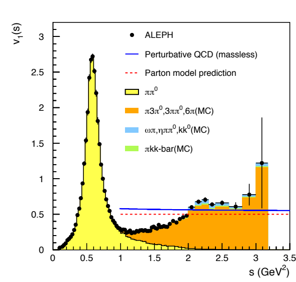
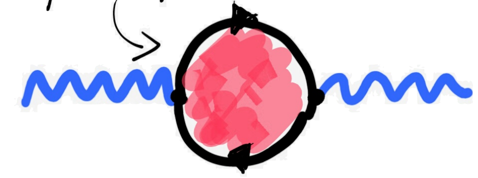


theory



Studying these sum rules one can extract the strong coupling and parameters of non-perturbative QCD.

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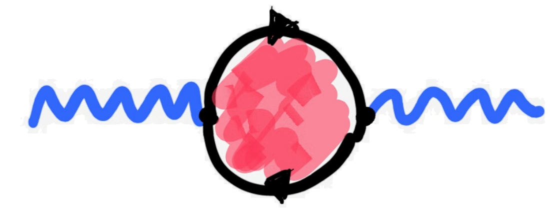
QCD in tau decays

$$J^{\mu}_{V/(A)}(x) = (\bar{u}\gamma^{\mu}(\gamma_5)d)(x)$$

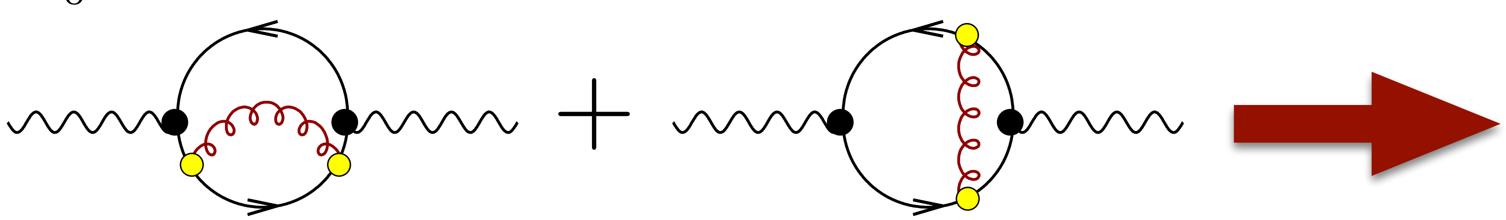
$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle 0|T\{J_\mu(x)J_\nu(0)^\dagger\}|0\rangle$$



Below charm one works with **massless** correlators.

$$\frac{-1}{2\pi i}\oint_{|z|=s_0} dz\, w(z)\,\tilde{\Pi}(z) \approx N_c(1+\delta^{(0)}+\delta_{\rm EW}+\delta_{\rm OPE}+\delta_{\rm DVs})$$

$$\delta^{(0)} = \sum_n^4 c_n\alpha_s^n$$



$$\frac{\alpha_s}{\pi} \approx \frac{0.3}{\pi} \sim 10\%$$

Gorishnii, Kataev, Larin '91
Surguladze&Samuel '91

Baikov, Chetyrkin, Kühn '08

5 loops!

	α_s^1	α_s^2	α_s^3	α_s^4	
$\delta^{(0)}_{\rm FO} =$	$0.1012 +$	$0.0533 +$	$0.0273 +$	0.0133	$= 0.1952$

pt. correction is ~20%

Moments dominated by perturbation theory

disclaimer!

Gorishnii, Kataev, Larin '91
Surguladze&Samuel '91

Baikov, Chetyrkin, Kühn '08

	α_s^1		α_s^2		α_s^3		α_s^4		5 loops!
$\delta^{(0)}_{\rm FO}$ =	0.1012	+	0.0533	+	0.0273	+	0.0133	=	0.1952
$\delta^{(0)}_{\rm CI}$ =	0.1375	+	0.0262	+	0.0104	+	0.0072	=	0.1814

pt. correction is ~20%

theoretical uncertainty

Contour Improved Perturbation Theory (CIPT)

Disclaimer: CIPT has a *different* Borel sum and requires a *different* OPE

Hoang and Regner, arXiv:2008.00578, arXiv: 2105.11222

I will focus on Fixed Order Perturbation Theory

testing the method

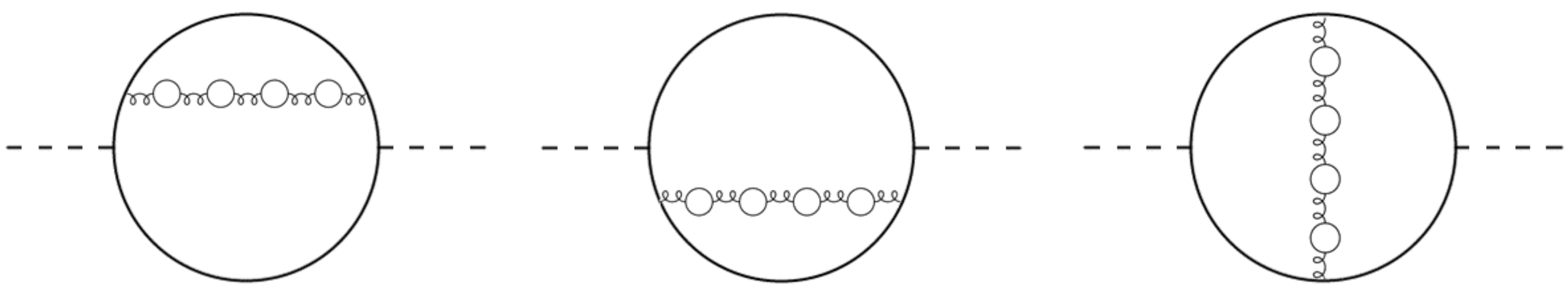
We validate our strategy with the help of a realistic model: large-β_0 limit

Gluon propagator with insertions of $q\bar{q}$ loops

$\alpha_s n_f$ $\qquad$ $(\alpha_s n_f)^2$

$$\alpha_s n_f \sim \mathcal{O}(1)$$

$$\beta_{0,f} = \frac{n_f}{6\pi}$$



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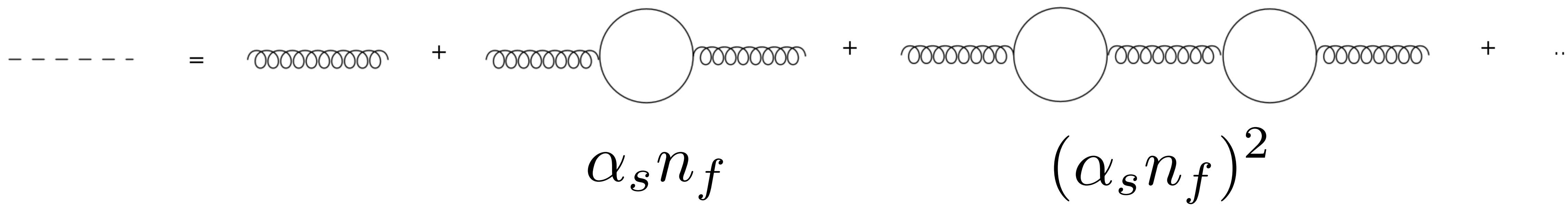

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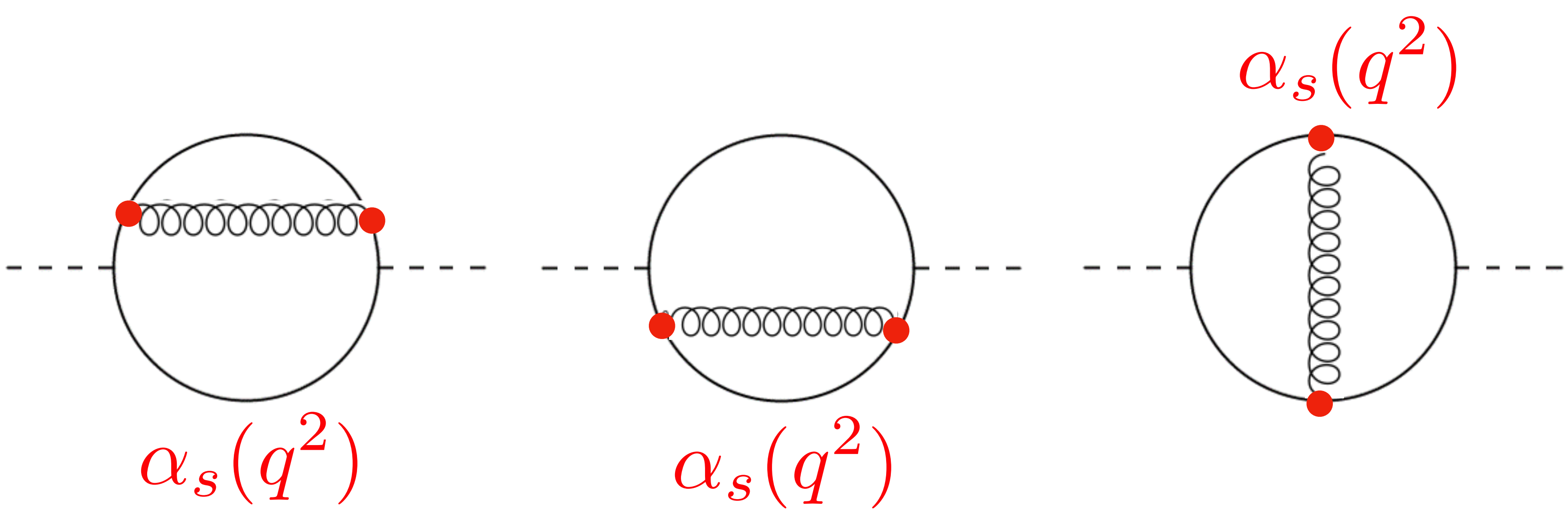


"Non-abelianization" of the result

$$n_f \to 6\pi\beta_0$$

A set of non-abelian diagrams is included (running coupling)

testing the method

We validate our strategy with the help of a realistic model: large-β_0 limit

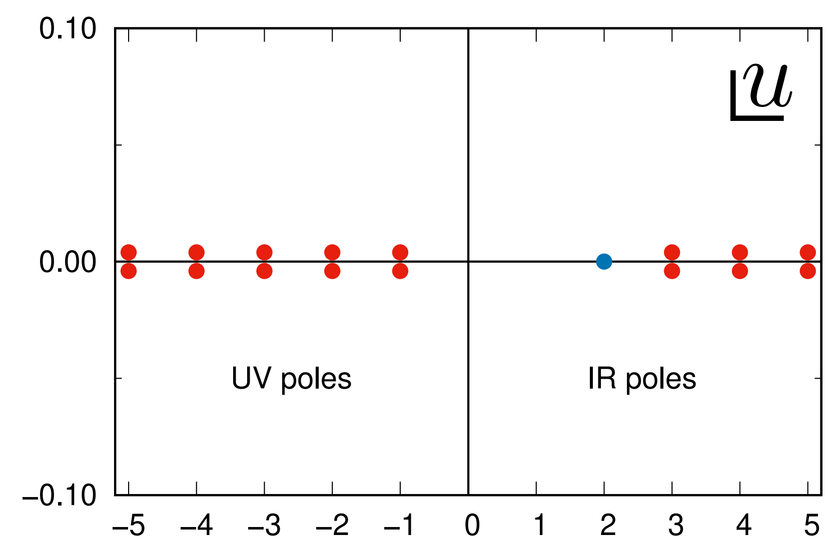
$$\delta^{(0)} = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} W(x) \widehat{D}^{(1+0)}_{\rm pert}(m_\tau^2 x)$$

$$D^{(1+0)}(s) = -s\frac{d}{ds}\left[\Pi^{(1+0)}(s)\right]$$

Borel transformed Adler function exactly known

$$B[\widehat{D}_{L\beta}](u) = \frac{32}{3\pi}\frac{e^{(C+5/3)u}}{2-u}\sum_{k=2}^{\infty}\frac{(-1)^k k}{[k^2-(1-u)^2]^2}$$

Beneke '93 Broadhurst '93



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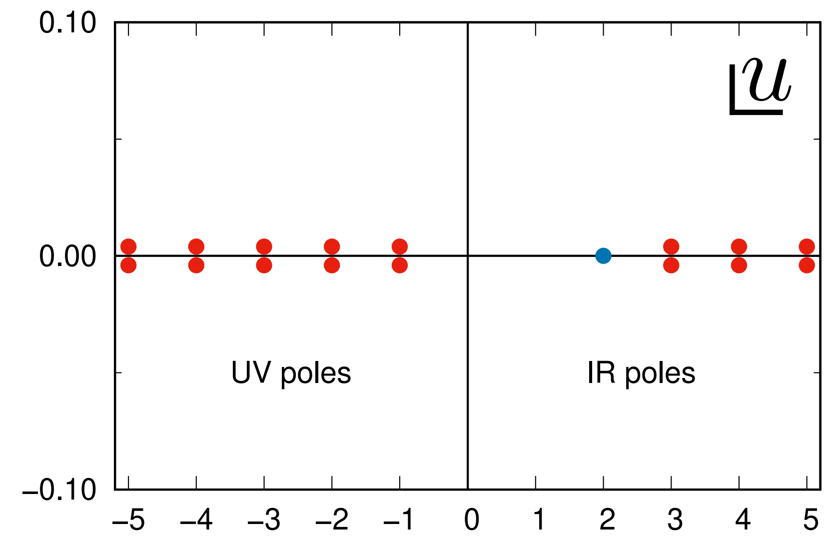


$$a_Q = \frac{\alpha_s(Q)}{\pi}$$

- $\delta^{(0)}$ function (FOPT):

$$\delta^{(0)}_{{\rm FO},l\beta}(a_Q) = a_Q + 5.119a_Q^2 + 28.78a_Q^3 + 156.7a_Q^4 + 900.8a_Q^5 + 4867a_Q^6 + \cdots .$$

- Borel transform of the $\delta^{(0)}$ function:

$$B[\delta^{(0)}](u) = \frac{-12}{(u-1)(u-3)(u-4)}\frac{\sin(\pi u)}{\pi u}B[\widehat{D}](u),$$

convergence

Systematic study of different strategies:

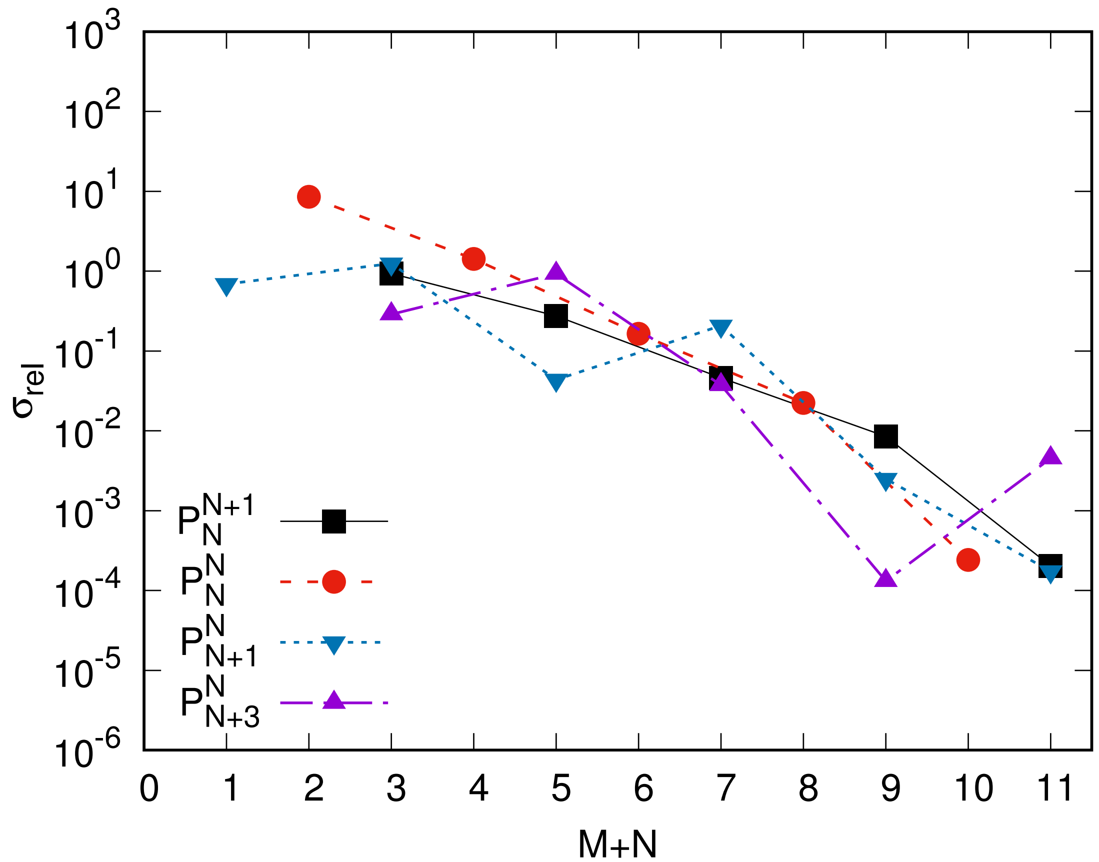
- PAs on $B[\widehat{D}](u)$;
- PAs on $\widehat{D}(\alpha_s)$;
- PAs on $\delta^{(0)}_{\rm FO}(\alpha_s)$;
- PAs on $B[\delta^{(0)}_{\rm FO}](u)$;
- DLog PAs on $B[\widehat{D}](u)$;
- DLog PAs on $B[\delta^{(0)}_{\rm FO}](u)$;
- $\cdots$

Check (and understand!) the convergence of the different Padé sequences

relative error of the 1st predicted coeff.

$$\sigma_{\rm rel} = \left| \frac{c^P_{n,1} - c_{n,1}}{c_{n,1}} \right|$$

With many input coefficients everything works well. Problem: optimize the method with only four coefficients.



optimal strategies

Systematic study of different strategies: optimal strategies

- PAs on $B[\widehat{D}](u)$;
- PAs on $\widehat{D}(\alpha_s)$;
- PAs on $\delta_{\rm FO}^{(0)}(\alpha_s)$; Padés to the series in alpha_s
- PAs on $B[\delta_{\rm FO}^{(0)}](u)$;
- DLog PAs on $B[\widehat{D}](u)$;
- DLog PAs on $B[\delta_{\rm FO}^{(0)}](u)$; D-log Padés to the Borel transform
- $\cdots$

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- $\cdots$

Why are D-log Padés best in this case?

- D-log method:

$$F(u) = \frac{\mathrm{d}}{\mathrm{d}u}\log\left(B[\delta^{(0)}](u)\right)$$
$$= C + \frac{5}{3} + \pi\cot(\pi u) - \frac{2}{1+u} + \frac{3}{3-u} + \frac{1}{4-u} + \frac{1}{1-u} + \frac{1}{2-u} - \frac{1}{u} + \cdots .$$

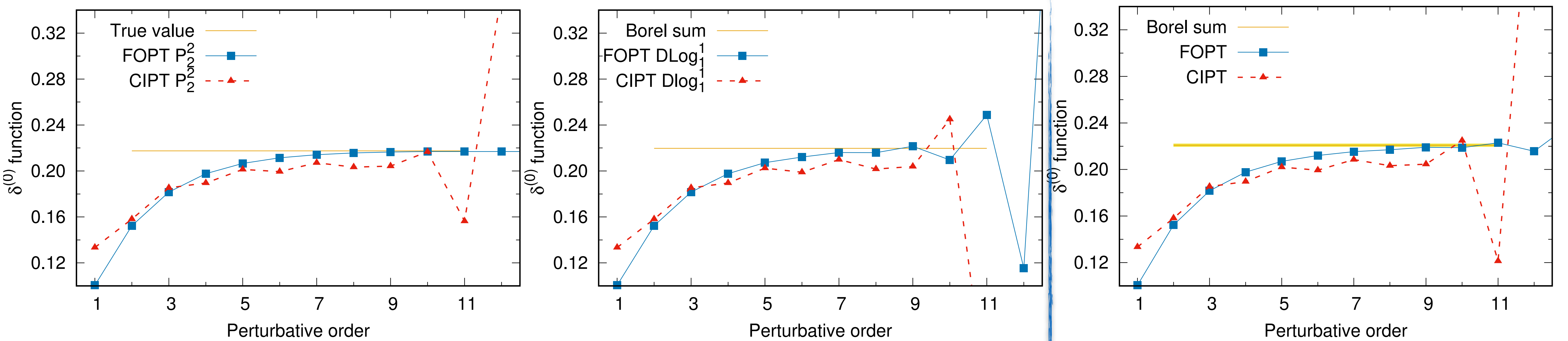
simplification of the analytic structure (no double poles, no scheme dependence, no cuts...)

results in large-beta0

the method can be tested in the so called “large-beta0” limit of QCD

	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	$c_{9,1}$
Large-β_0 (exact)	787.8	-1991	9.857×10^4	-1.078×10^6	2.775×10^7
P_2^2	749.3	-1444	8.169×10^4	-7.514×10^5	1.917×10^7
Dlog_1^1	818.7	-2738	1.189×10^5	-1.663×10^6	4.495×10^7

DB, Masjuan, Oliani '18



$P_2^2(\alpha_s)$ $\mathrm{Dlog}_1^1(u)$ Large-β_0 (exact)

Excellent reproduction of the series at high orders (up to ~10)

Apply the optimal strategies to QCD

Results in QCD

Results in QCD

- In QCD we know only the first four coefficients:

$$\widehat{D}(a_Q) = a_Q + 1.640a_Q^2 + 6.371a_Q^3 + 49.08a_Q^4 + c_{5,1}a_Q^5 + c_{6,1}a_Q^6 + \cdots$$

- Apply the optimal strategies (from the study of large-β_0):

PAs on $\delta^{(0)}$ function:

	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	$c_{9,1}$	Padé sum
P_1^2	55.62	276.1	3865	1.952×10^4	4.288×10^5	1.289×10^6	0.2080
P_2^1	55.53	276.5	3855	1.959×10^4	4.272×10^5	1.307×10^6	0.2079
P_1^3	input	304.7	3171	2.442×10^4	3.149×10^5	2.633×10^6	0.2053
P_3^1	input	301.3	3189	2.391×10^4	3.193×10^5	2.521×10^6	0.2051

D-log PAs on Borel transform of $\delta^{(0)}$ function:

	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	$c_{9,1}$	Borel sum
DLog_0^1	51.90	272.6	3530	1.939×10^4	3.816×10^5	1.439×10^6	0.2050
DLog_1^0	52.08	273.7	3548	1.953×10^4	3.840×10^5	1.456×10^6	0.2052
DLog_0^2	input	254.1	3243	1.725×10^4	3.447×10^5	1.186×10^6	0.2012
DLog_2^0	input	256.4	3270	1.769×10^4	3.493×10^5	1.258×10^6	0.2019

Results in QCD

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$$\widehat{D}(a_Q) = a_Q + 1.640a_Q^2 + 6.371a_Q^3 + 49.08a_Q^4 + c_{5,1}a_Q^5 + c_{6,1}a_Q^6 + \cdots$$

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Excellent post-diction of the 5-loop result

$$c_{4,1} = 53 \pm 4$$

Results in QCD

- In QCD we know only the first four coefficients:

$$\widehat{D}(a_Q) = a_Q + 1.640 a_Q^2 + 6.371 a_Q^3 + 49.08 a_Q^4 + c_{5,1} a_Q^5 + c_{6,1} a_Q^6 + \cdots$$

- Apply the optimal strategies (from the study of large-β_0):

PAs on $\delta^{(0)}$ function:

	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	$c_{9,1}$	Padé sum
P^2_1	55.62	276.1	3865	1.952×10^4	4.288×10^5	1.289×10^6	0.2080
P^1_2	55.53	276.5	3855	1.959×10^4	4.272×10^5	1.307×10^6	0.2079
P^3_1	input	304.7	3171	2.442×10^4	3.149×10^5	2.633×10^6	0.2053
P^1_3	input	301.3	3189	2.391×10^4	3.193×10^5	2.521×10^6	0.2051

D-log PAs on Borel transform of $\delta^{(0)}$ function:

	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	$c_{9,1}$	Borel sum
DLog^1_0	51.90	272.6	3530	1.939×10^4	3.816×10^5	1.439×10^6	0.2050
DLog^0_1	52.08	273.7	3548	1.953×10^4	3.840×10^5	1.456×10^6	0.2052
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DLog^0_2	input	256.4	3270	1.769×10^4	3.493×10^5	1.258×10^6	0.2019

Excellent post-diction of the 5-loop result

$$c_{4,1} = 53 \pm 4$$

Estimates from other methods proved less effective

$$c_{4,1} = 27 \pm 16$$

[Baikov et al, 03]
[Kataev, Starshenko 94,95]

Results in QCD

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$$c_{5,1} = 277 \pm 51$$

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$$c_{5,1} = 277 \pm 51 \qquad c_{6,1} = 3460 \pm 690$$

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$c_{5,1} = 277 \pm 51$ $\quad$ $c_{6,1} = 3460 \pm 690$ $\quad$ $c_{7,1} = (2.02 \pm 0.72) \times 10^4$

Results in QCD

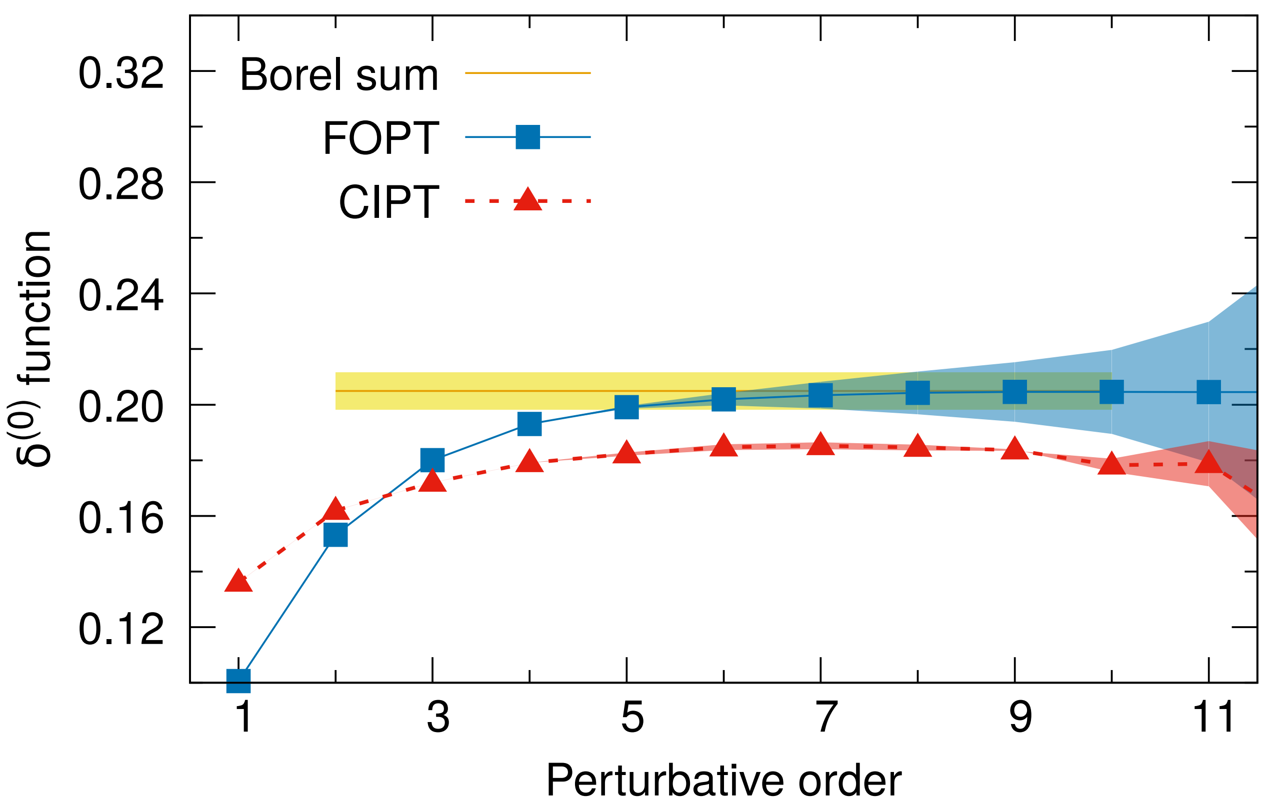
$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$
277 ± 51	3460 ± 690	$(2.02 \pm 0.72) \times 10^4$	$(3.7 \pm 1.1) \times 10^5$
$c_{9,1}$	$c_{10,1}$	$c_{11,1}$	$c_{12,1}$
$(1.6 \pm 1.4) \times 10^6$	$(6.6 \pm 3.2) \times 10^7$	$(-5 \pm 57) \times 10^7$	$(2.1 \pm 1.5) \times 10^{10}$

$c_{5,1} = 283 \pm 142$ [Beneke, Jamin, 08] (Using Borel Model)

$c_{5,1} = 140 \pm 100$ [Baikov et al, 03] (Using part of five loop info)

$c_{5,1} = 275$ [Baikov et al, 08] (Using the principle of Fastest Apparent Convregence)

$c_{5,1} = 287 \pm 40$ [Caprini '19] (Conformal mappings)



$\delta^{(0)} = 0.2050 \pm 0.0067 \pm 0.0130$

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$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$
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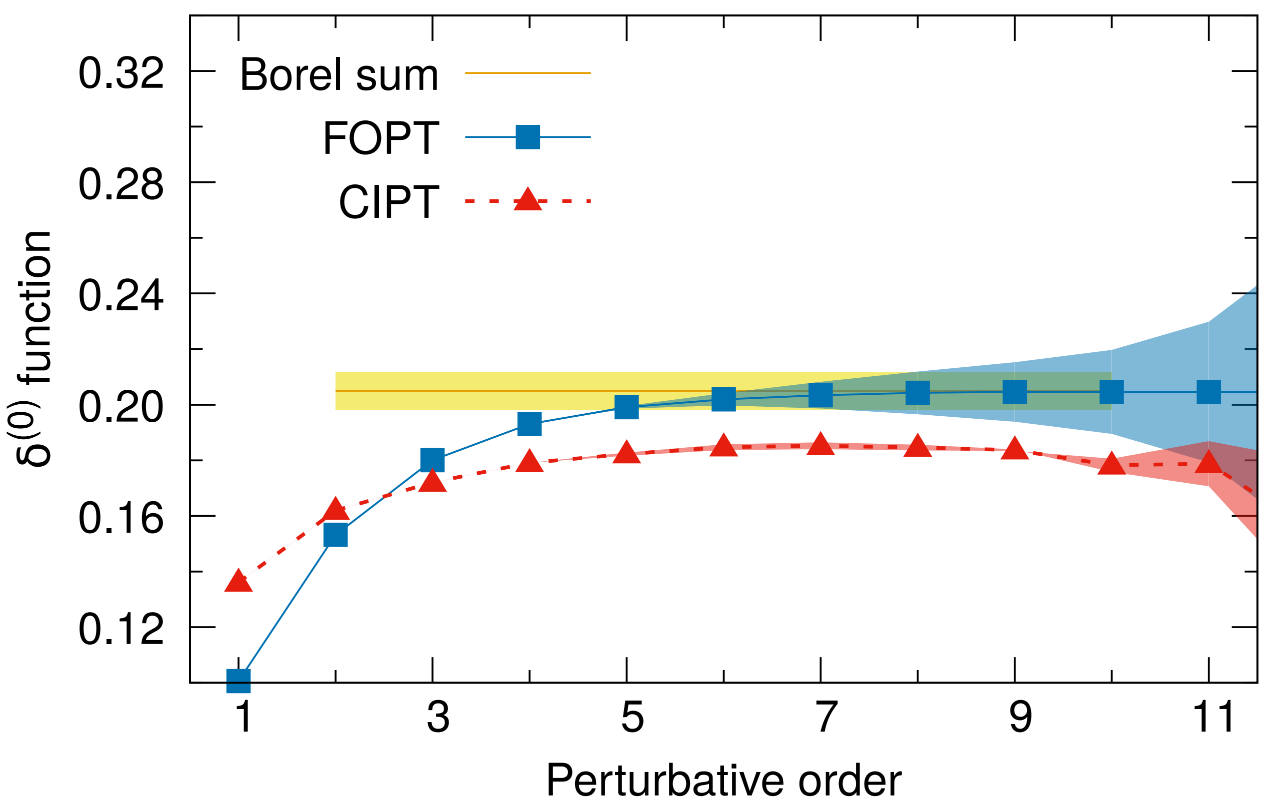


$$\delta^{(0)} = 0.2050 \pm 0.0067 \pm 0.0130$$

Impact on the strong coupling perturbative uncertainty from tau decays

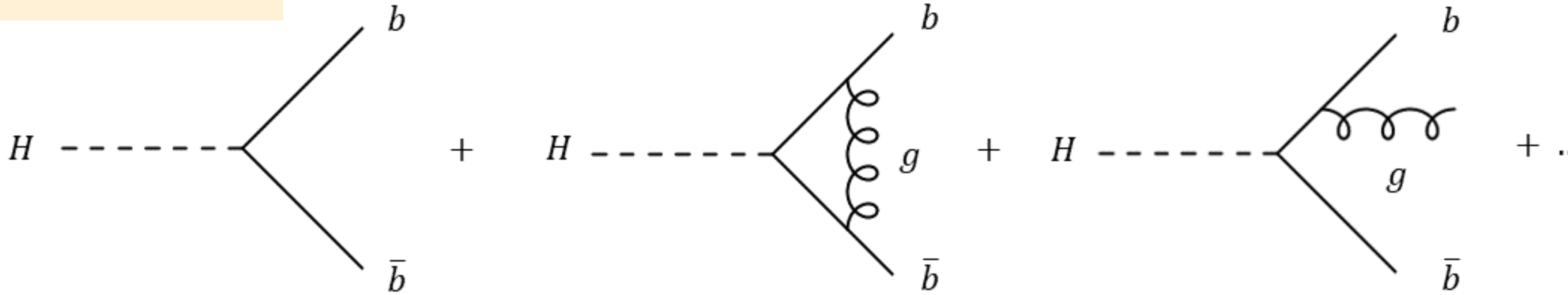
$$\tau \to (\text{hadrons}) + \nu_\tau$$

$$\alpha_s(m_Z) = 0.1171 \pm 0.0010$$

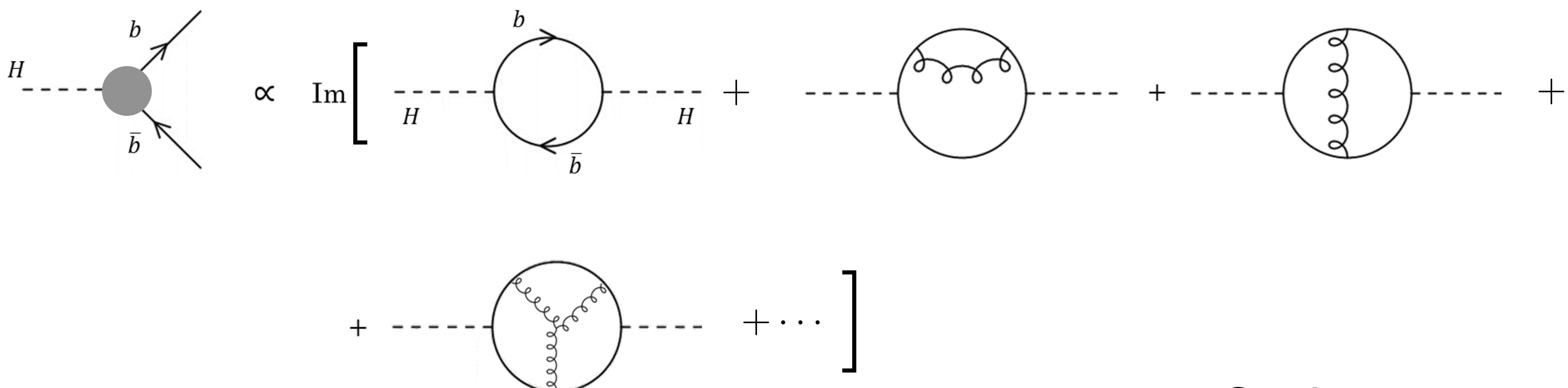
DB, Golterman, Maltman, Peris, Rodrigues, Schaaf, PRD '21 arXiv:[2012.10440]

Higgs decays to bottom quarks

Decay $H \to b\bar{b}$



Optical theorem



Optical theorem

$$\Gamma(H \to b\bar{b}) = \mathrm{Im}\,\Pi/m_H$$

(massless limit)

Scalar qq correlator

$$\Pi(p^2) \equiv i \int \mathrm{d}x \;\; \mathrm{e}^{ipx} \,\langle \Omega | T\{ j(x) j^{\dagger}(0) \} | \Omega \rangle$$

$$j(x) = m_q \;:\bar{q}_f(x)\, q_f(x):$$

$H \to b\bar{b}$

Decay $H \to b\bar{b}$

(massless case)

$$a_s = \frac{\alpha_s}{\pi}$$

$$\mathrm{Im}\,\Pi(s) = \frac{N_c}{8\pi}\, m_b^2\, s \left[1 + \sum_{n=0}^{\infty} c_n a_s^n\right]$$

$c_1 = \frac{17}{3}$	$c_2 = 29.1467$	$c_3 = 41.7576$	$c_4 = -825.747$
1980	**1990**	**1997**	**2006**
Braaten, Leveille Sakai	Gorishny et al	Chetyrkin	Baikov, Chetyrkin, Kühn
2-loop	3-loop	4-loop	5-loop
NLO	N2LO	N3LO	N4LO

Calibrating the strategies

large-β_0 limit

- Scalar correlator in large-β_0

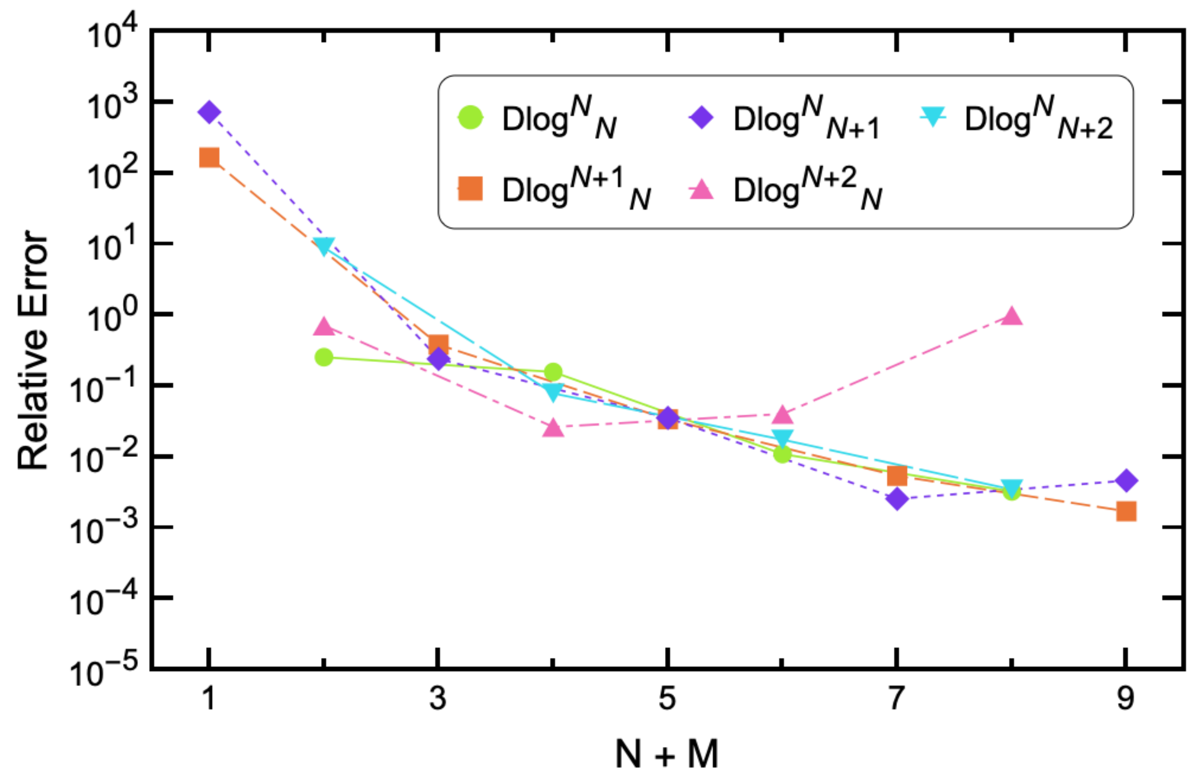
$$\Pi_{L\beta}(s) = \frac{N_c}{4\pi^2}\, m^2 s \left[1 - \frac{L}{2} - \frac{1}{9}\sum_{n=1}^{\infty}\left(-\frac{\beta_1}{2}\right)^{n-1} H_{n+1}(L)\, a_s^n\right], \qquad L \equiv \ln\left(-\frac{s}{\mu^2}\right)$$

[Broadhurst, Kataev, Maxwell '01]

- For $\mu^2 = -s$ and $N_f = 5$

$$\Pi_{L\beta}(s) = \frac{N_c}{4\pi^2}\, m^2 s\left[1 + 3.0542\, a_s + 17.990\, a_s^2 + 63.519\, a_s^3 + 443.45\, a_s^4 + 2958.45\, a_s^5 + \ldots\right]$$

D-log Padés (to the second derivative) again superior to ordinary Padés



Results in QCD

Second derivative of the massless scalar correlator

$$\Pi''(s) = -\frac{N_c}{8\pi^2}\,\frac{m^2}{s}\,\left[1 + 3.6667\,a_s + 12.8098\,a_s^2 + 39.6839\,a_s^3 + 153.955\,a_s^4 + \ldots\right]$$

	r_4	r_5	r_6	r_7	r_8	r_9
$P_1^1(u)$	184	1143	8850	82 240	891 707	1.10×10^7
$P_1^2(u)$	–	796	5149	39 954	361 671	3.74×10^6
$P_2^1(u)$	–	740	4297	29 376	231 963	2.08×10^6

	r_4	r_5	r_6	r_7	r_8	r_9
$\mathrm{Dlog}_1^0(u)$	107	247	473	716	802	581
$\mathrm{Dlog}_0^1(u)$	103	196	114	−956	−4287	−5728
$\mathrm{Dlog}_2^0(u)$	–	789	4877	36 178	307 326	2.99×10^6
$\mathrm{Dlog}_0^2(u)$	–	905	5605	34 626	262 295	2.21×10^6

Using all available coefficients we obtain stable results.

Results in QCD

We can then get the perturbative coefficients of the scalar correlator

	$d_{5,1}$	$d_{6,1}$	$d_{7,1}$	$d_{8,1}$
$P^2_1(u)$	40 968	549 637	8.37×10^6	1.43×10^8
$P^1_2(u)$	40 912	548 137	8.34×10^6	1.42×10^8
$\mathrm{Dlog}^0_2(u)$	40 961	549 284	8.36×10^6	1.43×10^8
$\mathrm{Dlog}^2_0(u)$	41 077	551 531	8.39×10^6	1.43×10^8
$P^3_1(a_s)$	40 769	544 499	8.26×10^6	1.41×10^8
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Final results

$d_{5,1}$	$d_{6,1}$	$d_{7,1}$	$d_{8,1}$
$(4.09 \pm 0.03) \times 10^4$	$(5.48 \pm 0.07) \times 10^5$	$(8.33 \pm 0.12) \times 10^6$	$(1.42 \pm 0.02) \times 10^8$

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…and of the imaginary part

$$\mathrm{Im}\,\Pi(s) = \frac{N_c}{8\pi}\, m_b^2\, s \left[1 + \sum_{n=0}^{\infty} c_n a_s^n\right]$$

Estimated Coefficients

$$c_5 = -8200 \pm 308$$

$$c_6 = (-2.80 \pm 0.69) \times 10^4$$

$$c_7 = (1.48 \pm 2.03) \times 10^5$$

$$c_8 = (2.39 \pm 4.92) \times 10^6$$

DB, P Masjuan, C London, in preparation

Results in QCD

Decay $H \to b\bar{b}$

(massless case)

$$a_s = \frac{\alpha_s}{\pi}$$

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Estimated 6-loop (N5LO)

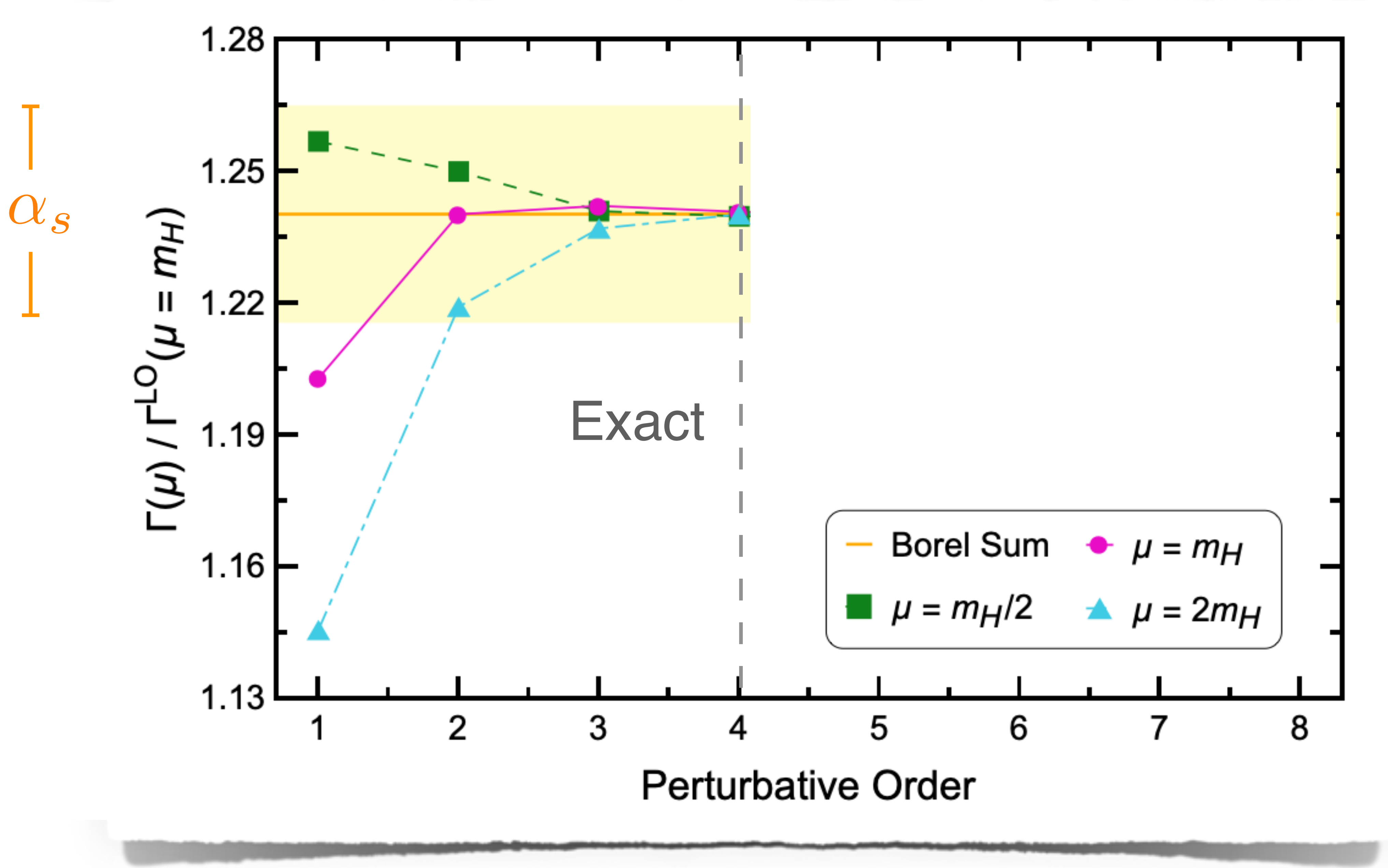
DB, P Masjuan, C London, in preparation

Diogo Boito

$H \to b\bar{b}$ theory uncertainties

Decay $H \to b\bar{b}$

Truncation error vs. strong coupling error



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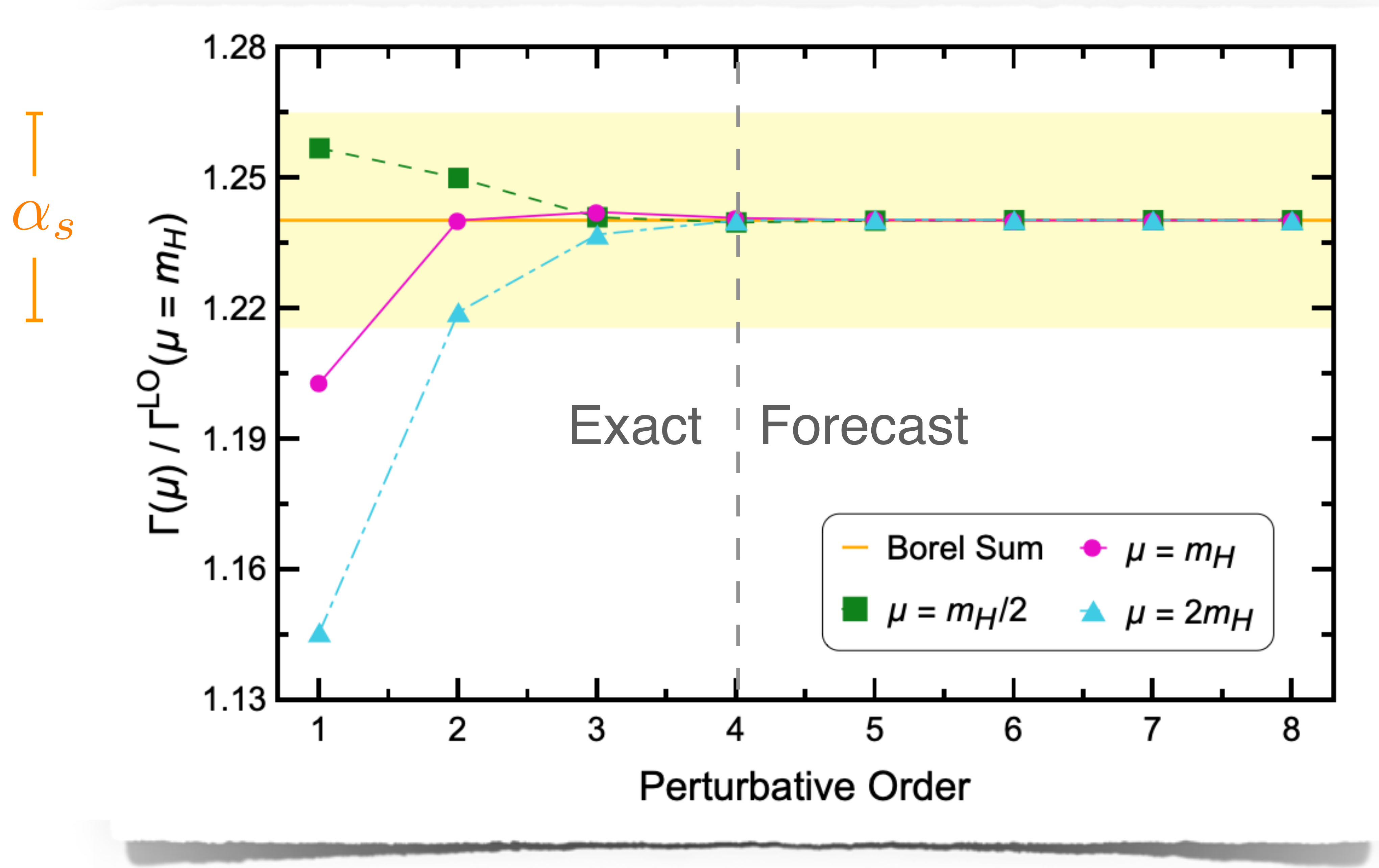
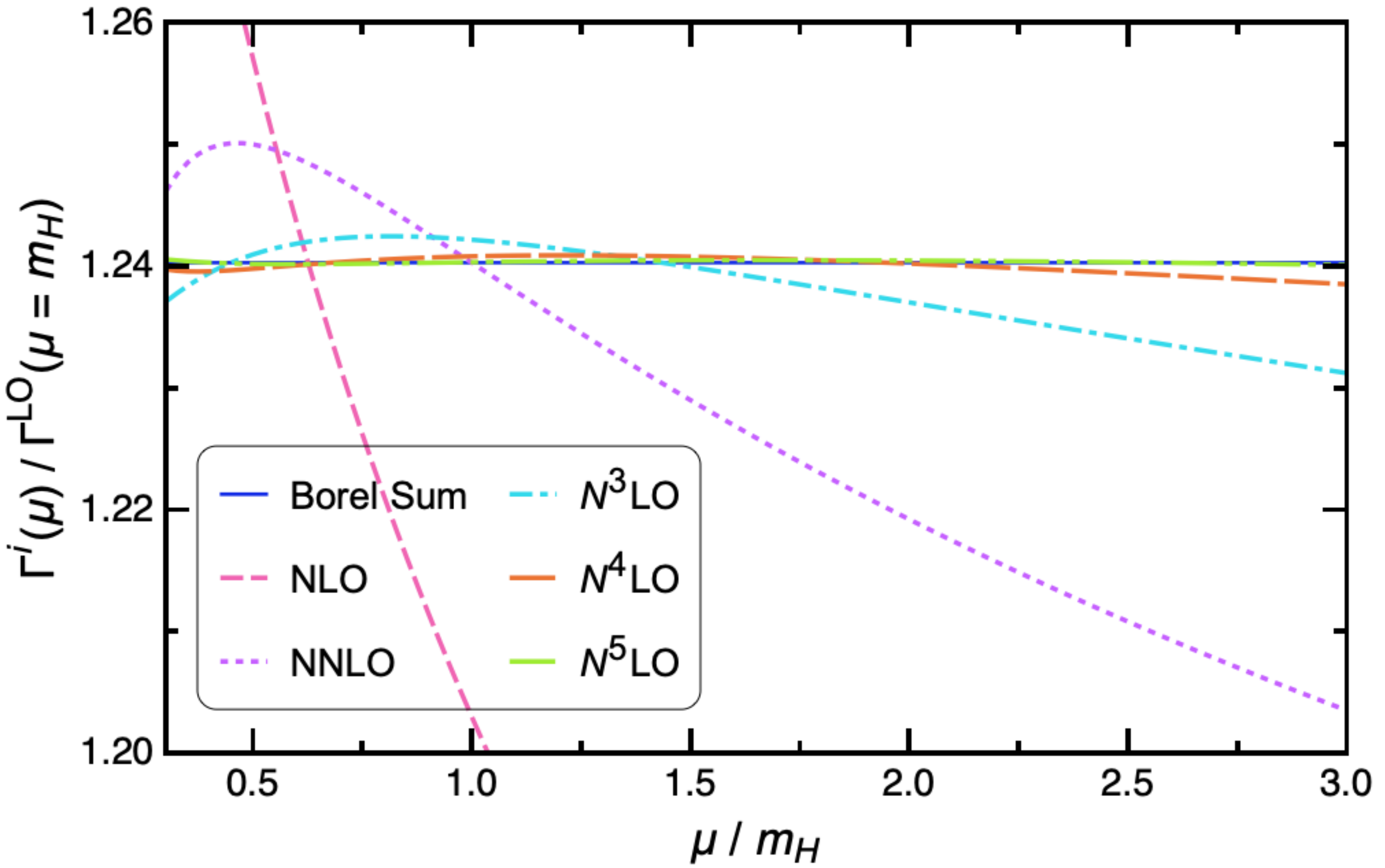


$H \to b\bar{b}$ theory uncertainties

Decay $H \to b\bar{b}$

Renormalization scale variation



At N4LO we already have a very stable perturbative series

Conclusions

- Method for a systematic, model ind., estimate of MHOs in pt. QCD.
- Borel Transform + D-log Padés very effective (faster convergence).
- Methods calibrated in the $\mathrm{large}\text{-}\beta_0$
- Very good post-diction of 5-loop results —> reliable 6-loop results.
- Excellent results for hadronic tau decays.
- Very good results for $H \to b\bar{b}$
- In $H \to b\bar{b}$ it is better to invest in QCD parameters than in more loops.