Extending the LCSR method within QCD for pion TFF to low momenta. Theory and phenomenology

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based on

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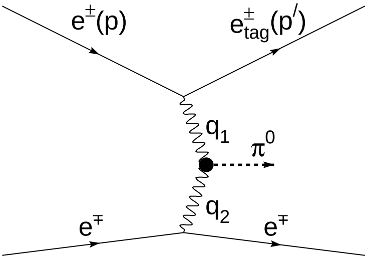
Experimental status of pion transition FF

Why it is interesting for QCD?

The measurements of TFF is the **clean** experiment with **a single pion in final state** that possesses **the best accuracy (BESIII!)** among others exclusive hard reactions ($q_2^2 \sim 0$). pQCD corrections N^2LO are also known.

BESIII (2019) Q^2 **: 0.3 — 3.1** GeV^2
Promising very precise data (PRELIMINARY, arXiv:1810.00654)

CELLO (1991) Q^2 **: 0.7 — 2.2** GeV^2
CLEO (1998) Q^2 **: 1.6 — 8.0** GeV^2
agrees with collinear QCD

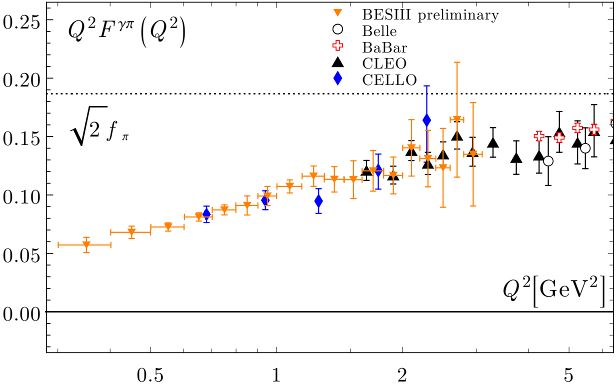


BaBar (2009) Q^2 **: 4 — 40** GeV^2
TFF has growing tendency with Q^2
creating the "BaBar puzzle"

Belle (2012) Q^2 **: 4 — 40** GeV^2
returns to collinear QCD

Experimental status of pion transition FF at $q_2^2 \sim 0$

Experimental Data on $F_{\gamma\gamma^*\pi}$: **BESIII (2019)**, **CELLO**, CLEO, **BaBar**



The low energy tail below 1 GeV^2 of **BESIII** data is **unreachable for pQCD**.

OUTLINE

1. **Intro**: Experimental and Theoretical motivations to modify fixed order pQCD (FOPT) calculation of transition FF for $\gamma\gamma^*(Q^2) \to \pi^0$ at low Q^2.

2. Current status of Light Cone SR **(LCSR)** predictions in **$N^2LO_{\beta_0}$ FOPT**

3. **Dispersive form for pion TFF + RG generates**
   a "New" perturbation theory - the known **fractional APT**.
   Behavior of **FAPT**couplings.

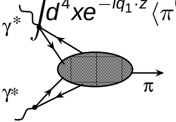
4. **Light cone SR with FAPT**, results of data processing:
   Determination of **twist-2 pion DA** and scales of **twist-4,6**
   new prediction for the **pion-photon TFF down to 0.35 GeV**^{2}

5. **Conclusions**

STORE

Review: APT $\Rightarrow$ **Generalized "Fractional" APT**= **FAPT**, coupling behavior.

Factorization $\gamma^*(q_1)\gamma^*(q_2) \to \pi^0(P)$ in pQCD

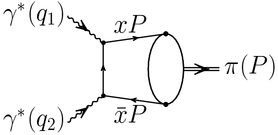


$$\int d^4x e^{-iq_1\cdot z}\langle\pi^0(P)|T\{j_\mu(z)j_\nu(0)\}|0\rangle = i\epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta \cdot \boldsymbol{F}^{\gamma^*\gamma^*\pi}(Q^2,q^2)\,,$$

where $-q_1^2 = Q^2 > 0, \;\; -q_2^2 = q^2 \geq 0$

$$\boldsymbol{F}^{\gamma^*\gamma^*\pi}(Q^2,q^2) = \boldsymbol{T(Q^2,q^2,\mu_F^2;x)} \otimes \boldsymbol{\varphi_\pi(x;\mu_F^2)} + O(\frac{1}{Q^4})\,,$$

Collinear factorization at $\boldsymbol{Q^2,q^2} \gg$ **(a hadron scale)^2**, for the leading twist, $\boldsymbol{\mu_F^2}$ – boundary between large scale Q^2 and hadronic $\boldsymbol{m_\rho^2}$. At the parton level



$$F^{\gamma^*\gamma^*\pi}(Q^2,q^2) = \frac{\sqrt{2}}{3}f_\pi \int_0^1 dx \frac{\boldsymbol{1}}{\boldsymbol{Q^2x+q^2\bar{x}}}\,\boldsymbol{\varphi_\pi(x)}$$

$+\boldsymbol{\alpha_s, \beta_0\alpha_s^2}$**–radiative corr.** $\sim -$**20**%(at $\sim$ **1GeV**)

+ twist 4, twist 6

We focus on the **radiative corr.** and their summation, this is **Main subject**.

Motivation, method of solution, phenomenological goals

- **To describe the low BESIII momentum domain in QCD the new perturbation theory is required.** Besides, the processing of data in this domain can get information about **the values of higher twists.**

- We rearrange the QCD perturbation theory following **RG** and **dispersion relation** in a way to extend the domain of applicability down to Λ^2_{qcd} and below. This **new PT** appears **by itself** and is a generalization of the known **Fractional Analytic PT - FAPT**.

- Based on the results of processing BESIII data **we reconcile all** of the main **pionic parameters oftwists 2, 4, 6 together** with the current results of lattice simulation.

Current status of Light Cone SR **(LCSR)** in **FOPT** (What was before).

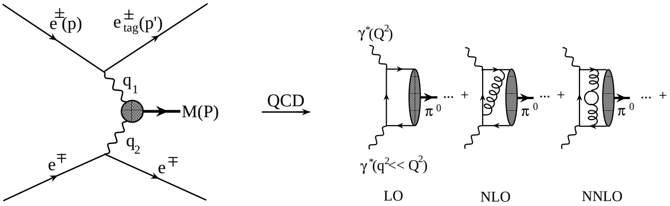
$$\gamma(q^2 \simeq 0)\gamma^*(Q^2) \to \pi^0$$

Status of Light Cone Sum Rules

at N^2LO of FOPT

M.S. &Pimikov A. & Stefanis N., PRD 93 (2016) 114018

Theoretical status of the pion TFF: **hard and hadronic parts**



Our Theoretical advances in both parts of **QCD factorization**:

- high order NNLO$_\beta$ contribution $O(\alpha_s^2\beta_0)$ **[Melic et al2003, MS2009]** to the hard part;
- distrib. amplit. **[BMS2001&QCDSR-NLC]** of twist-2 for pion part;
- contributions of tw-4**[BMS2003]**, corrections of**"tw-6"[Ageev etal2012]**.

3 steps to LCSR for TFF:

Perturbative & twist expansion;
Dispersive representation in q^2 for LCSR ,
LCSR with duality interval s_0 for $q^2 \to 0$.

The structure of pion TFF in QCD FOPT

Hard process at $-Q^2, -q^2 \gg m_\rho^2 \Rightarrow$ **collinear factorization**

$$F^{(\text{tw=2})}_{\text{FOPT}}(Q^2, q^2) = N_{\text{T}} \left(T_{\text{LO}} + a_s T_{\text{NLO}} + a_s^2 T_{\text{NNLO}} + \ldots\right) \otimes \varphi_\pi^{(2)}$$

$$\begin{aligned}
T_{\text{LO}} &= a_s^0(\mu_F^2)\, T_0(y) \equiv 1/\left(q^2\bar{y} + Q^2 y\right) \\
a_s T_{\text{NLO}} &= a_s^1(\mu_F^2)\, T_0(y) \otimes \left[\mathcal{T}^{(1)} + \underline{L\, V_0}\right](y,x)\,, \\
a_s^2 T_{\text{NNLO}} &= a_s^2(\mu_F^2)\, T_0(y) \otimes \Big[\mathcal{T}^{(2)} - \underline{L\, \mathcal{T}^{(1)}\beta_0} + \underline{L\, \mathcal{T}^{(1)} \otimes V_0} - \underline{\frac{L^2}{2}\, \beta_0 V_0} \\
&\quad + \underline{\frac{L^2}{2}\, V_0 \otimes V_0} + \underline{\underline{L\, V_1}}\Big](y,x)\,,
\end{aligned}$$

$L = L(y) = \ln\left[(q^2\bar{y} + Q^2 y)/\mu_F^2\right]$
Plain terms $\mathcal{T}^{(1)}, \mathcal{T}^{(2)} (\mathcal{T}_\beta^{(2)})$ - corrections to parton subprocess;
Underlined terms due to $\bar{a}_s(y)$ and **ERBL**, V_0 - kernel;
underlined term - two loops **ERBL**, V_1 - kernel.

Pion TFF in pQCD with **RG improvement**

Collecting all of the "underlined" terms of RG-evolution into
$a_s(\mu^2) \to \bar{a}_s(y) \equiv \bar{a}_s\left(q^2\bar{y} + Q^2 y\right)$ and **ERBL–factor** **[AMS2018]**.

$$F^{(\text{tw=2})}(Q^2,q^2) = N_{\text{T}} T_0(y) \underset{y}{\otimes} \left\{ \left[1 + \bar{a}_s(y)\mathcal{T}^{(1)}(y,x) + \bar{a}_s^2(y)\mathcal{T}^{(2)}(y,x) + \ldots\right] \underset{x}{\otimes} \exp\left[-\int_{a_s}^{\bar{a}_s(y)} d\alpha \frac{V(\alpha;x,z)}{\beta(\alpha)}\right]\right\} \underset{z}{\otimes} \varphi_\pi^{(2)}(z,\mu^2)\,,$$

$$\varphi_\pi^{(2)}(x,\mu^2) = \psi_0(x) + \sum_{n=2,4,\ldots}^{\infty} b_n(\mu^2)\,\psi_n(x) - \text{Gegenbauer basis}$$

$$F^{(\text{tw=2})}(Q^2,q^2) = F_0^{\text{RG}}(Q^2,q^2) + \sum_{n=2,4,\ldots}^{\infty} b_n(\mu^2) F_n^{\text{RG}}(Q^2,q^2)$$

One loop resumed result in leading Logs, $\nu_n = \gamma_n/2\beta_0$

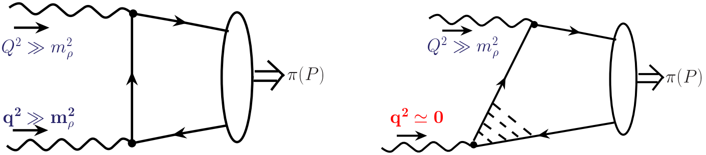
$$F_n^{\text{RG}}(Q^2,q^2) = N_{\text{T}} T_0(y) \underset{y}{\otimes} \left\{ \left[1 + \bar{a}_s(y)\mathcal{T}^{(1)}(y,x)\right] \left(\frac{\bar{a}_s(y)}{a_s(\mu^2)}\right)^{\nu_n} \right\} \underset{x}{\otimes} \psi_n(x)$$

But, the $\bar{a}_s(y) \equiv \bar{a}_s\left(q^2\bar{y} + Q^2 y\right)$ **is inapplicable** within factorization,
Resum formula **fall out** from the PT applicability domain at $q^2 = 0, y \ll 1$.

$\gamma^*\gamma \to \pi$: Light-Cone SR consideration at $q^2 \sim 0$ within FOPT

LCSR effectively accounts for long-distance effects of real photon using quark-hadron duality in vector channel and **dispersion relation** in q^2 **[Khodjamirian EJPC(1999)]**,

$$F_{\gamma\gamma^*\pi}(Q^2, q^2 \to 0) = \int_{s_0}^{\infty} \rho^{\rm PT}(Q^2, s)\frac{ds}{s} + \underbrace{\int_0^{s_0} \rho^{\rm PT}(Q^2, s) e^{(m_\rho^2 - s)/M^2}\frac{ds}{m_\rho^2}}_{F_{\rho\gamma^*\pi}},$$

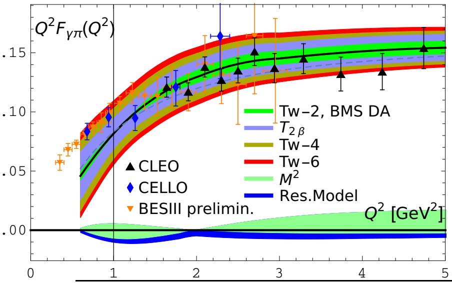


FOPT/twist contributions are given in form of convolution with pion DAs:

$$\rho^{\rm PT} \sim \frac{1}{\pi}{\rm Im}\left[T^{(2)}_{\rm LO} + a_s T^{(2)}_{\rm NLO} + a_s^2 T^{(2)}_{{\rm NNLO}_{\beta_0}} + \ldots\right] \otimes \varphi_\pi^{\rm tw2} + \frac{1}{\pi}{\rm Im}\left[T^{(4)}_{\rm LO}\right] \otimes \varphi_\pi^{\rm tw4} + \ldots$$

$${\rm Im}\left[\left\{T_0(y),\ T_0(y)L^n(y)\right\} \otimes \varphi\right]$$ – require **cumbersome calculations**

Pion TFF in LCSR in QCD FOPT vs exp. data



MS&Pimikov&Stefanis, PRD93(2016)114018

The predictions fall down arround 1 GeV^2

Challenge for low energy discription

Total rad. corrections	**-18% at 3 GeV^2**
Source	Uncertainty (%)
Unknown NNLO term $\mathcal{T}_c^{(2)}$	∓ 5
Range of Tw-2 BMS DAs	$-3.4 \div 4.1$
Tw-4 coupling $\delta^2 = [0.152 - 0.228]$ GeV^2	± 3.0
Tw-6 $\langle \bar{q}q \rangle^2 = (0.24 \pm 0.01)^6$ GeV^2	$-2.4 \div 3.0$
Total	$-13.6 \div 14.9$

$$\gamma(q^2 \simeq 0)\gamma^*(Q^2) \to \pi^0$$

Dispersive form for pion TFF $+$ RG

==

a “New” perturbation theory –

known Fractional APT.

Properties of FAPT couplings.

Pion TFF in pQCD with **RG improvement**

Collecting all of the terms of RG-evolution into
$a_s(\mu^2) \to \bar{a}_s(y) \equiv \bar{a}_s\left(q^2\bar{y} + Q^2 y\right)$ and **ERBL–factor.**

$$F_n^{(\text{tw=2})}(Q^2, q^2) = N_{\text{T}} T_0(y) \underset{y}{\otimes} \left\{ \left[1 + \bar{a}_s(y)\mathcal{T}^{(1)}(y,x) + \bar{a}_s^2(y)\mathcal{T}^{(2)}(y,x) + \ldots\right] \underset{x}{\otimes} \exp\left[-\int_{a_s(\mu^2)}^{\bar{a}_s(y)} d\alpha \, \frac{V(\alpha; x, z)}{\beta(\alpha)}\right]\right\} \underset{z}{\otimes} \psi_n(z)\,,$$

One loop resumed result, $\nu_n = \gamma_n/2\beta_0$, gives the simplest expression:

$$F_n^{\text{RG}}(Q^2, q^2) = \frac{N_{\text{T}}}{a_s(\mu^2)^{\nu_n}} T_0(Q^2, q^2; y) \underset{y}{\otimes} \left\{\bar{a}_s^{\nu_n}(y)\mathbb{1} + \bar{a}_s^{1+\nu_n}(y)\mathcal{T}^{(1)}(y,x)\right\} \underset{x}{\otimes} \psi_n(x)$$

All Logs $= L^n$ are accumulated into $\bar{a}_s^{\nu}(y)$

Dispersive form of TFF leads to fractional APT

$$\Big[F(Q^2,q^2)\Big]_{\text{an}} = \int_{m^2}^{\infty} \frac{\rho_F(Q^2,\sigma)}{\sigma+q^2-i\epsilon}\,d\sigma,\ \ \rho_F(\sigma) = \frac{\textbf{Im}}{\pi}\Big[F(Q^2,-\sigma)\Big]$$

Appear the known **FAPT** $\mathcal{A}_\nu, \mathfrak{A}_\nu$ couplings + a New one – $\mathcal{I}_\nu$ **[Ayala&M&S2018]**

$$\nu(0)=0;\ \boldsymbol{F}_{\mathbf{0}}^{\text{FAPT}}(Q^2,q^2)=N_T\,T_0(Q^2,q^2;y)\underset{y}{\otimes}\Big\{\mathbb{1}+\mathbb{A}_{\mathbf{1}}(\boldsymbol{y})\mathcal{T}^{(1)}(y,x)\Big\}\underset{x}{\otimes}\psi_0(x)$$

$$\nu(n)\neq 0;\ \boldsymbol{F}_{\boldsymbol{n}}^{\text{FAPT}}(Q^2,q^2)=\frac{N_T}{a_s^{\nu_n}(\mu^2)}T_0(Q^2,q^2;y)\underset{y}{\otimes}$$
$$\Big\{\mathbb{A}_{\boldsymbol{\nu_n}}(\boldsymbol{y})\mathbb{1}+\mathbb{A}_{\mathbf{1}+\boldsymbol{\nu_n}}(\boldsymbol{y})\mathcal{T}^{(1)}(y,x)\Big\}\underset{x}{\otimes}\psi_n(x)$$

$\{\mathbb{A}_\nu\}$ – **nonpower series instead of $\bar{a}_s^\nu$**

The same expression as for **RG**-case, $\mathbb{A}_\nu(\boldsymbol{y}) \Leftrightarrow \bar{a}_s^\nu(\boldsymbol{y})$

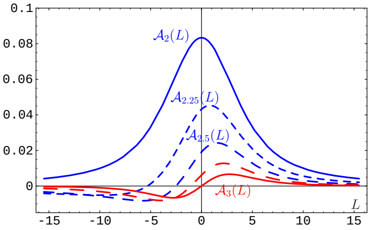
$\mathbb{A}_\nu(\boldsymbol{y}) = \mathcal{A}_\nu(\boldsymbol{Q}(\boldsymbol{y})) - \mathfrak{A}_\nu(\mathbf{0})$ – **are regular at $\boldsymbol{y} > \mathbf{0}$**

the certain kinematics enters by means of $\boldsymbol{Q}(\boldsymbol{y}) \equiv \boldsymbol{q}^2\bar{\boldsymbol{y}} + \boldsymbol{Q}^2\boldsymbol{y}$

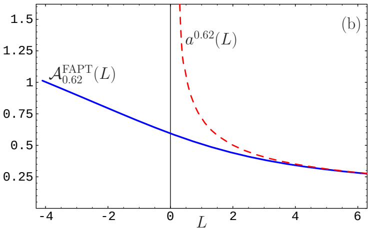
FAPT(Eucl): $\mathcal{A}_\nu[L]$ versus L

$$\mathcal{A}_\nu[L] = \frac{1}{\beta_0^\nu}\left(\frac{1}{L^\nu} - \frac{\text{Li}_{1-\nu}(e^{-L})}{\Gamma(\nu)}\right), L = \ln(Q^2/\Lambda_q^2),$$

Fractional $\nu \in [2,3]$:



Comparison with $\bar{a}_s^\nu[L]$:

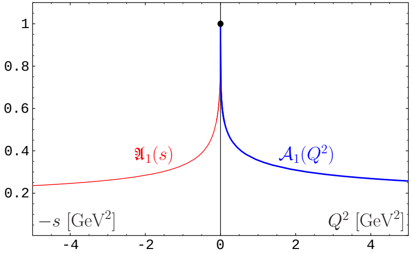


where $\nu = 0.62 = \gamma_2/2\beta_0$

$$\bar{a}_s^{1+\nu}[L] \gg \mathcal{A}_{1+\nu}[L] \gg \mathcal{A}_{2+\nu}[L] \text{ at } L \sim 1$$

$$\bar{a}_s^\nu[L] \gtrless (\mathcal{A}_\nu[L], \mathfrak{A}_\nu[L]) \xrightarrow{L\to\infty} a_s^\nu[L]$$

PT vs **FAPT** for partial TFF.



The original behavior in the vicinity of **$Q^2 = 0$ is not appropriate!** **$\mathfrak{A}_1(0), \mathcal{A}_1(0)$ should be equal to 0**
To hold the **correspondence with PT asymptotics** we put **"calibrated FAPT"** condition:

$$\mathcal{A}_\nu(0) = \mathfrak{A}_\nu(0) = 0 \text{ for } 0 < \nu \leqslant 1$$

Variety of coupling models where suggested to fulfill this property **[Ayala et al, 2017-20]**

Light Cone Sum Rules with FAPT,

New prediction for the pion TFF

$$\gamma(q^2 \simeq 0)\gamma^*(Q^2) \to \pi^0$$

Ayala C. & M.S. & Pimikov A. & Stefanis N.

PRD 103 (2021) 096003, EPJ Web Conf. 222 (2019) 0301, PRD 98 (2018) 09601

The partial TFF$_{\text{LCSR}}$ for zero harmonic ψ_0

$$\boldsymbol{Q^2 F^{\gamma\pi}_{\text{LCSR};0}\left(Q^2\right)} = \text{standard Born term + twist-4,6} \boxed{+\ \ldots}$$

$$N_{\text{T}}\left\{\int_0^{\bar{x}_0}\psi_0(x)\frac{dx}{\bar{x}} + \frac{Q^2}{m_\rho^2}\int_{\bar{x}_0}^1 \exp\left(\frac{m_\rho^2}{M^2} - \frac{Q^2}{M^2}\frac{\bar{x}}{x}\right)\psi_0(x)\frac{dx}{x} + \text{twist-4,6} + \right.$$

$$\left(\frac{\boldsymbol{\mathbb{A}_1(s_0;x)}}{\boldsymbol{x}}\right)\underset{x}{\otimes}\mathcal{T}^{(1)}(x,y)\underset{y}{\otimes}\psi_0(y) +$$

$$\left.\frac{Q^2}{m_\rho^2}\int_{\bar{x}_0}^1 \exp\left(\frac{m_\rho^2}{M^2} - \frac{Q^2}{M^2}\frac{\bar{x}}{x}\right)dx\,\frac{\boldsymbol{\Delta_1(\bar{x})}}{\boldsymbol{x}}\mathcal{T}^{(1)}(\bar{x},y)\underset{y}{\otimes}\psi_0(y) + O(\mathbb{A}_2)\right\},$$

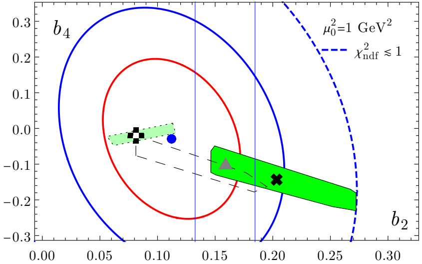
Appear specific couplings $\boldsymbol{\mathbb{A}_\nu(s_0;x), \Delta_\nu(x)}$ due to thresholds in LCSR, $x_0 = s_0/(s_0+Q^2)$,

$$\begin{aligned}
\boldsymbol{\mathbb{A}_\nu(s_0;x)} &= \theta\,(x \geqslant x_0)\,[\mathcal{A}_\nu(Q(x)) - \mathcal{A}_\nu(0)] + \\
&\quad \theta\,(x < x_0)\,[\mathcal{I}_\nu(s_0(x),Q(x)) - \mathfrak{A}_\nu(s_0(x))]\ , \\
\boldsymbol{\Delta_\nu(x)} &= \mathbb{A}_\nu(x) - \mathbb{A}_\nu(s_0;x), \\
s_0(x) &= s_0\bar{x} - Q^2x\,;\ s_0(x_0) = 0\,.
\end{aligned}$$

The processing of the BESIII+ data on $\mathbf{TFF_{LCSR}}$ up to 3.1 GeV^2

We **extract and reconcile** the hadronic characteristics presented in **TFF :**
twist-2 DA, b_2, b_4; the scales of twist-4,6 – $\delta^2_{\text{tw-4}}$, $\delta^2_{\text{tw-6}}$.

$$F^{\gamma\pi}_{\text{LCSR}}\left(Q^2\right) = F^{\gamma\pi}_{\text{LCSR;0}}\left(Q^2\right) + \sum\nolimits_{n=2,4} b_n(\mu^2)\, F^{\gamma\pi}_{\text{LCSR};n}\left(Q^2\right) + \text{Tw-4,6}$$



- best fit DA $\chi^2_{\text{ndf}} = 0.38$
- **red ellipse -** 1σ-region
- **blue solid ellipse -** 2σ-region
- **blue dashed ell.-**$\chi^2_{\text{ndf}} \leq 1$-region
- ✖center of **BMS domain[BMS2001]**
- ✜platykurtic **[Stefanis PLB738,2014]**

twist-2 DA $\in$ **BMS domain**: ▲ $\left(b_2(\mu_0^2) = 0.159, b_4(\mu_0^2) = -0.098\right)$

b_2 lattice (vert. blue lines) **[Bali et al.JHEP08,2019]**

twist-4 : $\delta^2_{\text{tw-4}}(\mu_0^2) = \mathbf{0.19 \pm 0.04}$ GeV^2

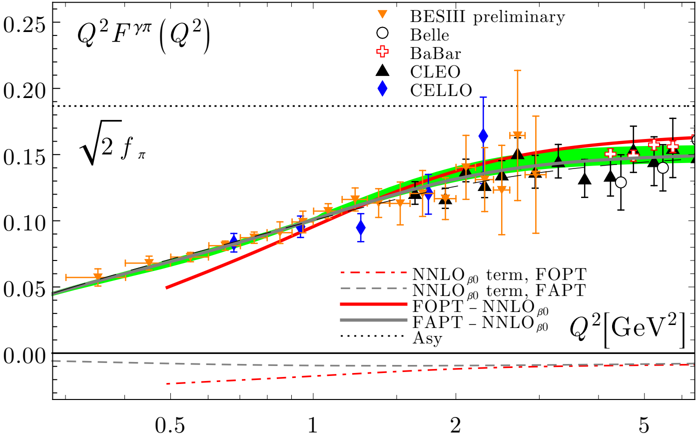
twist-6 : $\delta^2_{\text{tw-6}}(\mu_0^2) = \mathbf{1.61} \times 10^{-4}$ GeV^6

Processing BESIII, CELLO, Cleo data in window $[0.35 \leqslant Q^2 \leqslant 3.1]$ GeV^2,
[MS&Pimikov A.&Stefanis PRD103(2021)096003]

Predictions of $\mathbf{TFF_{LCSR}}$ in FAPT vs the experimental data

$$F^{\gamma\pi}_{\text{LCSR}}(Q^2) = F^{\gamma\pi}_{\text{LCSR;0}}(Q^2) + \sum_{n=2,4} b_n(\mu^2)\, F^{\gamma\pi}_{\text{LCSR};n}(Q^2) + \textbf{Tw-4,6}$$

(30% near lowest Q^2_{exp})



Green line&green strip around - **FAPT predictions** for $Q^2 F^{\gamma\pi}_{\text{LCSR}}$, $\chi^2_{pdf} = 0.57$

Red line - FOPT prediction at N^2LO for $Q^2 F^{\gamma\pi}_{\text{LCSR}}$ – **fall down**

The **fitted parameters** are the scale of Tw-6 $\langle\bar{q}q\rangle^2$ within its error bars and the certain pattern of **pion DA from BMS bunch.**

CONCLUSIONS

1. **LCSRs** augmented with **RG summation** of radiative corrections yield transition FF with improved $\boldsymbol{Q^2}$ behavior and **extends** the domain of **QCD applicability well below 1 GeV**^{2}

2. This composition of **RG sum** and **LCSRs** naturally leads to a generalization of **Fractional APT** that improves perturbative corrections to amplitudes.

3. The applicability of the **FAPT** to exclusive processes demands new conditions for the **FAPT** couplings, $\mathcal{A}_\nu(0) = \mathfrak{A}_\nu(0) = 0, \forall\nu$ as a "feedback"

4. The first time processing of low energy BESIII data + $[\mathbf{0.35} \leqslant \boldsymbol{Q^2} \leqslant \mathbf{3.1}]$ GeV^2 is performed.
   We have **reconciled all twist-2, -4, -"6" pion characteristics** and the lattice result.

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Fractional Analytic Perturbation Theory

FAPT couplings $\mathcal{A}_\nu, \mathfrak{A}_\nu$

Dispersive “Källen–Lehmann” representation

Different coupling images in **Euclidean, $\mathcal{A}_n$**, and **Minkowsk., $\mathfrak{A}_n$**, regions $\overline{\alpha}_s^n \to \{\mathcal{A}_n, \mathfrak{A}_n\}$ **[Shirkov&Solovtsov1997(534cit)-07]**–nonpower series

$$\left[f(Q^2)\right]_{\rm an} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma, \quad \rho_n(\sigma) = \frac{\mathbf{Im}}{\pi}\left[\overline{a}_s^n(-\sigma)\right]\beta_0$$

For 1 loop run, $L = \ln(Q^2/\Lambda^2)$, $L_s = \ln(s/\Lambda^2)$:

$$\rho_1(\sigma) \stackrel{1l}{=} \frac{1}{L_\sigma^2 + \pi^2}$$

$$\mathcal{A}_1[L] = \int_0^\infty \frac{\rho_1(\sigma)}{\sigma + Q^2}\, d\sigma \stackrel{1l}{=} \frac{1}{L} - \frac{1}{e^L - 1}$$

$$\mathfrak{A}_1[L_s] = \int_s^\infty \frac{\rho_1(\sigma)}{\sigma}\, d\sigma \stackrel{1l}{=} \frac{1}{\pi} \arccos \frac{L_s}{\sqrt{\pi^2 + L_s^2}}$$

Inequality:

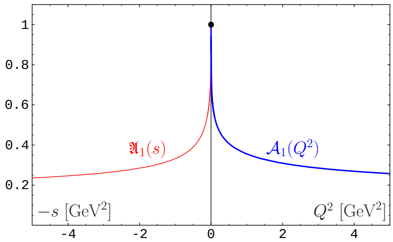
$$a_s^n[L] > (\mathcal{A}_n[L],\ \mathfrak{A}_n[L]) \stackrel{L\to\infty}{\longrightarrow} a_s^n[L]$$

Generalization of $(\mathcal{A}_n, \mathfrak{A}_n)$: $\mathcal{I}_n$ **[Ayala&MS&2018]**

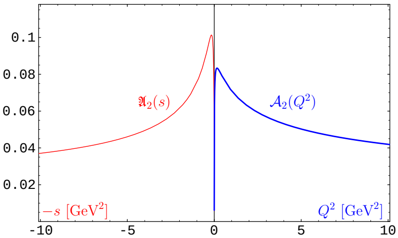
$$\mathcal{I}_n(s, Q^2) = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\, d\sigma$$

APT: Distorting mirror [Shirkov&Solovtsov1997-2007]

Coupling images: $\mathfrak{A}_1(s)$ & $\mathcal{A}_1(Q^2)$



Square-images: $\mathfrak{A}_2(s)$ & $\mathcal{A}_2(Q^2)$



FAPT(Eucl): Properties of $\mathcal{A}_\nu[L]$ [Bakulev,MS,Stefanis 2005-07]

Euclidean coupling :

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{e^{-L}\Phi(e^{-L}, 1-\nu, 1)}{\Gamma(\nu)} \equiv \frac{1}{L^\nu} - \frac{\mathrm{Li}_{1-\nu}(e^{-L})}{\Gamma(\nu)}$$

Here $\Phi(z, \nu, 1)$ is **Lerch's** transcendental, Li_ν - PolyLog functions.
They are analytic functions in ν. Properties:
The charge $\mathcal{A}_\nu(Q^2)$ **is Bounded** for $\nu \geqslant 1$,

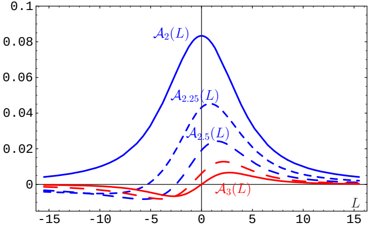
- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\mathcal{A}_\nu[\pm\infty] = 0$ for $\nu > 1$;

$\mathcal{A}_\nu[-\infty] = (\infty)^{1-\nu}$ for $\nu < 1$ *i.e.*,
$\mathcal{A}_\nu(Q^2 \to 0)$ becomes Unbounded for $\nu < 1$

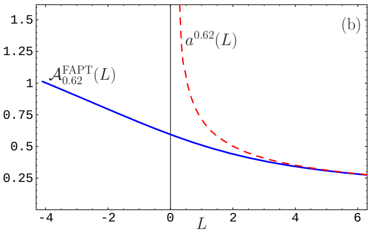
FAPT(Eucl): $\mathcal{A}_\nu[L]$ versus L

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{\mathrm{Li}_{1-\nu}(e^{-L})}{\Gamma(\nu)}$$

Fractional $\nu \in [2,3]$:



Comparison with $\bar{a}_s^\nu[L]$:



where $\nu = 0.62 = \gamma_2/2\beta_0$

$\bar{a}_s^{1+\nu}[L] \gg \mathcal{A}_{1+\nu}[L] \gg \mathcal{A}_{2+\nu}[L]$ at $L \sim 1$

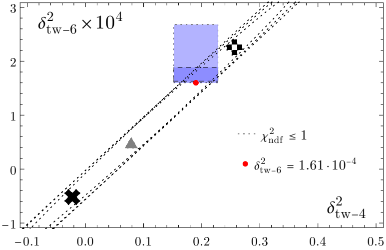
The processing of the experimental data on $\mathbf{TFF_{LCSR}}$ up to 3.1 GeV^2 (1)

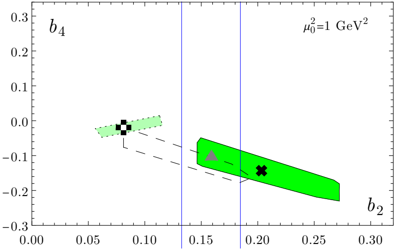
From BESIII, CELLO, Cleo data in window $0.35 \leq Q^2 \leq 3.1$ GeV^2, we **extract and reconcile** the hadronic characteristics presented in **TFF**:
twist-2 DA, the scales of twist-4,6 **[MS et.al PRD 103 (2021) 096003]**.

$$F^{\gamma\pi}_{\text{LCSR}}(Q^2) = F^{\gamma\pi}_{\text{LCSR};0}(Q^2) + \sum_{n=2,4} \boldsymbol{b_n}(\mu^2)\, F^{\gamma\pi}_{\text{LCSR};n}(Q^2) + \boldsymbol{\delta^2_{\text{tw-4}}}(\mu^2) F^{\gamma\pi}_{\text{tw-4}}(Q^2) + \boldsymbol{\delta^2_{\text{tw-6}}} F^{\gamma\pi}_{\text{tw-6}}(Q^2)$$

First, we consider three models of twist-2 DA given by $\boldsymbol{b_2}$, $\boldsymbol{b_4}$:

- DA from QCD SR with NLC **[BMS2001]**
- platykurtic DA ✜ **[Stefanis PLB738, 2014]**
- ▲: $\boldsymbol{b_2}$ from lattice (vert. blue lines) **[Bali et al. JHEP08,(2019]**, $\boldsymbol{b_4}$ from BMS





- Twist-4 from **[Bakulev et al. PRD67,2003]**
  $\boldsymbol{\delta^2_{\text{tw-4}}}(\mu_0^2) = 0.19(4)$ GeV$^2 \sim \langle \bar{q} D^2 q \rangle / \langle \bar{q} q \rangle$
- Twist-6 is extracted from data
  $\boldsymbol{\delta^2_{\text{tw-6}}}(\mu_0^2) = 1.61(26) \times 10^{-4}$ GeV$^6 \sim \alpha_S \langle \bar{q} q \rangle^2$

The processing of the experimental data on **$\mathbf{TFF_{LCSR}}$** up to 3.1 GeV^2 (2)

Second, we fit twist-2 DA given by b_2 , b_4: using obtained twist-4,-6 coeff:

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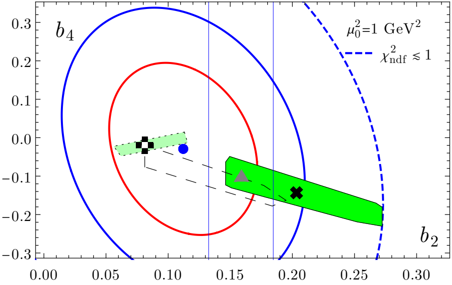

● best fit DA $\chi^2_{\text{ndf}} = 0.38$

red ellipse - 1σ-region

blue solid ellipse - 2σ-region

blue dashed ell.-χ^2_{ndf}≤1-region

NLC QCD SR **[BMS2001]**

✜ platykurtic DA **[Stefanis PLB738, 2014]**

▲ b_2 from lattice (vert. blue linef)
b_4 from BMS domain

- Pion DA model from BMS domain marked as ▲ is within 1σ-region and is suggested for postdictions of pion TFF.
- Low sensitivity of TFF to pion DA at low momenta
- Considered models are in a good agreement with data

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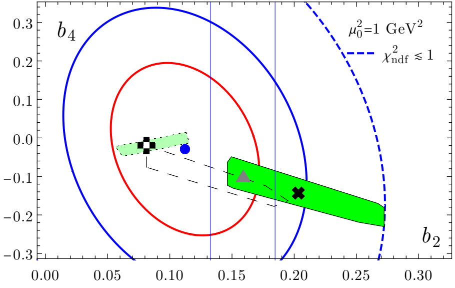

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