The pion LCDA from the pion electromagnetic form factor

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Pion light-cone distribution amplitudes (LCDAs)
* are used to describe hard exclusive processes with pions, e.g.:
\[ \gamma^*(Q^2)\pi^\pm \to \pi^\pm \quad \gamma\gamma^*(Q^2) \to \pi^0 \quad B \to \pi\ell\nu_\ell \quad B \to \pi\pi \]
* represent nonperturbative part of continuum-QCD methods: factorization theorems, QCDF, light-cone sum rules (LCSRs)

The leading twist-two LCDA: asymptotics of the pion e.m. form factor

\[
\lim_{Q^2 \to \infty} F_\pi(Q^2) = \frac{8\pi\alpha_s(Q^2)}{9Q^2} \left| f_\pi \int_0^1 \frac{\varphi_\pi(u)}{1-u} \right|^2
\]


We suggest to use data on the timelike e.m. pion form factor, measured in \( e^+e^- \to \pi^+\pi^- \) to determine/constrain the pion LCDA

[Shan Cheng, AK, Aleksey Rusov, 2007.05550]
Structure of the pion LCDA

• Definition:
\[
\langle \pi^+(p) | \bar{u}(x)[x,0] \gamma_\mu \gamma_5 d(0) | 0 \rangle \bigg|_{x^2 \to 0} = -i f_\pi p_\mu \int_0^1 du e^{iup \cdot x} \varphi_{\pi}(u) \bigg|_{x^2 \to 0} = -i f_\pi p_\mu \int_0^1 du e^{iup \cdot x} \varphi_{\pi}(u) + \ldots
\]

• Conformal expansion, Gegenbauer moments:
\[
\varphi_{\pi}(u, \mu) = 6u(1-u) \left[ 1 + \sum_{n=2,4,\ldots} a_n(\mu) C_n^{(3/2)}(2u-1) \right]
\]

* ERBL evolution: \( a_n(\mu) \to 0 \) at large \( \mu \)
* \( a_n(\mu \sim 1\text{GeV}) \) to be determined by nonperturbative methods
* models with \( \{a_2, a_4, \ldots, a_{n_{\text{max}}}\} \)
Pion electromagnetic form factor

- The standard definition:

\[ \langle \pi(p_2) | J_{\mu}^{em} | \pi(p_1) \rangle = F_\pi(q^2)(p_1 + p_2)_{\mu}, \quad q = p_2 - p_1, \]

- spacelike form factor: \( q^2 = -Q^2, \ Q^2 \geq 0 \), \( F_\pi(0) = 1 \)

\* \( e^- \pi^\pm \to e^- \pi^\pm \), measured only at small \( Q^2 \) [NA7 Collab. (1986)]

\* accessible indirectly in \( eN \to e\pi N \)
measured at \( Q^2 \lesssim 2.0 \ \text{GeV}^2 \)
[Jefferson Lab, G.Huber et al. (2008)]

- timelike form factor: \( q^2 = s \geq 4m^2_{\pi} \)

- \( F_\pi(q^2) \) is real valued at \( q^2 < 4m^2_{\pi} \)
and develops imaginary part at \( q^2 > 4m^2_{\pi} \)
(\( \rho \)-resonance poles and branch points of thresholds)
The timelike form factor: data

- Measured in $e^+e^- \rightarrow \pi^+\pi^-$

$$\sigma(e^+e^- \rightarrow \pi^+\pi^-)(s) \sim |F_\pi(s)|^2, \quad s = q^2 > 4m_{\pi}^2$$

- BaBar collaboration (using 232 fb$^{-1}$ data, ISR) [J.Lees et al. (2012)]
The timelike form factor: parametrization

- The BaBar data at $\sqrt{s} < 3.0$ GeV - fitted using a $\rho$-resonance ansatz:

$$F_{\pi}^{\text{data}}(s) = \frac{\sum c_{\rho_n}^{\pi} \text{BW}_{\rho_n}^{\text{GS}}(s)}{1 + \sum_{\rho_n} c_{\rho_n}^{\pi}}, \quad \rho_n = \{\rho, \omega, \rho', \rho'', \rho^{'''}\}, \quad F_{\pi}(0) = 1$$

- the Gounaris-Sakurai resonance formula [Gounaris, Sakurai (1968)]

$$\text{BW}_{\rho_n}^{\text{GS}}(s) = \frac{m_{\rho_n}^2 + m_{\rho_n} \Gamma_{\rho_n} d(m_{\rho_n})}{m_{\rho_n}^2 - s + f(s, m_{\rho_n}, \Gamma_{\rho_n}) - i m_{\rho_n} \Gamma(s, m_{\rho_n}, \Gamma_{\rho_n})}$$

- we extrapolate above $\sqrt{s} > 3.0$ GeV, employing the dual-QCD model [C.Dominguez, (2001)] [C.Bruch, AK, J.Kühn (2005)]

$$F_{\pi}^{\text{dQCD}}(s) = \sum_{n} c_{\rho_n}^{\pi} \text{BW}_{\rho_n}^{\text{dQCD}}(s)$$

$$\text{BW}_{\rho_n}^{\text{dQCD}}(s) = \frac{m_{\rho_n}^2}{m_{\rho_n}^2 - s - i m_{\rho_n} \Gamma_{\rho_n}}, \quad c_{\rho_n}^{\pi} = \frac{(-1)^n \Gamma(\beta - 1/2)}{\alpha' m_{\rho_n}^2 \sqrt{\pi} \Gamma(n + 1) \Gamma(\beta - 1 - n)}$$
"Standard" dispersion relation (unsubtracted, due to QCD asymptotics)

\[ F_\pi(q^2) = \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\text{Im} F_\pi(s)}{s - q^2}, \quad s_0 = 4m_\pi^2 \]

* Depends on a model for imaginary part at \( s > 4m_\pi^2 \)

A different form: modulus representation [B.Geshkenbein (2000)]

\[ F_\pi(q^2) = \exp \left[ \frac{q^2 \sqrt{s_0 - q^2}}{2\pi} \int_{s_0}^{\infty} ds \frac{\ln |F_\pi(s)|^2}{\sqrt{s - s_0} s (s - q^2)} \right], \quad q^2 < s_0 \]

valid if \( F_\pi(q^2) \) has no zeros in the \( q^2 \) plane [H.Leutwyler (2002)]

* Depends on the directly measured \( |F_\pi(s)|^2 \)
Pion form factor: timelike vs spacelike

- Spacelike form factor measurement in agreement with dispersion relation
- Onset of “duality”: asymptotics of $F(Q^2)$ and $F_\pi(s)$ coincide above few first $\rho$ resonances
LCSR for the spacelike form factor


• The correlation function: **vacuum-pion transition amplitude**

\[ F_{\mu\nu}(p, q) = i \int d^4x e^{iqx} \langle 0 | T \{ j_{\mu 5}(0) j^{em}_\nu(x) \} | \pi^+(p) \rangle = F((p - q)^2, Q^2) p_\mu p_\nu + ... \]

\[ j_{\mu 5} = \bar{d} \gamma_\mu \gamma_5 u \] - the pion interpolating current,

* Operator-product expansion near \( x^2 \sim 0 \) in terms of pion DAs:

\[ F_{\mu\nu}(p, q) = \sum_{t=2,4,..} \int D u_i T_t((p - q)^2, Q^2; u_i, \mu) \otimes \phi_t(u_i, \mu), \]

• factorization of the correlation function at \( |(p - q)^2|, Q^2 \gg \Lambda^2_{QCD} \):

\[ F_{\mu\nu}^{OPE}((p - q)^2, Q^2) = \sum_{t=2,4,..} \int D u_i T_t((p - q)^2, Q^2; u_i, \mu) \otimes \phi_t(u_i, \mu), \]
$F_\pi(Q^2)$ from LCSR

- Matching the OPE to the hadronic dispersion relation in $(p - q)^2$:

\[
F^{\text{OPE}}((p - q)^2, Q^2) = 2i \frac{f_\pi F_\pi(Q^2)}{m_\pi^2 - (p - q)^2} + \int_0^\infty ds \frac{\rho^h(Q^2, s)}{s - (p - q)^2} \left(3m_\pi^2\right) \text{pion pole} + \text{heavier states}
\]

- quark-hadron duality: $\rho^h(Q^2, s) \to \rho^{\text{OPE}}(Q^2, s) \theta(s - s_0^{\pi})$

- $s_0^{\pi} = 0.7 \pm 0.1$ GeV$^2$ (QCD sum rule for $f_\pi$) \[M.\text{Shifman}, A.\text{Vainshtein}, V.\text{Zakharov (1979)}\]

- the LCSR after Borel transform: $u_0 = Q^2/(s_0^{\pi} + Q^2)$ \[V.\text{Braun}, AK, M.\text{Maul (2000)}\]

\[
F_\pi(Q^2) = \int_{u_0}^{1} du \varphi_\pi(u, \mu) \exp\left(-\frac{\bar{u}Q^2}{uM^2}\right) + \{\text{tw2 NLO}\} + \{\text{tw 4, 6}\}
\]

- contains both hard-scattering (factorizable) and soft end-point (nonfactorizable) contributions to $F_\pi(Q^2)$

- the tw2 NLO term $\sim \alpha_s/Q^2$ at $Q^2 \to \infty$, and reproduces the QCD asymptotics

- subleading OPE terms determined by $\delta^2_\pi$ (twist 4), and $\langle 0|\bar{q}q|0\rangle$ (fact. twist 6)

\[
\delta^2_\pi(2\text{GeV}) = (0.18 \pm 0.06) \text{ GeV}^2 \text{ (QCD SR)} \quad \text{in agreement with lattice QCD [G.Bali et al (2018)]}
\]
Probing Gegenbauer moments with $F_\pi(Q^2)$

- Isolating the dependence on Gegenbauer coefficients at $\mu_0 \sim 1$ GeV:

$$F_{\pi\text{LCSR}}(Q^2) = F_\pi^{(tw2,as)}(Q^2) + \sum_{n=2,4,\ldots} a_n^\pi(\mu_0) f_n(Q^2, \mu, \mu_0)$$

$$+ F_\pi^{tw4,LO}(Q^2) + F_\pi^{tw6,fact}(Q^2)$$

$$f_n = 6 \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{\gamma_2n/\beta} \int_{0}^{1} du \frac{u(1-u)e^{-\bar{u}Q^2/(uM^2)}}{C_{2n}^{(3/2)}(2u - 1) + f_{nNLO}}$$

- assuming purely asymptotic $\varphi_\pi(u)$, comparing with dispersion relation:

- nonasymptotic terms are important!
Fit of the Gegenbauer moments

<table>
<thead>
<tr>
<th>Model</th>
<th>$a_2(1\text{ GeV})$</th>
<th>$a_4(1\text{ GeV})$</th>
<th>$a_6(1\text{ GeV})$</th>
<th>$a_8(1\text{ GeV})$</th>
<th>$\chi^2_{\text{min}}/\text{ndf}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${a_2}$</td>
<td>0.302 ± 0.046</td>
<td></td>
<td></td>
<td></td>
<td>0.302 ± 0.046</td>
</tr>
<tr>
<td>${a_2, a_4}$</td>
<td>0.279 ± 0.047</td>
<td>0.189 ± 0.060</td>
<td></td>
<td></td>
<td>0.279 ± 0.047</td>
</tr>
<tr>
<td>${a_2, a_4, a_6}$</td>
<td>0.270 ± 0.047</td>
<td>0.179 ± 0.060</td>
<td>0.123 ± 0.086</td>
<td></td>
<td>0.270 ± 0.047</td>
</tr>
<tr>
<td>${a_2, a_4, a_6, a_8}$</td>
<td>0.269 ± 0.047</td>
<td>0.185 ± 0.062</td>
<td>0.141 ± 0.096</td>
<td>0.049 ± 0.116</td>
<td>0.049 ± 0.116</td>
</tr>
</tbody>
</table>

**FIG. 6.** The pion spacelike form factor calculated from the dispersion relation (39) (magenta curve; central input) and from the LCSR with the fitted Gegenbauer moments (orange band; the model $\{a_2, a_4, a_6, a_8\}$) compared with the measurements of NA7.
Comparison of the second and fourth Gegenbauer moments obtained with various methods

<table>
<thead>
<tr>
<th>Method</th>
<th>$a_2$(1 GeV)</th>
<th>$a_4$(1 GeV)</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lattice QCD</td>
<td>0.135 ± 0.032</td>
<td>–</td>
<td>[1]</td>
</tr>
<tr>
<td>QCD sum rule</td>
<td>0.28 ± 0.08</td>
<td>–</td>
<td>[2]</td>
</tr>
<tr>
<td>QCD sum rule with nonlocal condensate</td>
<td>0.203$^{+0.069}_{-0.057}$</td>
<td>$^{+0.094}_{-0.087}$</td>
<td>[3]</td>
</tr>
<tr>
<td>LCSR fitted to Jlab data</td>
<td>0.17 ± 0.08</td>
<td>0.06 ± 0.10</td>
<td>[4]</td>
</tr>
<tr>
<td>LCSR fitted to dispersion relation</td>
<td>0.22 - 0.33</td>
<td>0.12 - 0.25</td>
<td>this work</td>
</tr>
</tbody>
</table>

[1] G.Bali et al. (2019) see the talk by Gunnar Bali
Outlook

- New method to assess the pion twist-2 LCDA: combining LCSR, dispersion relation and data on the timelike $F_\pi$

- Global fit involving other methods:
  - LCSR and data on the $F_{\pi\gamma\gamma}$* form factor
  - Lattice QCD

- Future applications:
  - the pion vector/isovector form factor in $\tau \rightarrow \pi^- \pi^0 \nu_{\tau}$ decays,
  - kaon DA (with $a_1^K, a_3^K \neq 0$) from the kaon timelike e.m. form factor,

- Comparing the modulus representation with $F_\pi(Q^2)$ on the lattice

[HPQCD, J.Koponen et al (2017)]
In conclusion, let me mention the recently published book, containing an introduction to the subjects of this talk:

see: https://www.routledge.com/Hadron-Form-Factors-From-Basic-Phenomenology-to-QCD-Sum-Rules/Khodjamirian/p/book/9781138306752
Backup
Fit in detail

\[
\{ a_2, a_4, a_6, a_8 \} \quad \text{vs} \quad \{ a_2, a_4, a_6 \}
\]

\[
\{ a_2, a_4 \} \quad \text{vs} \quad \{ a_2 \}
\]
Different version of the fit

Fixing $a_2^\pi(1\text{ GeV}) = 0.135 \pm 0.032$
Using a value \(a_2^\pi(2\text{ GeV}) = 0.101 \pm 0.024\) from LQCD [Bali et al. (2019)]

\[\Rightarrow a_2^\pi(1\text{ GeV}) = 0.135 \pm 0.032\]

<table>
<thead>
<tr>
<th>Model</th>
<th>(a_4(1\text{ GeV}))</th>
<th>(a_6(1\text{ GeV}))</th>
<th>(a_8(1\text{ GeV}))</th>
<th>(\chi^2_{\text{min}}/\text{ndf})</th>
</tr>
</thead>
<tbody>
<tr>
<td>({a_2, a_4})</td>
<td>0.218 (\pm) 0.059</td>
<td></td>
<td></td>
<td>3.93</td>
</tr>
<tr>
<td>({a_2, a_4, a_6})</td>
<td>0.203 (\pm) 0.060</td>
<td>0.157 (\pm) 0.086</td>
<td></td>
<td>2.81</td>
</tr>
<tr>
<td>({a_2, a_4, a_6, a_8})</td>
<td>0.210 (\pm) 0.061</td>
<td>0.179 (\pm) 0.095</td>
<td>0.062 (\pm) 0.116</td>
<td>2.71</td>
</tr>
</tbody>
</table>