Charting the scaling region of the Ising universality class in finite temperature QCD

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Based on:
M. Caselle and M. Sorba, Phys. Rev. D 102.1 (2020) 014505

A Virtual Tribute to Quark Confinement and the Hadron Spectrum
August 2nd-6th 2021, online

The Ising model

Ising spin model on lattice: $E(\{\sigma_i\}) = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - \hat{H} \sum_{i=1}^{N} \sigma_i \,, \quad \sigma_i = \pm 1$

$(t, H) = (0,0)$ critical point $\Leftrightarrow \mathbb{Z}_2$ SSB

Ising Field Theory: $S = S_{CFT} + t \int d^d x \, \epsilon(x) + H \int d^d x \, \sigma(x) \,, \quad (d = 2, 3)$

Thanks to integrability ($d = 2$) and Montecarlo simulations ($d = 3$) lots of results are known at the **critical point** or when **only one** perturbing operator is present.

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Thanks to integrability ($d = 2$) and Montecarlo simulations ($d = 3$) lots of results are known at the **critical point** or when **only one** perturbing operator is present.

- Give a description of the whole critical region.
- Valid for any realization of the Ising universality class.
- Identify the directions corresponding to the two relevant perturbations.

$\longrightarrow$ Application to the QCD phase diagram.

Observable to study

Define a combination of magnetic susceptibility χ and correlation length ξ

$$\boxed{\Omega = \left(\frac{\chi(t,H)}{\Gamma_-}\right)\left(\frac{\xi_-}{\xi(t,H)}\right)^{\gamma/\nu}}$$

where $\left.\begin{array}{l} \chi \approx \Gamma_- \, (-t)^{-\gamma} \\ \xi \approx \xi_- \, (-t)^{-\nu} \end{array}\right\} \; t < 0 \, , H = 0$

$\gamma, \, \nu =$ critical exponents

Why this choice?

1) Easy to evaluate both numerically and in experiments:
their definitions involve only the order parameter

$$\chi = \frac{\partial \langle \sigma \rangle}{\partial H} \quad \text{and} \quad \langle \sigma(x)\sigma(0)\rangle_c \underset{|x|\to\infty}{\sim} e^{-|x|/\xi}$$

2) Allow a description of the **whole scaling region** (t, H) close to the critical point, where both relevant perturbations are present.

Here we use the scaling variable $\eta = \frac{t}{|H|^{1/\beta\delta}} \to \Omega(\eta)$

What is known…

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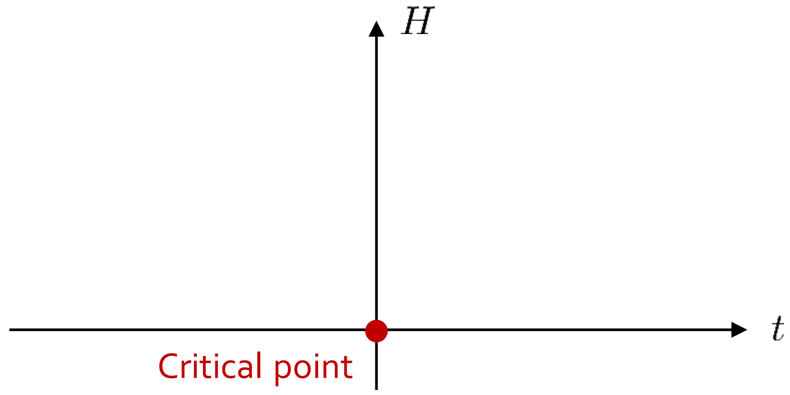

When **only one** of the two perturbations is present $\Omega(\eta)$ can be written in terms of the standard universal amplitude ratios $Q_2,\ \Gamma_+/\Gamma_-,\ \xi_+/\xi_-$:

$$\eta = -\infty \qquad \Omega(\eta) = 1$$

$$\eta = 0 \qquad \Omega(\eta) = \frac{1}{Q_2}\left(\frac{\Gamma_+}{\Gamma_-}\right)\left(\frac{\xi_-}{\xi_+}\right)^{\gamma/\nu}$$

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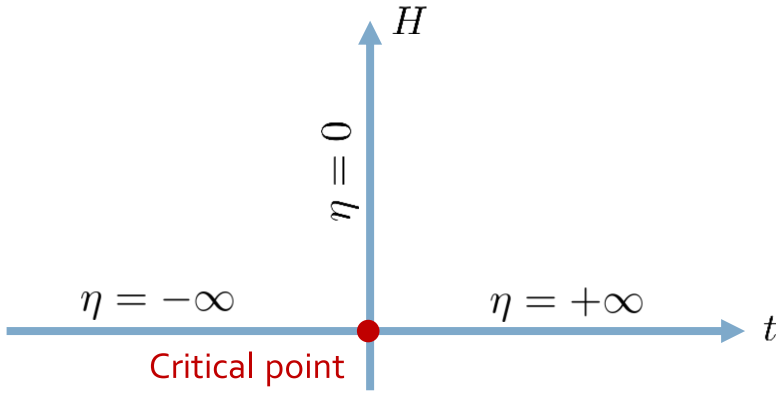

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$\eta = +\infty \qquad \Omega(\eta) = \left(\frac{\Gamma_+}{\Gamma_-}\right)\left(\frac{\xi_-}{\xi_+}\right)^{\gamma/\nu} = 11.2063897\ldots \quad 1.355(71)$

$d = 2$

$d = 3$

[G. Delfino, Phys. Lett. B 419, 291 (1998)]
[M. Hasenbusch, Nucl. Phys. B82, 174434 (2010)]

...what we want to know

$$\boxed{\Omega = \left(\frac{\chi(t,H)}{\Gamma_-}\right)\left(\frac{\xi_-}{\xi(t,H)}\right)^{\gamma/\nu}}$$

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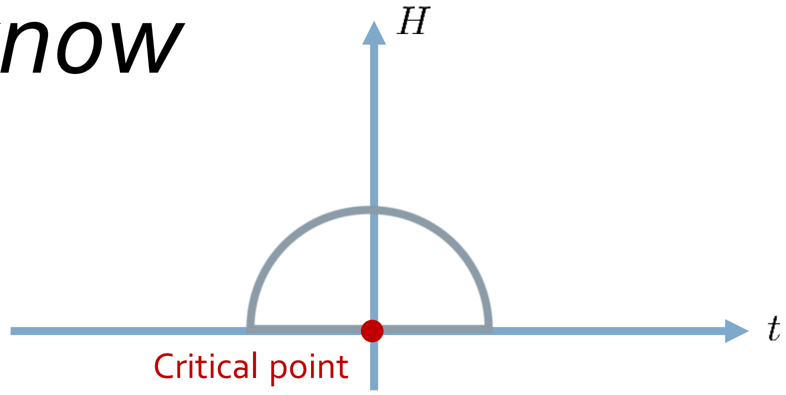

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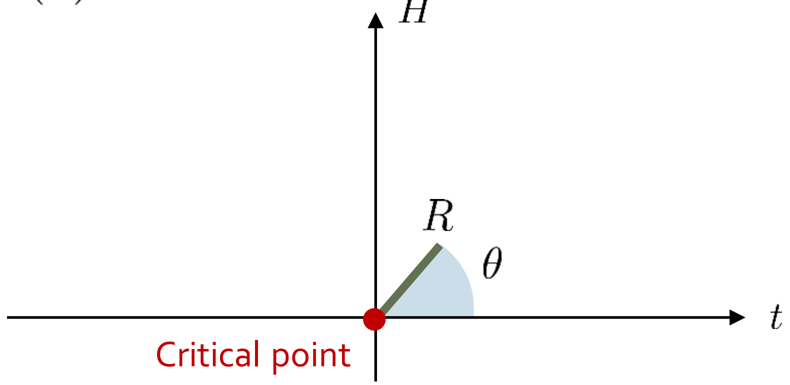
$d = 2$ (red values)
$d = 3$ (orange values)

These values can be used as benchmarks to test our results.

Parametric representation

Critical equation of state: $\begin{cases} M = m_0 R^\beta \theta, \\ t = R(1-\theta^2), \\ H = h_0 R^{\beta\delta} h(\theta) \end{cases}$ with $R \geq 0,\ 0 \leq \theta \leq \theta_0$



[P. Schofield, Phys. Rev. Lett. 22, 606 (1969)]

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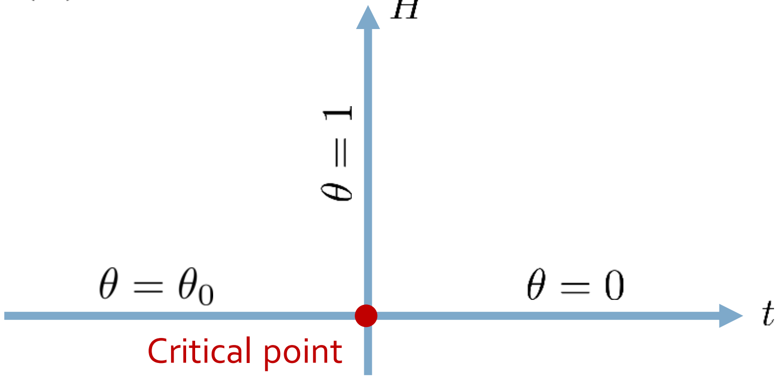


- Use the scaling variable $\theta \longrightarrow \Omega(\theta)$
- Make a polynomial approximation for the function $h(\theta)$
- Introduce also a parametric representation for ξ
- All coefficients are fixed from the universal amplitude ratios

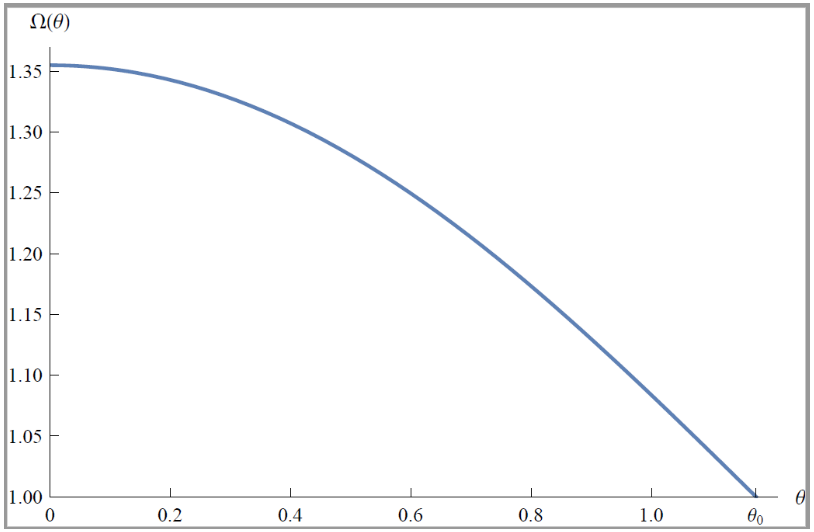
$$\boxed{\Omega(\theta) = \Omega_0 \frac{(1-\theta^2+2\beta\theta^2)(1+c\theta^2)^{\frac{\gamma}{2\nu}}}{2\beta\delta\theta h(\theta) + (1-\theta^2)h'(\theta)}}$$

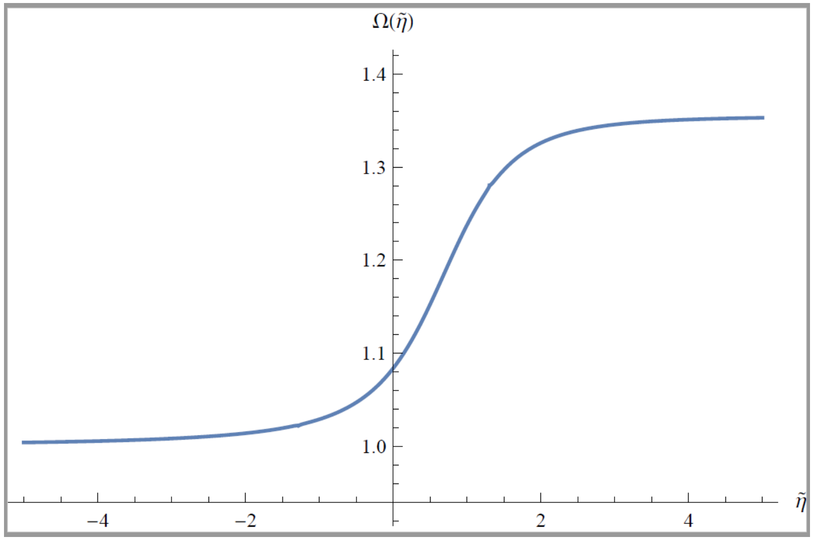
[H. B. Tarko and M. E. Fisher, Phys. Rev. B 11, 1217 (1975)]
[P. Schofield, Phys. Rev. Lett. 22, 606 (1969)]

Results in $d = 3$

$$h(\theta) = \theta + h_3\,\theta^3 + h_5\,\theta^5 + h_7\,\theta^7 + O(\theta^9)$$

We obtain:





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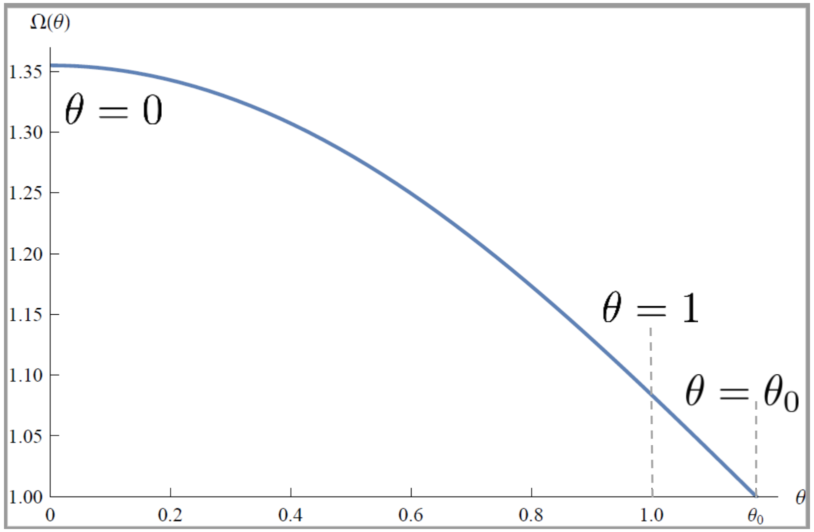
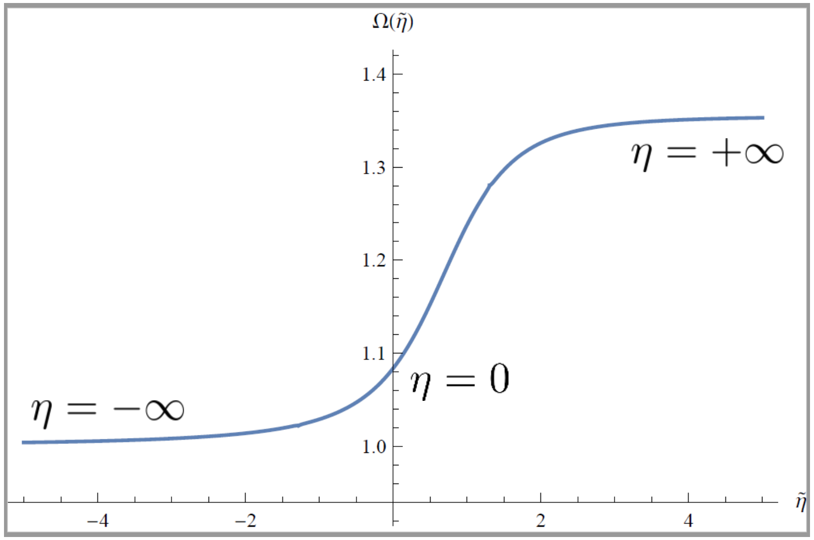


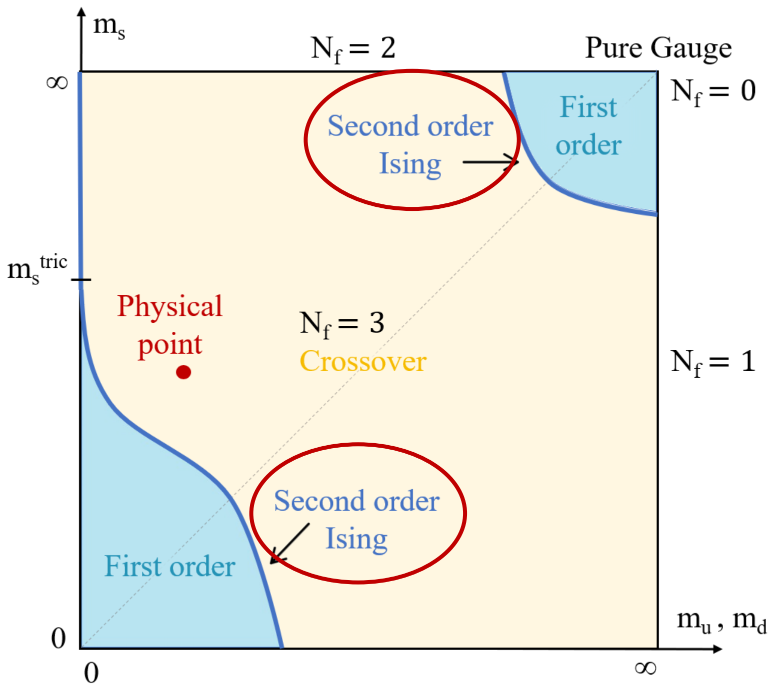


The function is monotonic: from an experimental estimate of Ω a precise value of the scaling variable may be extracted.

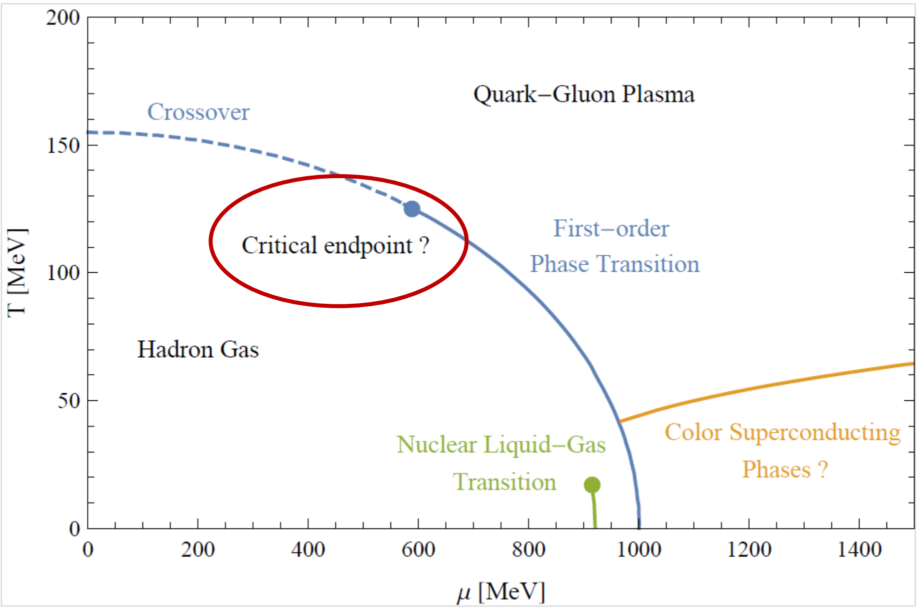
$\rightarrow$ **Chart the phase diagram close to the critical point.**

Finite temperature QCD

Thanks to universality, the previous results hold for all physical realizations of the 3-dimensional Ising model.



Columbia plot



Phase diagram

Results in $d = 2$

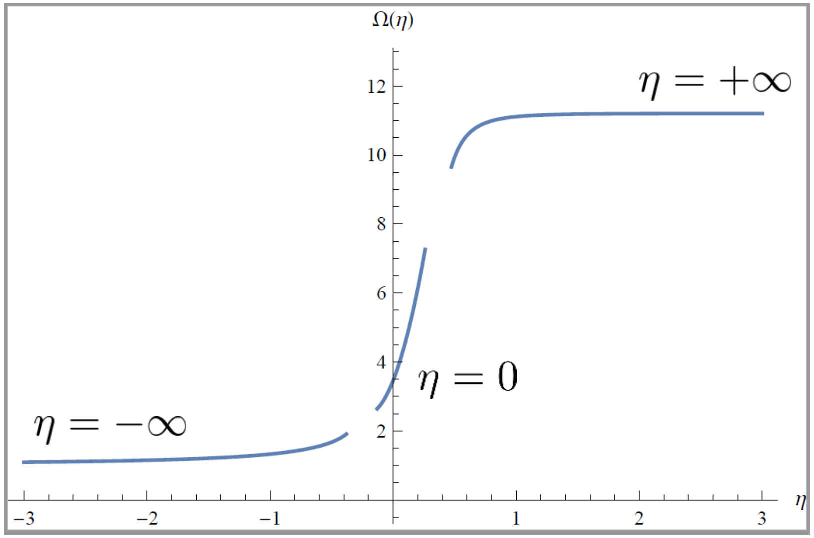
The parametric approach does not give good results.

But we can exploit the integrability of the model for $t = 0$ and/or $H = 0$
$\longrightarrow$ construct a perturbative expansion around these two axes.

$$\eta \to -\infty \qquad \Omega(\eta) = \sum_n \frac{\Omega_n^-}{(-\eta)^{\frac{5}{8}n}}$$

$$\eta \sim 0 \qquad \Omega(\eta) = \sum_n \Omega_n^0 \; \eta^n$$

$$\eta \to +\infty \qquad \Omega(\eta) = \sum_n \frac{\Omega_n^+}{\eta^{\frac{5}{8}n}}$$



The expansion coefficients are obtained from the Ising free energy and the lowest mass of the spectrum.

[P. Fonseca and A. Zamolodchikov, J. Stat. Phys. 110, 527 (2003)]

Conclusions

- We discuss the behavior of a universal combination of susceptibility and correlation length in the Ising model in presence of both magnetic and thermal perturbations, in the neighborhood of the critical point.

- In three dimensions we address the problem using a parametric representation of the equation of state.

- In two dimensions we make use of the exact integrability of the model along the thermal and the magnetic axes.

- Our results can be used as a sort of “reference frame” to chart the critical region of the model. We address in particular, instances of Ising behavior in finite temperature QCD related in various ways to the deconfinement transition.

Thank you
for the attention !

M. Caselle and M. Sorba, Phys. Rev. D 102.1 (2020) 014505

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