Overview of lattice calculations of the $x$-dependence of PDFs, GPDs and TMDs

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Outline:
Introduction/motivation
Review of results:
Nucleon twist-2 PDFs/GPDs
  twist-3 PDFs/GPDs
Pion PDFs/DAs
TMDs
Prospects/conclusion

Many thanks to my Collaborators:
C. Alexandrou, M. Bhat, S. Bhattacharya, Y. Chai, M. Constantinou
L. Del Debbio, J. Dodson, X. Feng, T. Giani, J. Green,
K. Hadjiyiannakou, K. Jansen, G. Koutsou, Y. Li, Ch. Liu,
F. Manigrasso, A. Metz, A. Scapellato, F. Steffens, S.-C. Xia

Acknowledgment for discussions/material for this talk:
M. Constantinou, R. Sufian, S. Zafeiropoulos, Y. Zhao
One of the central aims of hadron physics: to understand better nucleon structure.

- This is one of the crucial expectations from the approved Electron-Ion Collider (EIC).
- In particular, we want to probe the 3D structure.
- Thus, we need to access new kinds of functions: GPDs, TMDs.
- Also higher-twist is of growing importance for the full picture.
- Both theoretical and experimental input needed.

Lattice can provide qualitative and eventually quantitative knowledge of different functions and their moments:

- 1D: form factors
- 1D: parton distribution functions (PDFs)
- 3D: generalized parton distributions (GPDs)
- 3D: transverse momentum dependent PDFs (TMDs)
- 5D: Wigner function / generalized TMDs
Lattice QCD – brief reminder

- needed because of non-perturbative aspects of QCD
- allows for a quantitative \textit{ab initio} study of QCD
- QCD d.o.f.'s put on a \textbf{Euclidean} lattice
  \begin{itemize}
  \item quarks \rightarrow sites, gluons \rightarrow links
  \end{itemize}
- \textbf{Euclidean} – no direct access to partonic distributions!
  only moments accessible with standard lattice methods
- various discretizations can be used for quarks and gluons
- typical lattice parameters:
  \begin{itemize}
  \item \( a \in [0.04, 0.15] \) \text{ fm}
  \item \( L/a = 32, 48, 64, 80, 96, 128 \) \Rightarrow \( L \in [2, 10] \) \text{ fm}
  \item \( m_\pi \in [1, 4] \times m_\pi^{\text{physical}}, \quad m_\pi L \geq 3 - 4 \)
  \item \Rightarrow \text{\infty-dim path integral} \rightarrow 10^8 - 10^9\text{-dim integral}
  \end{itemize}
- Monte Carlo simulations to evaluate the discretized path integral
- feasible, but still requires huge computational resources of \( \mathcal{O}(1 - 1000) \) \text{ million core-hours, depending on the question asked}
- formally, evaluation of a thermodynamic expectation value with respect to the Boltzmann factor \( e^{-S_{\text{QCD}}} \)
- lattice regulates IR and UV divergences; the regulator needs to be removed \( \Rightarrow L \rightarrow \infty, \ a \rightarrow 0 \)
- prior to regulator removal – (non-perturbative) renormalization
- \textbf{key aspect}: control over various systematic effects
- \Rightarrow exploratory studies vs. precision studies
Approaches to $x$-dependence

- Recent years (since $\approx 2013$): breakthrough in accessing $x$-dependence.
  

- The common feature of all the approaches is that they rely to some extent on the factorization framework:

  $$Q(x, \mu_R) = \int_{-1}^{1} \frac{dy}{y} C \left( \frac{x}{y}, \mu_F, \mu_R \right) q(y, \mu_F),$$

  some lattice observable

- Matrix elements: $\langle N|\bar{\psi}(z)\Gamma F(z)\Gamma'\psi(0)|N\rangle$
  
  with different choices of $\Gamma, \Gamma'$ Dirac structures and objects $F(z)$.

  - hadronic tensor – K.-F. Liu, S.-J. Dong, 1993
  - auxiliary scalar quark – U. Aglietti et al., 1998
  - auxiliary heavy quark (HOPE) – W. Detmold, C.-J. D. Lin, 2005
  - auxiliary light quark – V. Braun, D. Müller, 2007
  - quasi-distributions – X. Ji, 2013
  - pseudo-distributions – A. Radyushkin, 2017
  - “OPE without OPE” – QCDSF, 2017
M. Constantinou et al., *Parton distributions and LQCD calculations: toward 3D structure*, 2006.08636

Some studies already advanced, but still full systematics needs to be investigated
Many exploratory directions: GPDs, twist-3 PDFs/GPDs, singlet PDFs, TMDs
Introduction

Nucleon structure

Lattice QCD

$x$-dependence

Results

Prospects

Progress of approaches to $x$-dependence

Theoretical idea

$(\Lambda)q$TMDs

Theoretical challenges

$(\Lambda)qGPDs$

Lattice challenges

$(\pi)q$DAs, $(\pi)q$PDFs, $(\Lambda,\pi)q$PDFs($g$), $(\Lambda)p$PDFs

$(\Lambda)ht$PDFs, $(\Lambda)ope$PDFs, $(\pi)$ahqDAs

$(\pi)alq$DAs, $(\pi)lcs$DAs, $(\pi)lcs$PDFs

Exploratory studies

Theoretical challenges

Advanced studies

$(\Lambda)q$PDFs

Lattice challenges

Precision calculations

K.C., M. Constantinou
arXiv: 1811.07248 [hep-lat]
Progress of approaches to $x$-dependence

**Theoretical idea**

- $\langle N \rangle qTMDs$, $\langle N \rangle qGPDs$
- $\langle N, \pi \rangle pPDFs(g)$

**Exploratory studies**

- $\langle N \rangle qPDFs(g)$, $\langle N \rangle htPDFs$
- $\langle \pi \rangle ahqDAs$, $\langle \pi \rangle alqDAs$

**Advanced studies**

- $\langle N \rangle qPDFs$, $\langle N \rangle pPDFs$, $\langle N \rangle opePDFs$
- $\langle \pi \rangle pPDFs$, $\langle \pi \rangle qDAs$, $\langle \pi \rangle qPDFs$, $\langle \pi \rangle lcsPDFs$

**Precision calculations**

K.C., M. Constantinou
arXiv: 1811.07248 [hep-lat]

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Krzysztof Cichy
Lattice calculations of the $x$-dependence of PDFs GPDs and TMDs – vCONF21 – 7 / 27
Quasi-distribution approach:

Quasi-distribution approach:


Main idea:

\[
\xi^- \quad \xi^0 \equiv t \quad \xi^+ \quad \xi^3 \equiv z
\]
Quasi-distribution approach:


Main idea:

\[ \xi^0 \equiv t \]

Correlation along the \( \xi^- \)-direction:

\[
q(x) = \frac{1}{2\pi} \int d\xi^- e^{-ixp^+\xi^-} \langle N|\overline{\psi}(\xi^-)\Gamma A(\xi^-, 0)\psi(0)|N\rangle
\]

\( |N\rangle \) – nucleon at rest in the light-cone frame
Quasi-distribution approach:


Main idea:

\[
\begin{align*}
\xi^- & \equiv z \\
\xi^0 & \equiv t \\
\xi^+ & \equiv \xi^3 \equiv z \\
\end{align*}
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Correlation along the $\xi^-$-direction:

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$|N\rangle$ – nucleon at rest in the light-cone frame

Correlation along the $\xi^3 \equiv z$-direction:

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\bar{q}(x) = \frac{1}{2\pi} \int dz e^{ixP_3z} \langle N|\bar{\psi}(z)\Gamma A(z, 0)\psi(0)|N\rangle
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$|N\rangle$ – nucleon at rest in the standard frame
Quasi-PDFs

Quasi-distribution approach:


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Correlation along the $\xi^3$-direction:

$$ \bar{q}(x) = \frac{1}{2\pi} \int dz e^{ixP_3z} \langle P | \bar{\psi}(z) \Gamma A(z, 0) \psi(0) | P \rangle $$

$|P\rangle$ – boosted nucleon
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Matching (Large Momentum Effective Theory (LaMET))


→ brings quasi-distribution to the light-cone distribution, up to power-suppressed effects:

\[ \tilde{q}(x, \mu, P_3) = \int_{-1}^{1} \frac{dy}{|y|} C \left( \frac{x}{y}, \frac{\mu}{P_3} \right) q(y, \mu) + \mathcal{O} \left( \frac{\Lambda_{QCD}^2}{P_3^2}, \frac{M_N^2}{P_3^2} \right) \]
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\[ \text{quasi-PDF} \]
Quasi-PDFs

Quasi-distribution approach:


Main idea:

\[ \xi^- \equiv \xi_0 \equiv t \]

\[ \xi^0 \equiv t \]

\[ \xi^- \]

\[ \xi^+ \]

\[ \xi^3 \equiv z \]

matching

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Krzysztof Cichy Lattice calculations of the \( x \)-dependence of PDFs GPDs and TMDs – vCONF21 – 8 / 27
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$$
The same matrix elements that define quasi-distributions can also be used to construct pseudo-distributions. A. Radyushkin, Phys. Rev. D96 (2017) 034025

Recent state-of-the-art unpolarized PDFs @ phys.pt.

B. Joó et al. (HadStruc)

\[ \overline{\text{MS}}(2\text{GeV}) \]
\[ u - d \]

M. Bhat et al. (ETMC)
Phys. Rev. D103 (2021) 034510

<table>
<thead>
<tr>
<th>PSEUDO</th>
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<th>( m_\pi = 358, 278, 172 \text{ MeV} )</th>
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| PSEUDO | TMF | \( m_\pi = 130 \text{ MeV} \) | \( a = 0.094 \text{ fm} \) |

\[
\pi = 358, 278, 172 \text{ MeV} \quad a = 0.094 \text{ fm}
\]

**Qualitative agreement with pheno**

**Quantitative agreement if including "plausible systematics"**

**True systematics to be investigated**
PDFs reconstruction from actual lattice data

JAM framework

unpolarized
2930 datapoints

helicity
650 datapoints

unpolarized: significant tension \text{lat} \leftrightarrow \text{exp}
much improved precision of \text{lat} needed for any impact
(rather benchmark case)
helicity: promising agreement \text{lat} \leftrightarrow \text{exp}
current precision of \text{lat} provides significant constraints

NNPDF framework
NNPDF (L. Del Debbio, T. Giani)
+ K.C., JHEP10(2019)137

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\text{QUASI} \quad \text{TMF} \quad m_\pi = 130 \text{ MeV} \quad a = 0.094 \text{ fm}

\text{PSEUDO} \quad \text{clover} \quad m_\pi = 415, 358, 278, 172 \text{ MeV} \quad a = 0.091, 0.094, 0.127 \text{ fm}
Other works for isovector nucleon PDFs

- **continuum limit**
  theoretical framework for $O(a)$-improvement
  J. Green, K. Jansen, F. Steffens, PRD101(2020)074509

- **lattice results**
  C. Alexandrou et al. (ETMC), Phys. Rev. D103 (2021) 094512 (quasi)
  J. Karpie, K. Orginos, A. Radyushkin, S. Zafeiropoulos (HadStruc), 2105.13313 (pseudo)

- **superfine lattice with $a = 0.042$ fm**
  Z. Fan et al. (BNL+MSULat), Phys. Rev. D102 (2020) 074504

- **parametrization of systematic uncertainties with Jacobi polynomials + Bayesian fits**
  J. Karpie, K. Orginos, A. Radyushkin, S. Zafeiropoulos (HadStruc), 2105.13313

- **distillation with momentum smearing**
  C. Egerer et al. (HadStruc), Phys. Rev. D103 (2021) 034502
  C. Egerer et al. (HadStruc), 2107.05199

- **forward Compton amplitude – extraction of 4 lowest moments of nucleon’s structure function $F_1$ using Feynman-Hellmann method**

- **PDFs of the $\Delta^+$ baryon – can shed light on the sea quark asymmetry of the nucleon**
  Y. Chai et al. (Beijing+ETMC), PRD102(2020)014508
Theoretical/methodological developments

- 2-loop matching
  V. Braun, K. Chetyrkin, B. Kniehl, JHEP 07 (2020) 161

- Developments in non-perturbative renormalization
  hybrid scheme
  X. Ji et al., Nucl. Phys. B964 (2021) 115311
  residual power divergence
  K. Zhang et al. (χQCD), 2012.05448
  self-renormalization
  Y.-K. Huo et al. (LPC), Nucl. Phys. B969 (2021) 115443

- Origin and resummation of threshold logarithms

- Renormalon effects in quasi- and pseudo-distributions

- Chiral perturbation theory for LaMET
  W.-Y. Liu, J.-W. Chen, 2011.13536

- FVE for non-local current-current operators
  R. Briceño, C. Monahan, Phys. Rev. D103 (2021) 094521

- Parton distributions in nongauge theories
  L. Del Debbio, T. Giani, C. Monahan, JHEP 09(2020)021

- Bayesian determination of OPE Wilson coefficients from lattice and pheno data
  N. Karthik, R. Sufian, 2106.03875

- Bayes-Gauss-Fourier transform for PDF reconstruction
  C. Alexandrou et al. (ETMC), Phys. Rev. D102 (2020) 094508

- Pion (pseudo-)PDFs QCD3 with 0,2,4,8 flavors
  N. Karthik, Phys. Rev. D103 (2021) 074512
Flavor decomposition

Most studies up to date were for the flavor non-singlet $u - d$ combination. Important direction: flavor decomposition.

C. Alexandrou et al. (ETMC), Phys. Rev. Lett. 126 (2021) 102003; 2106.16065

- disconnected diagrams (hierarchical probing, one-end trick)
- mixing with gluon PDFs neglected

Strange and charm contributions also in: R. Zhang, H.-W. Lin, B. Yoon, 2005.01124
Recent computation of gluon PDFs with crucial role of distillation
T. Khan et al. (HadStruc), 2107.08960

Key aspects of the calculation:
- distillation combined with momentum smearing
- summed GEVP to access smaller temporal separations
- gradient flow to improve signal (extrapolate to $\tau = 0$)

Matching – take only $gg$ part
(mixing with singlet quark neglected)

Fit to pheno-inspired ansatz
(2 or 3-param., incl./excl. cutoff effects term)

Gluon PDFs/ITDs model motivated by counting rules based on pQCD analyses at large-$x$ + pheno. behavior at low-$x$

Other work for gluon pseudo-PDFs:
Z. Fan et al. (MSULat), Int. J. Mod. Phys. A36(2021)13 (nucleon)
Z. Fan et al. (MSULat), 2104.06372 (pion)
Generalized parton distributions (GPDs)

First studies also for GPDs

C. Alexandrou et al. (ETMC), Phys. Rev. Lett. 125 (2020) 262001

- nucleon boosts up to 1.67 GeV

Challenges:

- momentum transfer lowers the signal-to-noise ratio
- 2 or 4 GPDs \((H, E, \tilde{H}, \tilde{E})\) contribute to MEs at \(Q^2 \neq 0\) ⇒ need to disentangle them using different projectors
- standard GPDs need Breit frame: \(P_i^\perp = -P_f^\perp\)
- needs optimization of momentum smearing for each \(\vec{Q}\)

Important insights from models:


Independent work towards GPDs: H.-W. Lin, 2008.12474
PDFs can be classified according to their twist, which describes the order in $1/Q$ at which they appear in the factorization of structure functions.

**LT: twist-2** – probability densities for finding partons carrying fraction $x$ of the hadron momentum.

**Twist-3:**
- no density interpretation,
- contain important information about $qgq$ correlations,
- appear in QCD factorization theorems for a variety of hard scattering processes,
- have interesting connections with TMDs,
- important for JLab’s 12 GeV program + for EIC,
- however, measurements very difficult.

**Exploratory studies:**
- matching for twist-3 PDFs: $g_T$, $h_L$, $e$
  

BC-type sum rules
  
  S. Bhattacharya, A. Metz, 2105.07282

Note: neglected $qgq$ correlations
  
  see also: V. Braun, Y. Ji, A. Vladimirov, 2103.12105

- lattice extraction of $g_{T}^{u-d}(x)$ and $h_{L}^{u-d}(x)$
  
  + test of Wandzura-Wilczek approximation
  
  S. Bhattacharya et al., Phys. Rev. D102 (2020) 111501(R)  
  S. Bhattacharya et al., 2107.02574
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  S. Bhattacharya et al., Phys. Rev. D102 (2020) 111501(R)
  S. Bhattacharya et al., 2107.02574

- first exploration of twist-3 GPDs
  S. Bhattacharya et al., 2107.12818
Interest also in pion PDFs, using several approaches. Question: large-$x$ behavior $(1-x)^{-1}$ vs. $(1-x)^{-2}$ decay.

C. Alexandrou et al. (ETMC), 2104.02247

<table>
<thead>
<tr>
<th>Moments</th>
<th>TMF</th>
<th>$m_\pi = 260$ MeV</th>
<th>$a = 0.093$ fm</th>
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</table>


<table>
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<tr>
<th>Pseudo</th>
<th>Clover</th>
<th>$m_\pi = 415$ MeV</th>
<th>$a = 0.127$ fm</th>
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<tr>
<th>Current-</th>
<th>Current (LCS)</th>
<th>Clover</th>
<th>$m_\pi = 415$, $358,278$ MeV</th>
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X. Gao et al. (BNL), Phys. Rev. D102 (2020) 094513

| pseudo | Clover | $m_\pi = 160$ MeV | $a = 0.06$, $0.04$ fm |

ETMC: $\beta \approx 2$, pseudo: $\beta = 1.1(4)$, LCS: 2-param.: $\beta = 1.24(22)(7)$, 3-param.: $\beta = 2.12(56)(14)$

BNL – different analyses:

- $\beta$ from $0.66(34)(22)$ to $1.55(34)(27)$

Pheno analyses:

- FNAL E615: $\beta \approx 1$
- ASV: $\beta \approx 2$
- JAM: $\beta \approx 1.2$
- DSE: $\beta \approx 2$
- xFitter: $\beta \approx 1$
- NJL: $\beta \approx 1$

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- DSE: $\beta \approx 2$
- xFitter: $\beta \approx 1$
- NJL: $\beta \approx 1$
The standard procedure of quasi-PDF MEs renormalization:

\[ O_{\text{MS}}(z, \mu) = Z_{\text{MS}}(z, -p^2, \mu) \frac{O(z, a)}{Z(z, -p^2, a)} \]

is argued to contain non-perturbative effects at large-\( z \).

X. Ji et al., Nucl. Phys. B964 (2021) 115311

Proposed way out: **hybrid renormalization**.

- short distance \( z \leq z_S \approx 0.3 \text{ fm} \)
  - ratio scheme / RI-MOM,
- intermediate distance \( 0.3 \text{ fm} \approx z_S \leq z \leq z_L \approx \Lambda_{\text{QCD}}^{-1} \)
  - separate renormalization of log and linear divergences:
\[ Z(a, \mu) \exp(-\delta m|z|)O(z, a), \]
\( \delta m \) – Wilson line mass renormalization, e.g. from the static potential or from fitting MEs at large-\( z \)
- large distance \( z_L \approx \Lambda_{\text{QCD}}^{-1} \approx z_L \geq z \)
  - exponential/algebraic extrapolation (Regge-based),
- matching the different procedures at \( z_S \) and \( z_L \).

Used recently for extraction of pion PDFs.
Meson distribution amplitudes (DAs)

Meson distribution amplitudes (DAs) important for many exclusive decays.

\[ \rightarrow \]

represent momentum distribution of quarks/antiquarks in the leading \( q\bar{q} \) Fock state of the meson’s wave function.

Recent work:

- \( K^* \) and \( \phi \) mesons with quasi (physical point, continuum limit) J. Hua et al. (LPC), 2011.09788
- \( \pi \) and \( K \) mesons with quasi (continuum limit) R. Zhang et al., Phys. Rev. D102 (2020) 094519
- \( B \) meson DA formalism with
  - pseudo: S. Zhao, A. Radyushkin, Phys. Rev. D103 (2021) 054022

Theoretical progress in another method, recently dubbed HOPE (heavy OPE)

Compton tensor with an auxiliary heavy quark + OPE to relate to Mellin/Gegenbauer moments of PDFs/LCDAs W. Detmold, C.-J. D. Lin, Phys. Rev. D73 (2006) 014501
W. Detmold, A. Grebe, I. Kanamori, C.-J. D. Lin, R. Perry, Y. Zhao (HOPE), 2103.09529

- uses flavor-changing axial vector current: \( J_A^\mu = \bar{\Psi} \gamma^\mu \gamma^5 \psi + \bar{\psi} \gamma^\mu \gamma^5 \Psi \), \( \Psi (\psi) \) – heavy (light) quark
- all effects of the heavy quark in Wilson coefficients, no power-divergent mixings, suppressed HTE
- recent: shown relation to other approaches, analytic structure of HOPE amplitudes, convergence radius, calculation of 1-loop Wilson coefficients for unpolarized/helicity PDFs and LCDA
PDFs provide information only on the longitudinal momentum distributions, while in many cases important effects also from transverse momentum.

- Important for wide kinematical ranges in Drell-Yan, $e^+e^-$ annihilation, SIDIS
- Example: unpolarized
  \[
  f(x, \vec{k}_\perp) = \frac{1}{2P^+} \int \frac{d\lambda}{2\pi} \frac{d^2\vec{b}_\perp}{(2\pi)^2} e^{-i\lambda x + i\vec{k}_p \cdot \vec{v}_\perp} \langle P|\bar{\psi}(\lambda n/2 + \vec{b}_\perp)\gamma^+W_n(\lambda n/2 + \vec{b}_\perp)\psi(-\lambda n/2)|P\rangle
  \]
- Crucial new aspect: rapidity divergences from soft gluon radiation
  \[\Rightarrow\text{rapidity regulator } \delta + \text{UV renormalization scale } \mu\]
- Rapidity divergences can be incorporated in the soft function $S(\vec{b}_\perp, \mu, \delta^+, \delta^-)$ represents soft gluon radiation effects of a fast-moving charged particle
- Physical renormalized TMD: $f^{\text{TMD}} = f/\sqrt{S}$
- Soft function:
  * intrinsic part (rapidity-independent)
  * rapidity-dependent part defining Collins-Soper kernel $K(\vec{b}_\perp, \mu)$ – log-derivative of $f^{\text{TMD}}$.
- $f^{\text{TMD}}(x, \vec{b}_\perp, \mu, \zeta)$ – final desired object with evolution in the 2 last arguments governed by:
  * CS kernel for rapidity $\zeta$
  * $\gamma/\mu$ anomalous dimension (consisting of cusp and hard anomalous dimension) for renormalization scale $\mu$
- also: single transverse-spin asymmetry & Sivers Function from LaMET
  X. Ji, Y. Liu, A. Schäfer, F. Yuan, Phys. Rev. D103 (2021) 074005
  light-front wave functions from LaMET X. Ji, Y. Liu, 2106.05310

From: X. Ji et al., 2004.03543
The soft function can be extracted from a pseudoscalar meson form factor

\[ F_\Gamma(b_\perp, P^z) = \langle \pi(-P^z) | \bar{u} \Gamma u(t, b_\perp) \bar{d} \Gamma d(t, 0) | \pi(P^z) \rangle \]

\[ F_\Gamma(b_\perp, P^z) \] can be factorized into:

- intrinsic soft function
- quasi-TMDWF \( \approx \) pion LCDA with a staple-shaped operator

2 groups followed this strategy

Y. Li, S.-C. Xia et al. (Beijing+ETMC), 2106.13027

LPC calculation:

- \( \Gamma = I \) – best signal, leading-twist
- renormalization of bare \( S_I(b_\perp, 1/a) \):
  \[ S_I^{\text{MS}}(b_\perp, 1/a) = \frac{S_I(b_\perp, 1/a)}{S_I(b_\perp, 0, 1/a)} S_I^{\text{MS}}(b_\perp, 0, \mu) \]
  \( (S_I^{\text{MS}}(b_\perp, 0, \mu) \) from 1-loop PT)
- leading-order matching: \( 1/2N_c + O(\alpha_s) \)
Intrinsic soft function

Beijing+ETMC calculation  Y. Li, S.-C. Xia et al. (Beijing+ETMC), 2106.13027

- $\Gamma = I, \gamma_1, \gamma_2, \gamma_5, \gamma_5 \gamma_1, \gamma_5 \gamma_2$
  - found significant higher-twist contamination!
- considered combinations to reduce HTE using Fierz identities
- ratio scheme renormalization:
  $$C^{\text{ratio}}(b, l, P_3) = \frac{C(b, l, P_3)}{C(b, l, 0)} \frac{C^\text{MS}(0, 0, 0)}{C^\text{MS}(0, 0, 0)}$$
  ($C^\text{MS}(0, 0, 0)$ – standard local RI’ renormalization)
- leading-order matching: $1/2N_c + O(\alpha_s)$
- test of convergence in hadron boost $P_3$

| QUASI | TMF | $m_{\text{stn}} = 350$ MeV | $m_{\text{val}} = 350-827$ MeV | $a = 0.093$ fm |
The CS kernel governs the rapidity evolution of TMDs

Two approaches:

- ratio of TMDs at different rapidities
  M. Ebert, I. Stewart, Y. Zhao, PRD99(2019)034505

- ratios of first Mellin moments of TMDs
  M. Schlemmer et al., 2103.16991

### MOMENTS

<table>
<thead>
<tr>
<th>QUASI</th>
<th>clover</th>
<th>$m_\pi^{val} = 1.2$ GeV</th>
<th>$m_\pi^{sea} = 333$ MeV</th>
<th>$a = 0.06$ fm</th>
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<td>clover (quench.)</td>
<td>clover on HISQ</td>
<td>$m_\pi^{val} = 538$ MeV</td>
<td>$m_\pi^{sea} = 350-827$ MeV</td>
<td>$a = 0.12$ fm</td>
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### Q.-A. Zhang et al. (LPC), Phys. Rev. Lett. 125 (2020) 192001

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### Y. Li, S.-C. Xia et al. (Beijing+ETMC), 2106.13027

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### Data Points

- **2103.16991**
  - LPC $P_1^Z/P_2^Z = 4/3$
  - this work $P_1/P_2 = 4/3$
  - LPC $P_1^Z/P_2^Z = 4/2$
  - this work $P_1/P_2 = 5/4$
  - this work $P_1/P_2 = 6/5$

- **2017.11930**
  - M. Ebert, I. Stewart, Y. Zhao, PRD99(2019)034505

### Parameters

- $a = 0.085$ fm
- $\alpha = 0.098$ fm
Key prospects for the future

1. Robustness and reliability of the lattice extraction of $x$-dependent distributions
   $\Rightarrow$ towards precision studies
   improvements of lattice techniques
   study and removal of systematic effects

2. Exploration of new directions
   new kinds of distributions higher-twist, GPDs, TMDs, LFWFs
   other hadrons?
   can be phenomenologically relevant, e.g. $K^*$, $\phi$
   can shed light on the nucleon, e.g. $\Delta^+$

3. Synergy between lattice and phenomenology
   unpolarized PDFs – benchmark
   other distributions – potentially crucial impact
Robustness/reliability of lattice extraction

- Lattice-specific systematics:
  - isolation of the ground state hadron
  - discretization effects
  - finite volume effects
  - pion mass dependence (if not working at the physical point)

Note: hierarchy of systematics needs to be observed

- Broader systematics of the lattice calculation:
  - reconstruction of the $x$-dependence
  - non-perturbative renormalization
  - truncation effects: conversion, evolution, matching
  - higher-twist effects

Key challenges:
- **lattice:** reliably reach large hadron boosts
- **lattice:** control all lattice-specific systematics
- **pheno:** insights into HTE?
Conclusions

- **Message of the talk:** enormous progress in lattice calculations of $x$-dependent distributions with very encouraging results!

- Increasing number of distribution types accessible for lattice.
- However, there are still major challenges related to control of several sources of systematics.
- Expect:
  - slow, but consistent progress,
  - complementary role of LQCD and phenomenology.
Conclusions

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Thank you for your attention!