Light cone distribution amplitudes from lattice QCD

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(RQCD)
Recent DA calculations

- Introduction
- Moments of pseudoscalar twist-2 DAs
- Moments of octet baryon DAs
- DAs from $X$-space and $\delta_2^\pi$
- Normalization of the photon DA
- Outlook

1) RQCD: GB, VM Braun, M Göckeler, M Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, 1705.10236; 1811.06050;
   RQCD: GB, VM Braun, S Bürger, M Göckeler, M Gruber, F Hutzler, P Korcyl, A Schäfer, A Sternbeck, P Wein, 1903.08038 (with "addendum").

2) GB, V Braun, S Collins, A Schäfer, J Simeth, 2106.05398.


Distribution amplitudes

Light front wavefunction (distribution amplitude, DA) of a hadron parameterizes the distribution of the longitudinal momentum among its partons.

Momentum fractions $0 \leq x_i \leq 1$, $\sum_{i \in \{q, \bar{q}, g\}} x_i = 1$.

Wavefunction of a hadron (here pion) near the infinite momentum frame, written as a superposition of different Fock states:

$$|\pi\rangle = c_1 |\bar{q}q\rangle + c_2 |\bar{q}gq\rangle + c_3 |\bar{q}q\bar{q}q\rangle + \ldots$$

At leading twist (twist-2) only the valence quarks contribute (unlike in PDFs):

$$x = x_q = 1 - x_{\bar{q}}, \quad \xi = x_q - x_{\bar{q}} = 2x - 1 \in [-1, 1].$$

In hard processes higher Fock states are power suppressed (higher twist).

PDFs are (within the parton model) single particle probability densities and can directly be extracted from fits to DIS and SIDIS data.

DAs are wavefunctions and only appear within convolutions in hard exclusive processes. Due to other hadronic uncertainties (and experimental techniques), it is much harder to extract these reliably from experimental data.
Distribution amplitudes 2

DAs are needed for the theoretical description of hard exclusive processes. Example: collinear factorization of the $\gamma\gamma^* \rightarrow \pi^0$ photoproduction formfactor ($Q \gg \mu \gg \Lambda$)

$$F_{\pi\gamma}(Q^2) = \frac{2F_{\pi}}{3} \int_0^1 dx \left[ H_{\bar{q}q\gamma}(x, \mu_F, Q^2) \cdot \phi_{\pi}(x, \mu_F) + \right. \left. \text{higher twist} \cdot F_{\pi} \cdot O(1/Q^4) \right]$$

$F_{\pi} \approx 92 \text{ MeV}$. Factorization scale $\mu_F \sim Q^2/4$. Renormalization of $H$ usually at $\mu^2_R = Q^2$. Also $O(1/Q^4)$ non-factorizable soft contribution.
Definition of meson DAs

Non-local light front matrix element at a separation \( n (n^2 = 0) \):

\[
\langle 0 \left| \bar{d} \left( \frac{n}{2} \right) \not{n} \gamma_5 \left[ \frac{n}{2}, -\frac{n}{2} \right] u \left( -\frac{n}{2} \right) \right| \pi^+(p) \rangle
\]

\[
= i \sqrt{2} F_\pi n \cdot p \int_0^1 dx \exp \left\{ i \left[ x - (1 - x) \right] \frac{(n \cdot p)}{2} \right\} \phi_\pi(x, \mu)
\]

\([n/2, -n/2]\) above denotes a gauge covariant connection.

The DA cannot directly be computed in Euclidean spacetime but its moments can:

\[
\langle \xi^n \rangle = \int_0^1 dx (2x - 1)^n \phi_\pi(x, \mu), \quad \langle \xi^0 \rangle = 1, \quad \langle \xi^1 \rangle = 0.
\]

\(\langle \xi^{0,2} \rangle\) can be extracted from local matrix elements \( \langle 0 \left| O^{\pm}_{\mu\nu\rho} \right| \pi^+(p) \rangle \) with

\[
O^{\pm}_{\mu\nu\rho} = \bar{d} \left\{ \left[ \not{D}_{(\mu} \not{D}_{\nu)} \pm 2 \not{D}_{(\mu} \not{D}_\nu + \not{D}_{(\mu} \not{D}_\nu \right] \gamma_\rho \right\} u,
\]

where \( \cdots \) gives a traceless, symmetrized expression.
Gegenbauer expansion (only even moments for the pion):

\[ \phi_\pi(x, \mu) = 6x(1 - x) \left[ 1 + \sum_{n \in \mathbb{N}} a_{2n}^\pi(\mu) C_{2n}^{3/2}(2x - 1) \right] \]

Collinear conformal symmetry: \( C_n^{3/2}(\xi) \) in SL(2, \( \mathbb{R} \)) analogous to \( Y_{\ell m}(\theta, \phi) \) in SO(3). \( \langle \xi^{2n} \rangle \) and \( a_{2n}^\pi \) are related by simple algebraic expressions (\( n = 1 \) example):

\[ a_2^\pi(\mu) = \frac{7}{12} \left( 5\langle \xi^2 \rangle - 1 \right) = \frac{7}{12} \left( 5\langle \xi^2 \rangle - \langle \xi^0 \rangle \right) \]

\( a_{2n}^\pi(\mu) \to 0 \) as \( \mu \to \infty \): At large scales the lower moments will dominate.

Note the difference in the counting: 2nd DA-moment \( \sim \) 3rd PDF-moment.
CLS ensemble overview

![Graph showing ensemble overview with points colored by various ensembles such as E, J, D, N, C, S, H, B, and U, and with $m_s + 2m_\ell = \text{const}$ and $\hat{m}_s \approx \text{const}$ formulas and values for E, J, D, N, C, S, H, B, and U ensembles.]

$m_s + 2m_\ell = \text{const}$

$\beta$

$3.85$ $3.70$ $3.55$ $3.46$ $3.4$

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$\hat{m}_s \approx \text{const}$

$\exists$ additional ensembles with $m_s = m_\ell$. 

E: $192 \cdot 96^3$, J: $192 \cdot 64^3$, D: $128 \cdot 64^3$, N: $128 \cdot 48^3$, C: $96 \cdot 48^3$, S: $128 \cdot 32^3$, H: $96 \cdot 32^3$, B: $64 \cdot 32^3$, U: $128 \cdot 24^3$. 

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Results 1: test of \( \langle \xi^0 \rangle \to 1 + \mathcal{O}(\alpha_s^4) \) \((a \to 0)\)

\( \zeta_{ij} \): renormalization from lattice to \( \overline{\text{MS}} \) [RQCD: GB et al, 2012.06284].

Lattice \( \to \) RI’-SMOM non-perturbatively, RI’-SMOM \( \to \overline{\text{MS}} \) at 3-loop (N\(^3\)LO).

Test of renormalization: \( \langle \xi^0 \rangle = 1 + \mathcal{O}(a) \sim (\zeta_{22} - 5\zeta_{12})O^+ \)
SU(3) NLO ChPT and linear lattice spacing effects (8 parameters):

\[ \xi^2_X(M_\pi, M_K, a) = \left[ \xi^2_0 + \bar{A}M^2 + A_X\delta M^2 \right] \left[ 1 + a\left( c_0 + \bar{c}\bar{M}^2 + c_X\delta M^2 \right) \right], \quad X \in \{ \pi, K, \eta_8 \}, \]

\[ A_\pi = -2A_K = -A_{\eta_8}, \quad \delta M^2 = 2M_K^2 - 2M_\pi^2, \quad \bar{M}^2 = \frac{1}{3} \left( 2M_K^2 + M_\pi^2 \right). \]
Continuum limit, at the physical point: \(a_2 \propto 5\langle \xi^2 \rangle - 1\)

\[
a_2^\pi,\overline{\text{MS}}(2 \text{ GeV}) = 0.116^{+16}_{-17}(4)_{r}(9)a(5)m, \quad \langle \xi^2 \rangle_\pi = 0.240^{+6}_{-6}(2)_{r}(3)a(2)m,
\]

\[
a_2^K,\overline{\text{MS}}(2 \text{ GeV}) = 0.106^{+10}_{-12}(4)_{r}(9)a(4)m, \quad \langle \xi^2 \rangle_K = 0.236^{+3}_{-4}(1)_{r}(3)a(1)m,
\]

\[
a_1^K,\overline{\text{MS}}(2 \text{ GeV}) = 0.0525^{+17}_{-19}(3)_{r}(20)a(17)m = \frac{5}{3} \langle \xi^1 \rangle_K.
\]
Summary of light pseudoscalar DAs

Model I: $\phi^{(I)}(x) = 6x(1 - x)\left[1 + a_1 C_1^{3/2}(\xi) + a_2 C_2^{3/2}(\xi)\right]$  

Model II: $\phi^{(II)}(x) = \frac{\Gamma(2+\alpha^+ + \alpha^-)}{\Gamma(1+\alpha^+)\Gamma(1+\alpha^-)} x^{\alpha^+}(1 - x)^{\alpha^-}$

with $a_1, a_2$ fixed from our analysis ($\overline{MS}$ at 2 GeV). $\pi$: $\alpha^+ = \alpha^-$, $a_1 = 0$.

Note that in model I $\langle \xi^4 \rangle_\pi = 0.109^{+5}_{-5}$ and in model II $\langle \xi^4 \rangle_\pi = 0.112^{+7}_{-6}$. Discrimination requires very high precision of $\langle \xi^4 \rangle_\pi$!

Is a 1% error on $\langle \xi^4 \rangle_\pi$ achievable? The present error on $\langle \xi^2 \rangle_\pi$ is 3.0%. Higher twist may also matter at low $Q^2$. 
Outlook: towards moments of $\eta$ and $\eta'$ DAs

$\eta/\eta'$ decay constants (normalization of twist-2 DAs) \[\text{RQCD: 2106.04398}\]

Data shifted to the continuum limit. $F_\pi \approx 92$ MeV.

Singlet decay constants given in \(\overline{\text{MS}}\) scheme at the scale $\mu = \infty$. 

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Baryons
Baryon distribution amplitudes

\( f, g, h: \) quark flavours. \( p: \) \( fgh = uud, \) \( \Sigma^0, \Lambda: \) \( fgh = sud, \) \( \Xi^0: \) \( fgh = ssu. \)

\[
\langle 0 | f_\alpha(a_1 n)g_\beta(a_2 n)h_\gamma(a_3 n) | B_p, \lambda \rangle = \frac{1}{4} \int [dx] e^{-ip \cdot n} \sum_i a_i x_i [v_{\alpha \beta; \gamma}^B V^B(x_1, x_2, x_3) + a_{\alpha \beta; \gamma}^B A^B(x_1, x_2, x_3) + t_{\alpha \beta; \gamma}^B T^B(x_1, x_2, x_3) + \ldots]
\]

\[
\int [dx] = \int_0^1 dx_1 \, dx_2 \, dx_3 \, \delta(1 - x_1 - x_2 - x_3)
\]

\[
\Phi^B_+(x_1, x_2, x_3) = 120 x_1 x_2 x_3 [\varphi^B_{00} + \varphi^B_{11} 7(x_1 - 2x_2 + x_3) + \ldots] \propto (V - A)^B_{123} + (V - A)^B_{132}
\]

\[
\Phi^B_-(x_1, x_2, x_3) = 120 x_1 x_2 x_3 [\varphi^B_{10} 21(x_1 - x_3) + \ldots] \propto (V - A)^B_{123} - (V - A)^B_{132}
\]

\[
\Pi^B\neq\Lambda(x_1, x_2, x_3) = 120 x_1 x_2 x_3 [\pi^B_{00} + \pi^B_{11} 7(x_1 - 2x_2 + x_3) + \ldots] \propto T^B_{132}
\]

\[
\Pi^\Lambda(x_1, x_2, x_3) = 120 x_1 x_2 x_3 [\pi^\Lambda_{10} 21(x_1 - x_3) + \ldots] \propto T^B_{132}
\]

Normalizations \( f^B = \varphi^B_{00}, f^B\neq\Lambda = \pi^B_{00} \) similar to decay constants.

\( \varphi^B_{10}, \pi^\Lambda_{10}, \varphi^B_{11}, \pi^B\neq\Lambda \) are first moments of BDAs (operators contain a derivative).

Additional twist-4 BDA normalizations \( \lambda^B_1, \lambda^B_2, \lambda^\Lambda_T \).
BDA results 1: the continuum limit

\begin{align*}
0.00 & 0.02 & 0.04 & 0.06 & 0.08 & 0.10 & 0.12 \\
\begin{array}{c}
a \text{[fm]} \\
0.0035 \\
0.0040 \\
0.0045 \\
0.0050 \\
0.0055 \\
0.0060 \\
0.0065 \\
\text{[GeV}^2\text{]} \\
f_N \\
f_\Sigma \\
f_\Sigma \\
f_\Xi \\
f_\Xi \\
f_\Lambda \\
f_\Lambda
\end{array}
\end{align*}

\begin{align*}
0.00 & 0.02 & 0.04 & 0.06 & 0.08 & 0.10 & 0.12 \\
\begin{array}{c}
a \text{[fm]} \\
-0.0001 \\
0.0000 \\
0.0001 \\
0.0002 \\
0.0003 \\
0.0004 \\
\text{[GeV}^2\text{]} \\
\phi_N^{10} \\
\phi_\Sigma^{10} \\
\phi_\Xi^{10} \\
\phi_\Lambda^{10} \\
\pi_\Lambda^{10}
\end{array}
\end{align*}
BDA results 2: twist-4 normalizations

\[ a \text{ [fm]} \]

\[ \lambda_1^N, \lambda_1^\Sigma, \lambda_1^\Xi, \lambda_1^\Lambda, \lambda_1^\Lambda_T \]

\[ \lambda_2^N, \lambda_2^\Sigma, \lambda_2^\Xi, \lambda_2^\Lambda \]

\[ \text{GeV}^2 \]

\[ a \text{ [fm]} \]
BDA results 3: “barycentric plots”

\[
\left[ \frac{V-A}{f_{N}} \right]^{N}_{\text{as}} \quad \left[ \frac{V-A}{f_{\Xi}} \right]^{\Xi}_{\text{as}} \\
\left[ \frac{V-A}{f_{\Sigma}} \right]^{\Sigma}_{\text{as}} \quad \left[ \frac{V-A}{\sqrt{\frac{3}{2}} f^{\Lambda}} \right]^{\Lambda}_{\text{as}}
\]

\begin{align*}
\frac{1}{f_{N}} & : u \uparrow \, d \uparrow \\
\frac{1}{f_{\Xi}} & : s \uparrow \, u \uparrow \\
\frac{1}{f_{\Sigma}} & : d \uparrow \, s \uparrow \\
\frac{1}{\sqrt{\frac{3}{2}} f^{\Lambda}} & : u \uparrow \, d \downarrow \, s \uparrow
\end{align*}
Photon distribution amplitude: the tensor coefficient

Constant background magnetic (not chromomagnetic) field $F_{xy}$. A quark of flavour $f$ carries the charge $q_f \in \{-e/3, 2e/3\}$.

$$\langle \bar{f} \sigma_{xy} f \rangle = -q_f \tau_f F_{xy} + \mathcal{O}(F^3).$$

Tensor coefficient $\tau_f$ undergoes additive and multiplicative renormalization:

$$\tau_{fr} = Z_T \tau_f - \tau^\text{div}_f, \quad \tau^\text{div}_f = m_f \frac{3}{4\pi^2} \log\left(\mu^2_{\text{QED}} a^2\right) + \mathcal{O}(\alpha_s).$$

It turns out that $f_{\gamma, f}^\perp = \tau_{fr}$ is the normalization of the distribution amplitude of finding $\bar{f}f$ inside a photon.

Using the background field method, it is also possible to compute higher moments, see [GB, G Endrődi, S Piemonte, 2004.08778] for details.

“Susceptibility of the quark condensate”:

$$X_u = \tau_{ur}/\langle \bar{u}u \rangle, \quad X_u(1 \text{ GeV}) = [542(11) \text{ MeV}]^{-2}$$

(see the next slide for $\tau_{ur}$).
Normalization of the photon distribution amplitude

\( \tau_f \) diverges for \( m_u > 0 \) as discussed above.

\[
f_{\perp}^{\gamma} = \tau^{\overline{\text{MS}}} (\mu = 2 \text{ GeV}) = \lim_{a \to 0} \lim_{m_u \to 0} Z_T \tau_u = -44.7(1.2) \text{ MeV}
\]

One can also obtain \( f_{\perp}^{\gamma} \) for massive quarks, where the difference between the lattice scheme and the \( \overline{\text{MS}} \) scheme is \((\text{finite #}) \cdot m_f\).

Applications: short distance hadronic light-by-light correction to \( g_\mu - 2 \) \cite{Bijnens:2019pel}, \( B \to \gamma l \bar{\nu}_l \) decay \cite{Janowski:2021gew} etc.
Gegenbauer/Mellin moments: problem solved. For the first time a controlled continuum and physical quark mass limit was taken, with renormalization to $\overline{\text{MS}}$ at $N^3\text{LO}$. What next?

- $\exists [\text{RQCD: 2106.04398}]$ $\eta$ and $\eta'$ decay constants as well as $\langle 0 | F \tilde{F} | \eta^{(f)} \rangle$. What about higher twist-2 and twist-4 DA moments?
- Higher photon DA moments?
- Combine with X-space methods to explore higher twist contributions?

$$\delta_2^\pi(2 \text{ GeV}) = 0.22(3) \text{ GeV}^2 \ [\text{RQCD:1807.06671}]$$

$$(\epsilon^{\mu\nu\rho\sigma} \langle 0 | \bar{d} \gamma_\nu \overleftarrow{D}_\rho \overleftarrow{D}_\sigma u | \pi^+ \rangle) = 4 p^\mu \sqrt{2} F_\pi \delta_2^\pi).$$

- Higher Gegenbauer moments are almost impossible to obtain within meaningful errors.
- The future: combine different approaches within global fits.