Light cone distribution amplitudes from lattice QCD

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Recent DA calculations

- **Introduction**
- **Moments of pseudoscalar twist-2 DAs**[1,2]
- **Moments of octet baryon DAs**[3]
- DAs from X-space and δ_2^{π}[4]
- **Normalization of the photon DA**[5]
- **Outlook**

1) RQCD: GB, VM Braun, M Göckeler, M Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, 1705.10236; 1811.06050;
RQCD: GB, VM Braun, S Bürger, M Göckeler, M Gruber, F Hutzler, P Korcyl, A Schäfer, A Sternbeck, P Wein, 1903.08038 (with "addendum").

2) GB, V Braun, S Collins, A Schäfer, J Simeth, 2106.05398.

3) RQCD: GB, VM Braun, S Bürger, S Collins, M Göckeler, M Gruber, F Hutzler, P Korcyl, A Schäfer, W Söldner, A Sternbeck, P Wein, 1903.12590.

4) RQCD: GB, VM Braun, B Gläßle, M Göckeler, M. Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, P Wein, J-H Zhang, 1709.04325; 1807.06671.

5) GB, G Endrődi, S Piemonte, 2004.08778.

Distribution amplitudes

Light front wavefunction (distribution amplitude, DA) of a hadron parameterizes the distribution of the longitudinal momentum among its partons.

Momentum fractions $0 \leq x_i \leq 1$, $\sum_{i\in\{q,\bar{q},g\}} x_i = 1$.

Wavefunction of a hadron (here pion) near the infinite momentum frame, written as a superposition of different Fock states:

$$|\pi\rangle = c_1|\bar{q}q\rangle + c_2|\bar{q}gq\rangle + c_3|\bar{q}q\bar{q}q\rangle + \dots$$

At leading twist (twist-2) only the valence quarks contribute (unlike in PDFs):

$$x = x_q = 1 - x_{\bar{q}}, \quad \xi = x_q - x_{\bar{q}} = 2x - 1 \in [-1, 1].$$

In hard processes higher Fock states are power suppressed (higher twist).

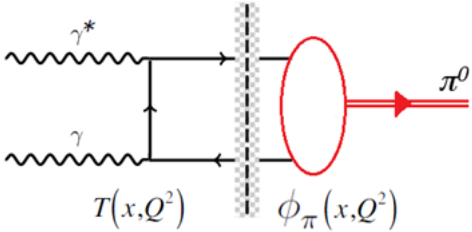
PDFs are (within the parton model) single particle probability densities and can **directly be extracted from fits to DIS and SIDIS data**.

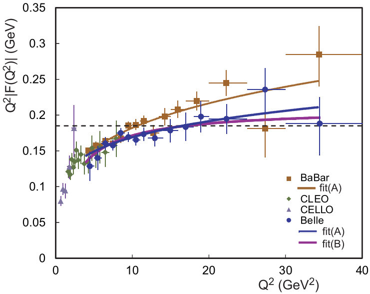
DAs are wavefunctions and only appear within convolutions in hard exclusive processes. Due to other hadronic uncertainties (and experimental techniques), it is much **harder to extract these reliably from experimental data**.

Distribution amplitudes 2

DAs are needed for the theoretical description of hard exclusive processes.

Example: collinear factorization of the $\gamma\gamma^* \to \pi^0$ photoproduction formfactor ($Q \gtrsim \mu \gg \Lambda$) [Belle, 1205.3249]





$$F_{\pi\gamma}(Q^2) = \frac{2F_\pi}{3}\int_0^1 \mathrm{d}x \underbrace{H_{\bar{q}q\gamma}(x,\mu_F,Q^2)}_{\text{hard coefficient function}} \cdot \underbrace{\phi_\pi(x,\mu_F)}_{\text{soft factor (DA)}} + \underbrace{\text{higher twist}}_{F_\pi\cdot\mathcal{O}(1/Q^4)}.$$

$F_\pi \approx 92\,\text{MeV}$. Factorization scale $\mu_F \sim Q^2/4$. Renormalization of H usually at $\mu_R^2 = Q^2$. Also $\exists\,\mathcal{O}(1/Q^4)$ non-factorizable soft contribution.

Definition of meson DAs

Non-local light front matrix element at a separation n ($n^2 = 0$):

$$\left\langle 0 \left| \bar{d}\left(\frac{n}{2}\right) \not{n}\gamma_5 \left[\frac{n}{2}, -\frac{n}{2}\right] u\left(-\frac{n}{2}\right) \right| \pi^+(p) \right\rangle$$
$$= i\sqrt{2}\, F_\pi\, n \cdot p \int_0^1 \mathrm{d}x\, \exp\left\{ i \underbrace{[x-(1-x)]}_{=\xi} \frac{(n\cdot p)}{2} \right\} \phi_\pi(x,\mu)$$

$[n/2, -n/2]$ above denotes a gauge covariant connection.

The DA cannot directly be computed in Euclidean spacetime but its moments can:

$$\langle \xi^n \rangle = \int_0^1 \mathrm{d}x\, (2x-1)^n \phi_\pi(x,\mu)\,, \quad \langle \xi^0 \rangle = 1\,, \quad \langle \xi^1 \rangle = 0.$$

$\langle \xi^{0,2} \rangle$ can be extracted from local matrix elements $\langle 0 | O^{\pm}_{\mu\nu\rho} | \pi^+(p) \rangle$ with

$$O^{\pm}_{\mu\nu\rho} = \bar{d}\left\{ \left[\overleftarrow{D}_{(\mu} \overleftarrow{D}_{\nu} \pm 2 \overleftarrow{D}_{(\mu} \overrightarrow{D}_{\nu} + \overrightarrow{D}_{(\mu} \overrightarrow{D}_{\nu} \right] \gamma_{\rho)} \gamma_5 \right\} u,$$

where $(\cdots)$ gives a traceless, symmetrized expression.

Mesons: Gegenbauer moments

Gegenbauer expansion (only even moments for the pion):

$$\phi_\pi(x,\mu) = 6x(1-x)\left[1 + \sum_{n\in\mathbb{N}} a_{2n}^{\pi}(\mu)\, C_{2n}^{3/2}(2x-1)\right]$$

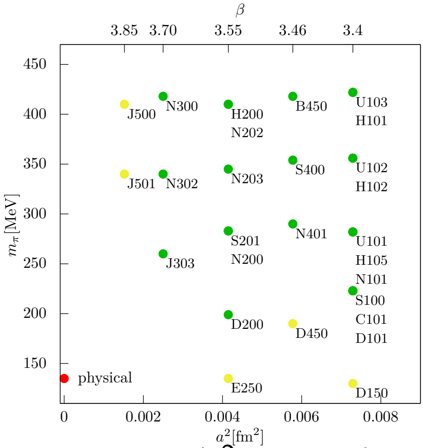
Collinear conformal symmetry: $C_n^{3/2}(\xi)$ in $SL(2,\mathbb{R})$ analogous to $Y_{\ell m}(\theta,\phi)$ in SO(3).
$\langle \xi^{2n}\rangle$ and a_{2n}^{π} are related by simple algebraic expressions ($n=1$ example):

$$a_2^{\pi}(\mu) = \frac{7}{12}\left(5\langle\xi^2\rangle - 1\right) = \frac{7}{12}\left(5\langle\xi^2\rangle - \langle\xi^0\rangle\right)$$

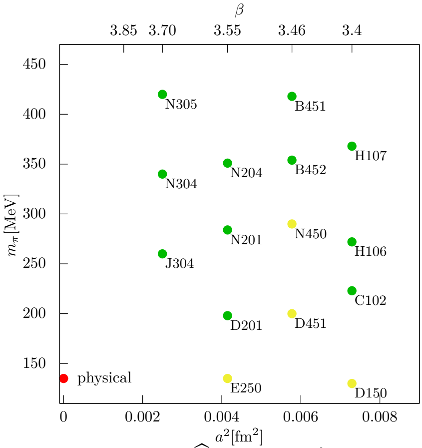
$a_{2n}^{\pi}(\mu) \to 0$ as $\mu \to \infty$: At large scales the lower moments will dominate.

Note the difference in the counting: 2nd DA-moment $\sim$ 3rd PDF-moment.

CLS ensemble overview



$m_s + 2m_\ell = \text{const}$



$\widehat{m}_s \approx \text{const}$

E: $192 \cdot 96^3$, J: $192 \cdot 64^3$, D: $128 \cdot 64^3$, N: $128 \cdot 48^3$, C: $96 \cdot 48^3$,
S: $128 \cdot 32^3$, H: $96 \cdot 32^3$, B: $64 \cdot 32^3$, U: $128 \cdot 24^3$.

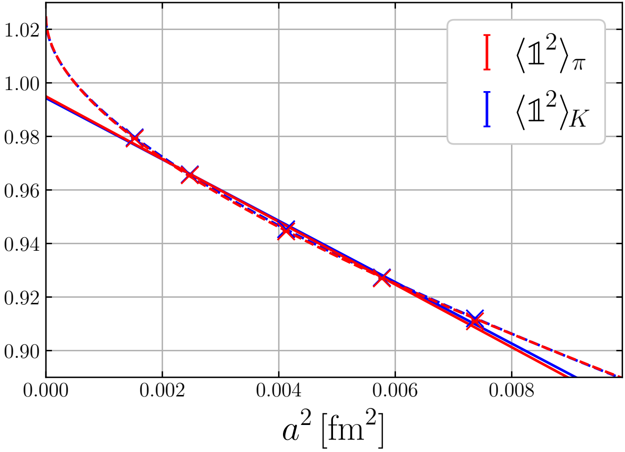
$\exists$ additional ensembles with $m_s = m_\ell$.

Results 1: test of $\langle \xi^0 \rangle \to 1 + \mathcal{O}(\alpha_s^4) \quad (a \to 0)$

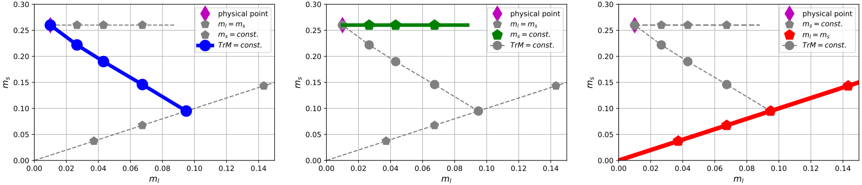
ζ_{ij}: renormalization from lattice to $\overline{\mathrm{MS}}$ [RQCD: GB et al, 2012.06284].

Lattice $\to$ RI'-SMOM non-perturbatively, RI'-SMOM $\to$ $\overline{\mathrm{MS}}$ at 3-loop (N^{3}LO).

Test of renormalization: $\langle \xi^0 \rangle = 1 + \mathcal{O}(a) \sim (\zeta_{22} - 5\zeta_{12}) O^+$



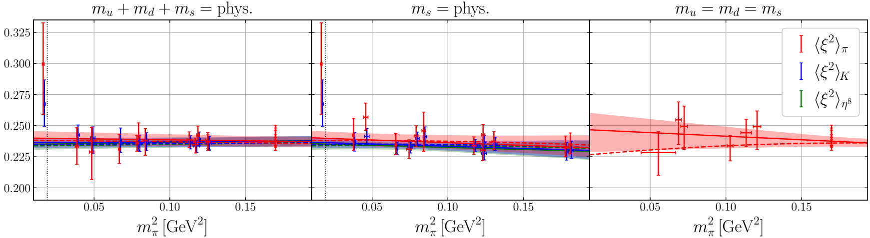
Results 2: quark masss extrapolation



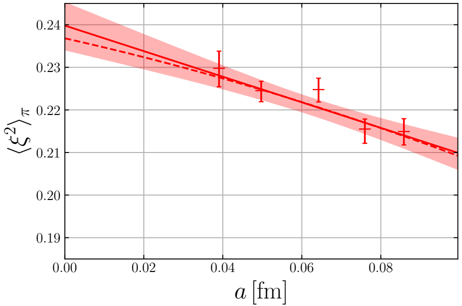
SU(3) NLO ChPT and linear lattice spacing effects (8 parameters):

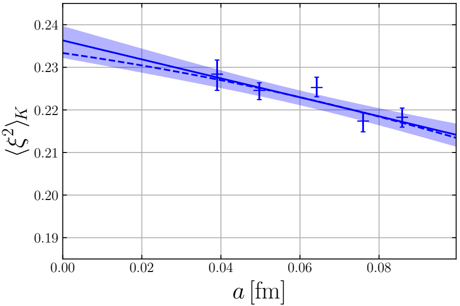
$$\xi_X^2(M_\pi, M_K, a) = \left[\xi_0^2 + \bar{A}\overline{M}^2 + A_X \delta M^2\right]\left[1 + a\left(c_0 + \bar{c}\overline{M}^2 + c_X \delta M^2\right)\right],\ X \in \{\pi, K, \eta_8\},$$

$$A_\pi = -2A_K = -A_{\eta_8},\ \ \delta M^2 = 2M_K^2 - 2M_\pi^2,\ \ \overline{M}^2 = \frac{1}{3}\left(2M_K^2 + M_\pi^2\right).$$



Results 3: the continuum limit





Continuum limit, at the physical point: $(a_2 \propto 5\langle \xi^2 \rangle - 1)$

$$a_2^{\pi,\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.116^{+16}_{-17}(4)_r(9)_a(5)_m, \qquad \langle \xi^2 \rangle_\pi = 0.240^{+6}_{-6}(2)_r(3)_a(2)_m,$$

$$a_2^{K,\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.106^{+10}_{-12}(4)_r(9)_a(4)_m, \qquad \langle \xi^2 \rangle_K = 0.236^{+3}_{-4}(1)_r(3)_a(1)_m,$$

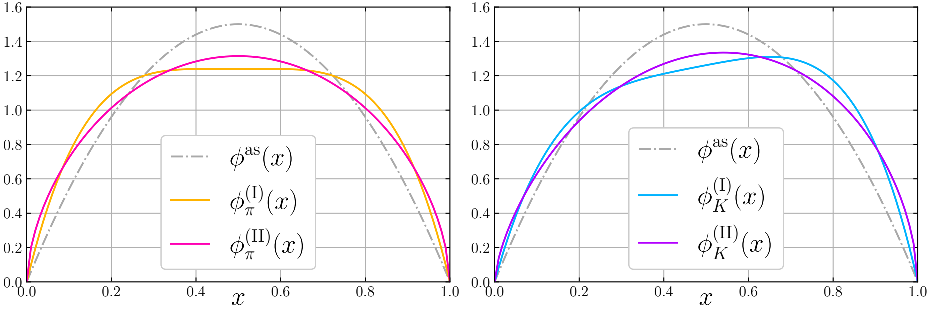
$$a_1^{K,\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.0525^{+17}_{-19}(3)_r(20)_a(17)_m = \tfrac{5}{3}\langle \xi^1 \rangle_K.$$

Summary of light pseudoscalar DAs

Model I: $\phi^{(\mathrm{I})}(x) = 6x(1-x)\left[1 + a_1 C_1^{3/2}(\xi) + a_2 C_2^{3/2}(\xi)\right]$

Model II: $\phi^{(\mathrm{II})}(x) = \frac{\Gamma(2+\alpha^+ + \alpha^-)}{\Gamma(1+\alpha^+)\Gamma(1+\alpha^-)}\, x^{\alpha^+}(1-x)^{\alpha^-}$

with a_1, a_2 fixed from our analysis ($\overline{\mathrm{MS}}$ at 2 GeV). π: $\alpha^+ = \alpha^-$, $a_1 = 0$.



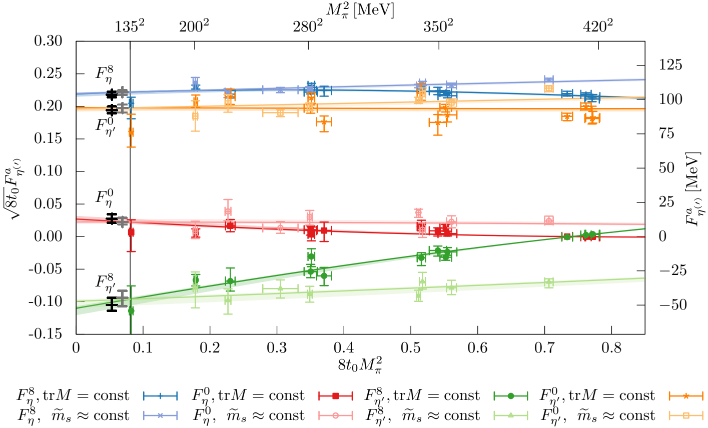
Note that in model I $\langle \xi^4 \rangle_\pi = 0.109^{+5}_{-5}$ and in model II $\langle \xi^4 \rangle_\pi = 0.112^{+7}_{-6}$. Discrimination requires very high precision of $\langle \xi^4 \rangle_\pi$!

Is a 1% error on $\langle \xi^4 \rangle_\pi$ achievable? The present error on $\langle \xi^2 \rangle_\pi$ is 3.0%.

Higher twist may also matter at low Q^2.

Outlook: towards moments of η and η' DAs

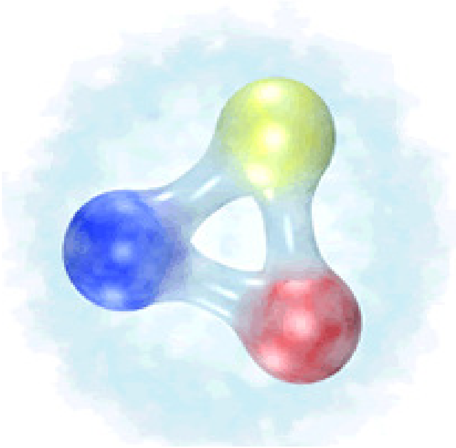
η/η' decay constants (normalization of twist-2 DAs) [RQCD: 2106.04398]



Data shifted to the continuum limit. $F_\pi \approx 92\,\mathrm{MeV}$.

Singlet decay constants given in $\overline{\mathrm{MS}}$ scheme at the scale $\mu = \infty$.

Baryons



Baryon distribution amplitudes

f, g, h: quark flavours. p: $fgh = uud$, Σ^0, Λ: $fgh = sud$, Ξ^0: $fgh = ssu$.

$$\langle 0|f_\alpha(a_1 n)g_\beta(a_2 n)h_\gamma(a_3 n)|B_{p,\lambda}\rangle = \frac{1}{4}\int[\mathrm{d}x]\, e^{-ip\cdot n\sum_i a_i x_i}\left[v^B_{\alpha\beta;\gamma}V^B(x_1,x_2,x_3)\right.$$
$$\left. + a^B_{\alpha\beta;\gamma}A^B(x_1,x_2,x_3) + t^B_{\alpha\beta;\gamma}T^B(x_1,x_2,x_3) + \ldots\right]$$

$$\int[\mathrm{d}x] = \int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\delta(1-x_1-x_2-x_3)$$

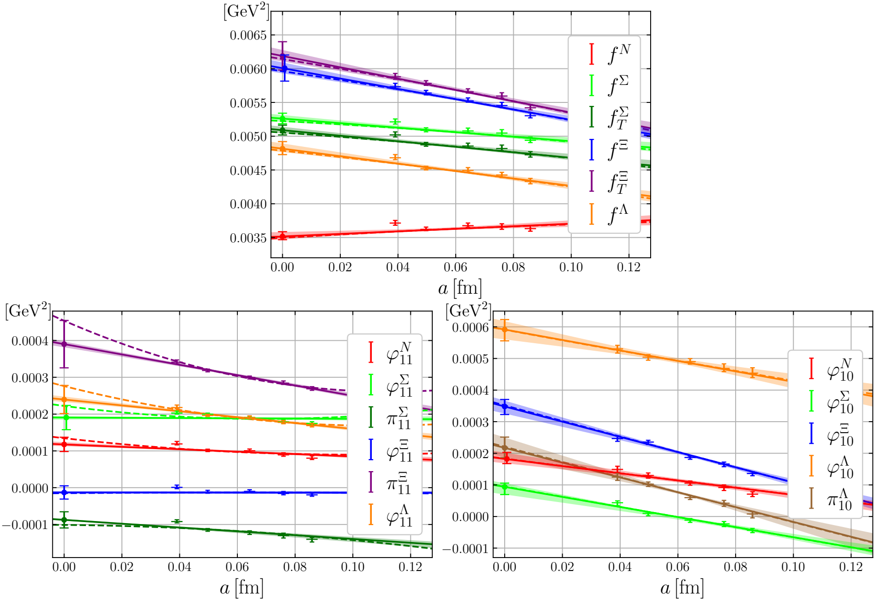
$$\begin{aligned}
\Phi^B_+(x_1,x_2,x_3) &= 120x_1x_2x_3[\varphi^B_{00} + \varphi^B_{11}7(x_1-2x_2+x_3)+\ldots] && \propto (V-A)^B_{123} + (V-A)^B_{132}\\
\Phi^B_-(x_1,x_2,x_3) &= 120x_1x_2x_3[\varphi^B_{10}21(x_1-x_3)+\ldots] && \propto (V-A)^B_{123} - (V-A)^B_{132}\\
\Pi^{B\neq\Lambda}(x_1,x_2,x_3) &= 120x_1x_2x_3[\pi^B_{00} + \pi^B_{11}7(x_1-2x_2+x_3)+\ldots] && \propto T^B_{132}\\
\Pi^\Lambda(x_1,x_2,x_3) &= 120x_1x_2x_3[\pi^\Lambda_{10}21(x_1-x_3)+\ldots] && \propto T^B_{132}
\end{aligned}$$

Normalizations $f^B = \varphi^B_{00}$, $f_T^{B\neq\Lambda} = \pi^B_{00}$ similar to decay constants.

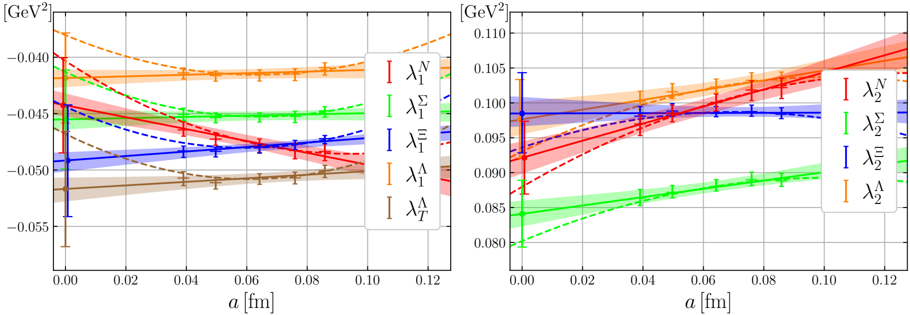
φ^B_{10}, π^Λ_{10}, φ^B_{11}, $\pi^{B\neq\Lambda}_{11}$ are first moments of BDAs (operators contain a derivative).

Additional twist-4 BDA normalizations λ^B_1, λ^B_2, λ^Λ_T.

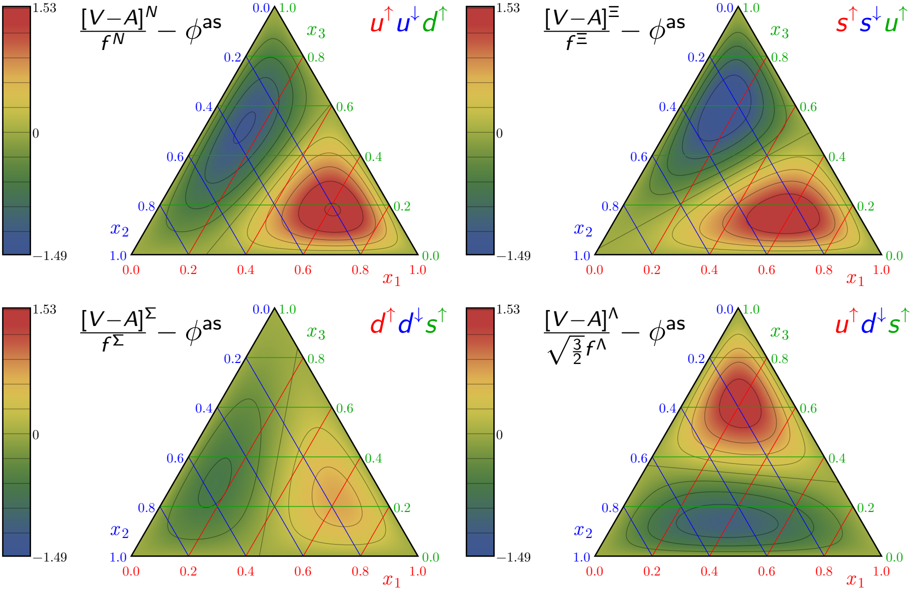
BDA results 1: the continuum limit



BDA results 2: twist-4 normalizations



BDA results 3: "barycentric plots"



Photon distribution amplitude: the tensor coefficient

Constant background magnetic (not chromomagnetic) field F_{xy}.
A quark of flavour f carries the charge $q_f \in \{-e/3, 2e/3\}$.

$$\left\langle \bar{f}\sigma_{xy} f \right\rangle = -q_f\, \tau_f F_{xy} + \mathcal{O}(F^3).$$

Tensor coefficient τ_f undergoes additive and multiplicative renormalization:

$$\tau_{fr} = Z_T \tau_f - \tau_f^{\mathrm{div}}, \quad \tau_f^{\mathrm{div}} = m_f \frac{3}{4\pi^2} \log(\mu_{\mathrm{QED}}^2 a^2) + \mathcal{O}(\alpha_s).$$

It turns out that $f_{\gamma,f}^{\perp} = \tau_{fr}$ is the normalization of the distribution amplitude of finding $\bar{f}f$ inside a photon.

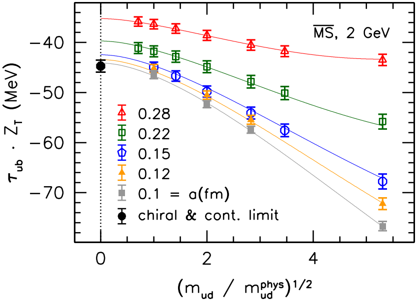
Using the background field method, it is also possible to compute higher moments, see [GB, G Endrődi, S Piemonte, 2004.08778] for details.

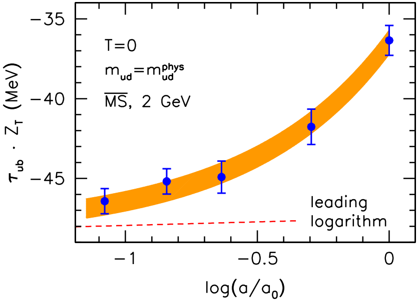
"Susceptibility of the quark condensate":

$X_u = \tau_{ur}/\langle \bar{u}u \rangle$, $X_u(1\,\mathrm{GeV}) = [542(11)\,\mathrm{MeV}]^{-2}$ (see the next slide for τ_{ur}).

Normalization of the photon distribution amplitude

τ_f diverges for $m_u > 0$ as discussed above.





$f_\gamma^\perp = \tau^{\overline{\mathrm{MS}}}(\mu = 2\,\mathrm{GeV}) = \lim_{a\to 0}\lim_{m_u\to 0} Z_T\tau_u = -44.7(1.2)\,\mathrm{MeV}$

One can also obtain $f_\gamma^\perp$ for massive quarks, where the difference between the lattice scheme and the $\overline{\mathrm{MS}}$ scheme is $(\text{finite}\,\#)\cdot m_f$.

Applications: short distance hadronic light-by-light correction to $g_\mu - 2$ [J Bijnens et al, 1908.03331],

$B \to \gamma\,\ell\bar{\nu}_\ell$ decay [T Janowski et al, 2106.13616] etc.

Outlook

Gegenbauer/Mellin moments: problem solved. For the first time a controlled continuum and physical quark mass limit was taken, with renormalization to $\overline{\mathrm{MS}}$ at N^{3}LO. What next?

- $\exists$ [RQCD: 2106.04398] η and η' decay constants as well as $\langle 0|F\widetilde{F}|\eta^{(\prime)}\rangle$. What about higher twist-2 and twist-4 DA moments?
- Higher photon DA moments?
- Combine with X-space methods to explore higher twist contributions? $\delta_2^\pi(2\,\mathrm{GeV}) = 0.22(3)\,\mathrm{GeV}^2$ [RQCD:1807.06671] $(\epsilon^{\mu\nu\rho\sigma}\langle 0|\bar{d}\gamma_\nu \overleftrightarrow{D}_\rho \overleftrightarrow{D}_\sigma u|\pi^+\rangle = 4p^\mu\sqrt{2}F_\pi\,\delta_\pi^2)$.
- Higher Gegenbauer moments are almost impossible to obtain within meaningful errors.
- The future: combine different approaches within global fits.