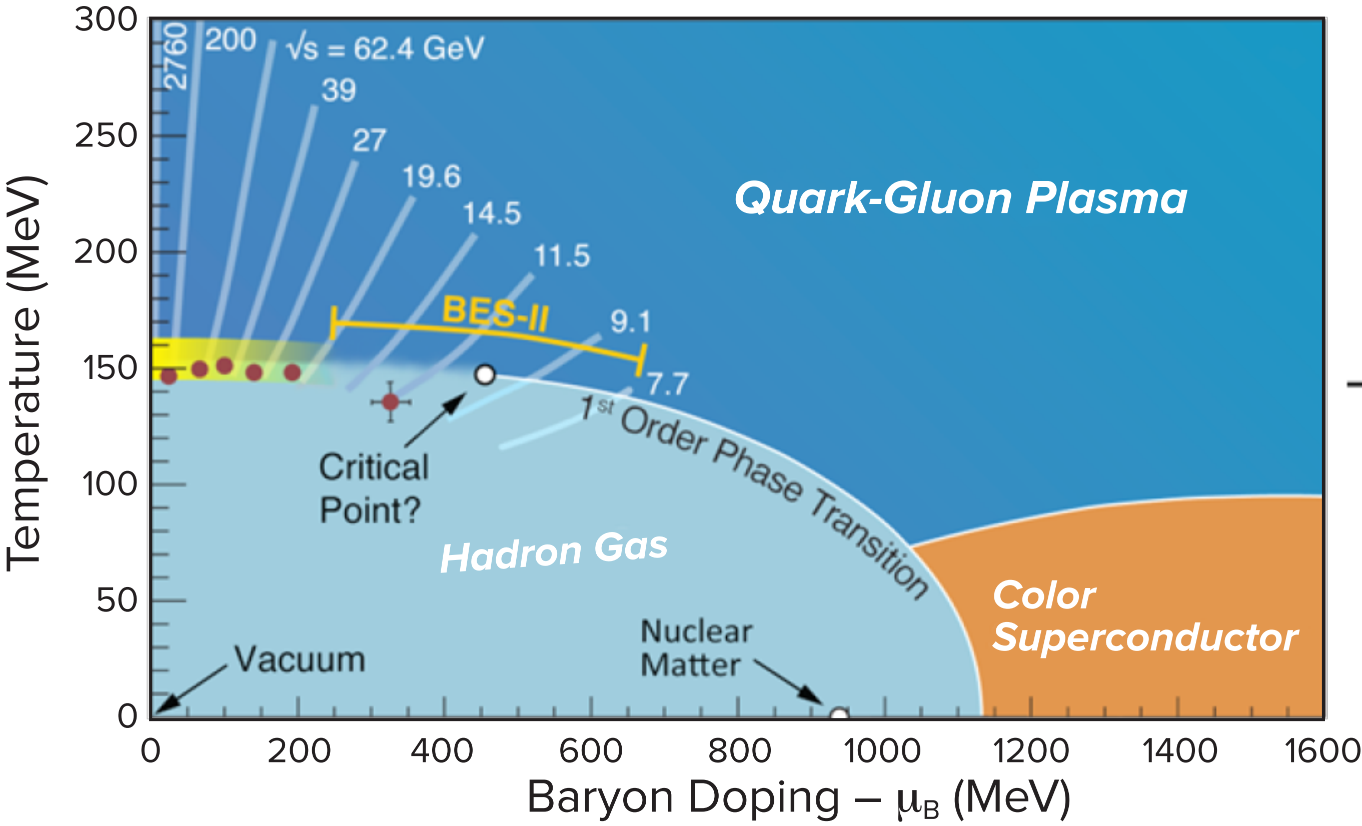
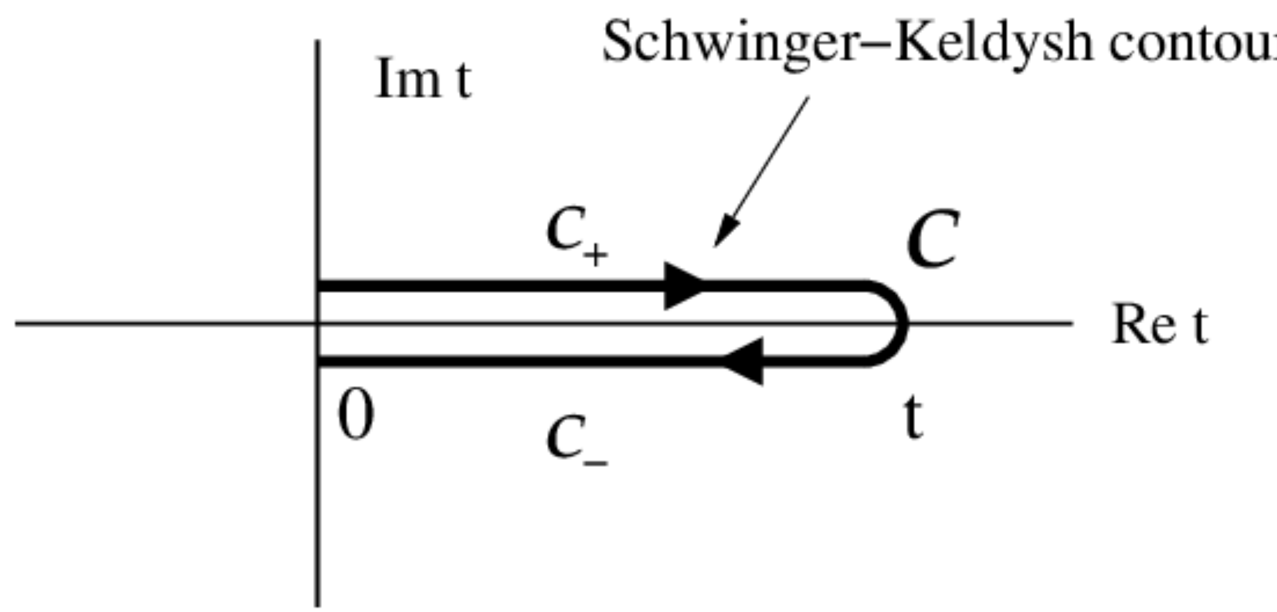
Resurgence: connecting perturbative and nonperturbative physics, with applications to QCD

Gerald Dunne
University of Connecticut

Quark Confinement and Hadron Spectrum Conference
virtually in Stavanger, August 2-6, 2021

Physical Motivation





- QCD phase diagram
- Non-equilibrium physics at strong-coupling
- Extreme hydrodynamics and kinetic theory

Basic toolkit of QFT:

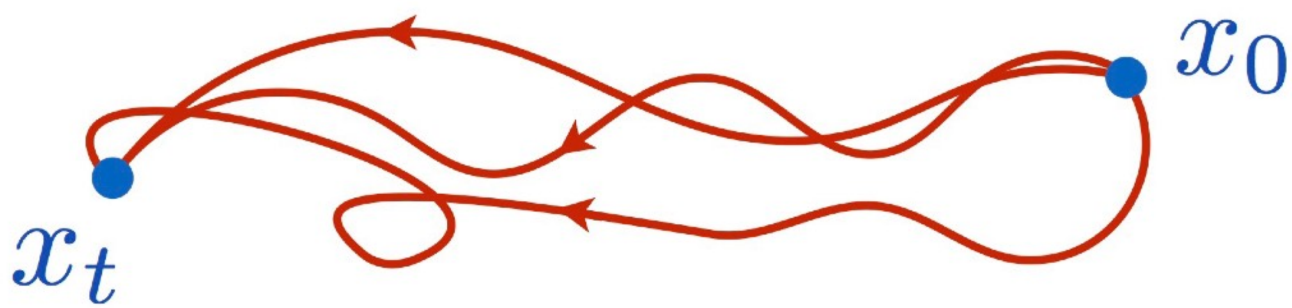
- perturbation theory
- non-perturbative semi-classical methods: “instantons"
- non-perturbative numerical methods: Monte Carlo
- asymptotics

“resurgence”: new form of asymptotics that unifies these approaches

technical problem: what does a quantum path integral really mean?

The Feynman Path Integral

$$\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle =$$



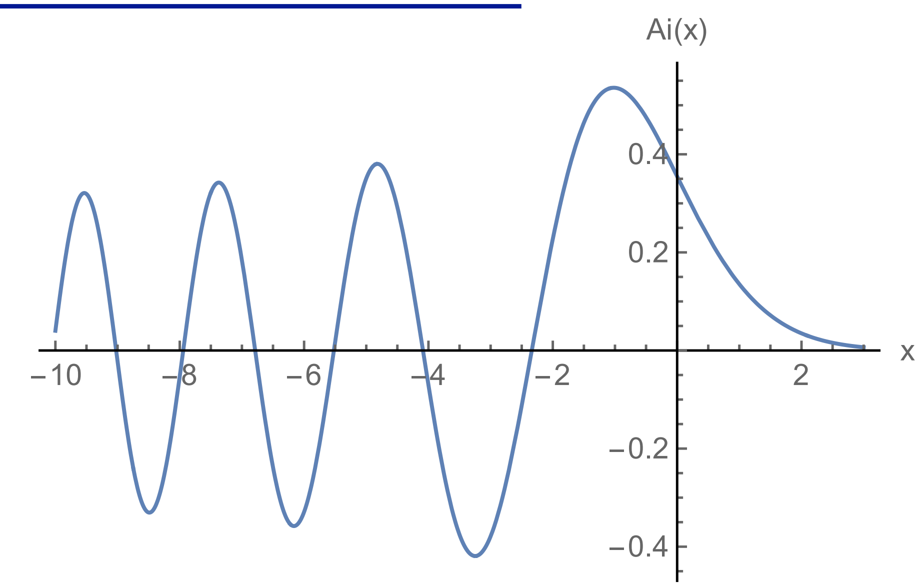
$$\text{QM:} \quad \int \mathcal{D}x(t)\, \exp\left[\frac{i}{\hbar} S[x(t)]\right]$$

$$\text{QFT:} \quad \int \mathcal{D}A(x^\mu)\, \exp\left[\frac{i}{g^2} S\left[A(x^\mu)\right]\right]$$

- stationary phase approximation: classical physics
- quantum perturbation theory: fluctuations about trivial saddle point
- other saddle points: non-perturbative physics
- resurgence: saddle points are related by analytic continuation, so perturbative and non-perturbative physics are *necessarily unified*

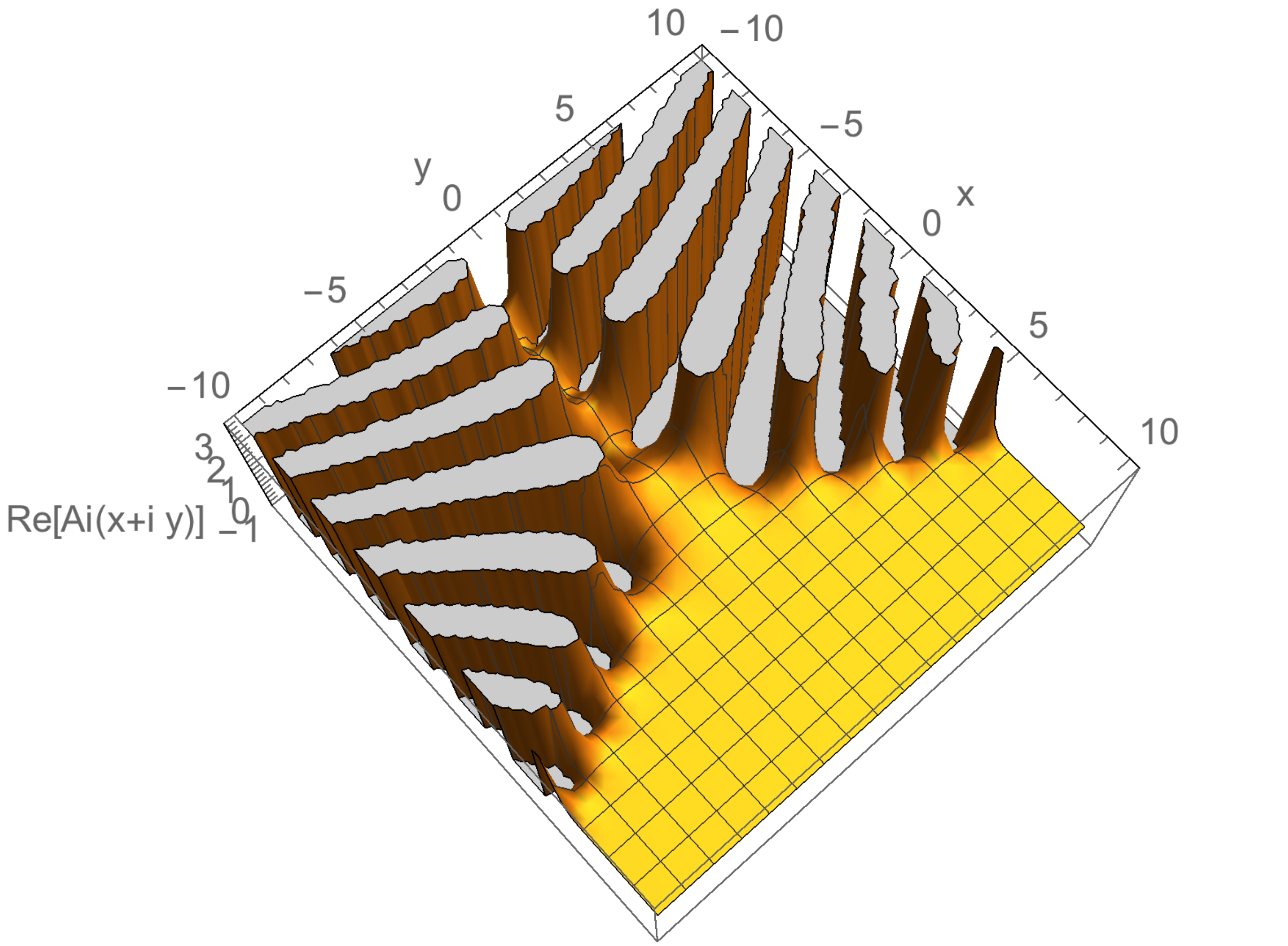
Stokes and the Airy Function: “Stokes Phenomenon”

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, e^{i\left(\frac{1}{3}t^3 + x\,t\right)}$$



- Real t integration is ill-posed
- Deform t contours as x changes
- Basis of steepest-descent contours (“thimbles”)

- Stokes transitions occur in the complex x plane



- Exact non-perturbative connection formulae connect sectors
- These are “hidden” in naive perturbation theory

Analytic Continuation of Path Integrals

since we need complex analysis and contour deformation to make sense of oscillatory exponential integrals, it is natural to explore similar methods for (infinite dimensional) path integrals

$$\int \mathcal{D}A(x^\mu)\, \exp\left[\frac{i}{g^2}\, S[A(x^\mu)]\right] \longleftrightarrow \int \mathcal{D}A(x^\mu)\, \exp\left[-\frac{1}{g^2}\, S[A(x^\mu)]\right]$$

goal: a satisfactory formulation of the functional integral should be able to describe Stokes transitions

Resurgent Trans-Series

resurgence: "new" idea in mathematics

Ecalle 1980s; Dingle 1960s; Stokes 1850

perturbative series $\longrightarrow$ "trans-series"

$$f(\hbar) = \sum_p c_{[p]}\, \hbar^p \quad \longrightarrow \quad f(\hbar) = \sum_k \sum_p \sum_l c_{[kpl]}\, e^{-\frac{k}{\hbar}}\, \hbar^p\, (\ln \hbar)^l$$

physics: • unifies perturbative and non-perturbative physics

mathematics:
- trans-series is well-defined under analytic continuation
- expansions about different saddles are related
- exponentially improved asymptotics

idea: seek a computationally viable constructive definition of the path integral using ideas from resurgent trans-series

Resurgent Functions

"resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities" J. Ecalle, 1980

implication: fluctuations about different singularities are related

conjecture: this structure occurs for all "natural problems"

Many examples have recently been found in QM, QFT & string theory

Resurgence and Perturbation Theory

Perturbation theory works, but it is generically divergent

Perturbation theory encodes non-perturbative information

Resurgence can be viewed as a formalism for doing this encoding & decoding

Borel Summation: the Physics of Divergent Series

Borel transform of a divergent series with $c_n \sim n!$

$$f(g) \sim \sum_{n=0}^{\infty} c_n \, g^n \quad \rightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \, t^n$$

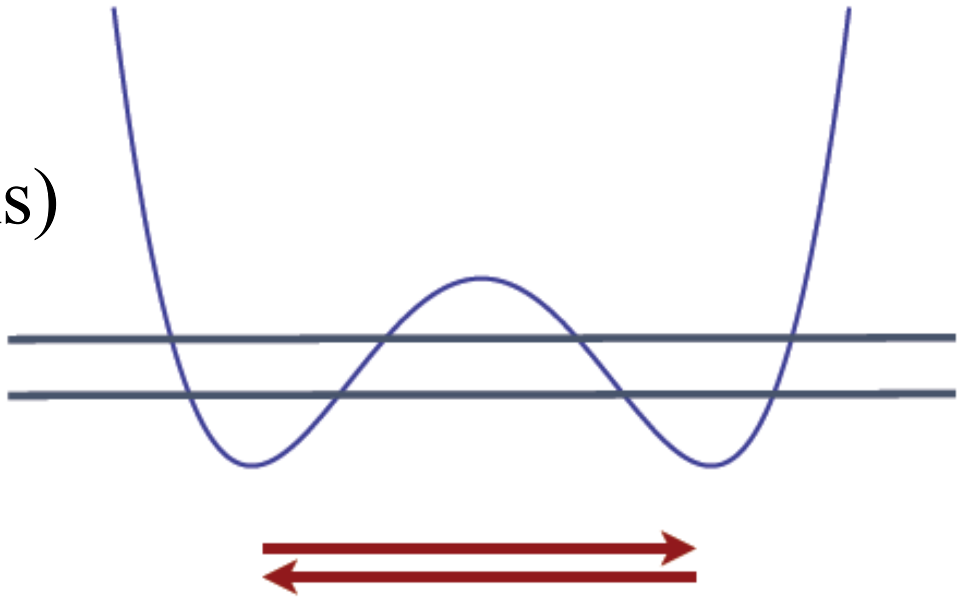
Borel sum of the divergent series:

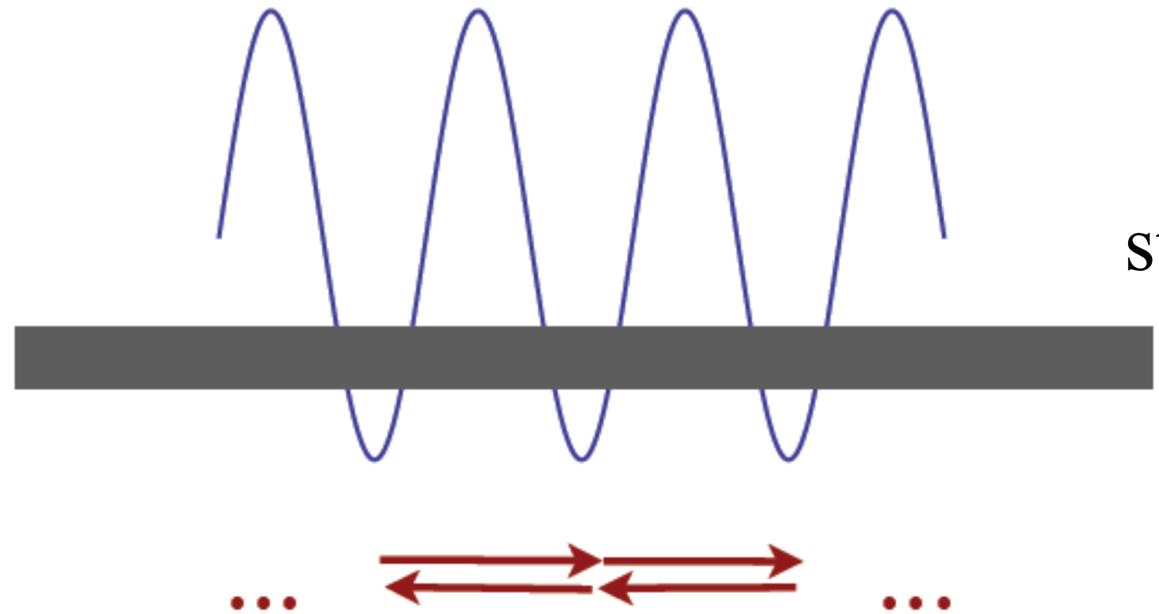
$$\mathcal{S}[f](g) = \frac{1}{g} \int_0^\infty dt \, e^{-t/g} \, \mathcal{B}[f](t)$$

- the singularities of B[f](t) provide a physical encoding of the global asymptotic behavior of f(g), which is also much more mathematically efficient than the asymptotic series
- singularities of Borel transform = non-perturbative physics
- singularities on positive Borel t axis: exponentially small imaginary part
- imaginary part is frequently associated with *physical instability* (Dyson)

Beyond Instantons for Non-Perturbative Physics

(phase transitions)





(band structure)

- exponentially small non-perturbative splitting due to tunneling
- Yang-Mills theory and QCD have aspects of both systems
- Borel analysis gives exponentially small (& ambiguous) imaginary part ???
- Resolution: Bogomolnyi/Zinn-Justin (BZJ) cancellation

perturbation theory + Borel: $\longrightarrow \quad +i\,\exp\left[-\frac{2\,S_I}{\hbar}\right]$

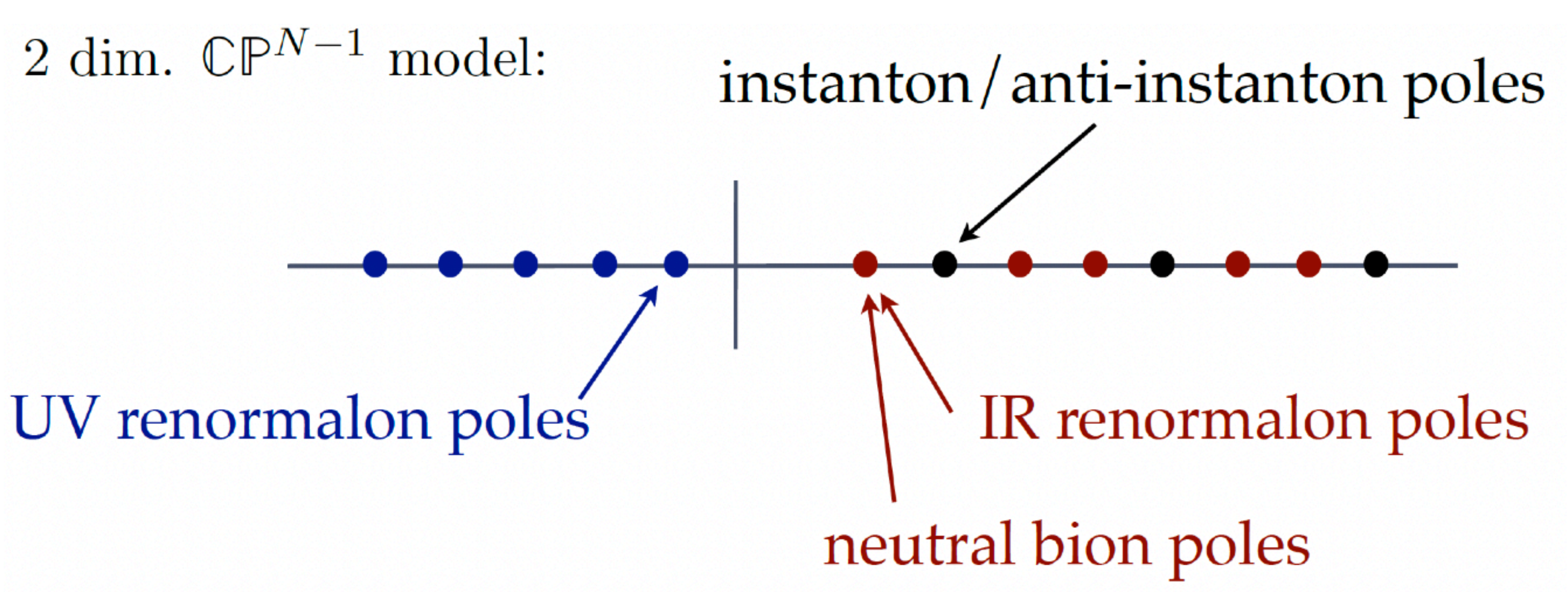
non-perturbative instanton & anti-instanton interaction: $\longrightarrow \quad -i\,\exp\left[-\frac{2\,S_I}{\hbar}\right]$

unphysical imaginary parts exactly cancel

- combined as a trans-series, energy is consistent (to all orders)

Renormalons and Resurgence in QFT

- QM: divergence of PT is due to diagram combinatorics ("instantons")
- QFT: divergence from classes of diagrams (e.g. bubble chains, "renormalons")
- Asymptotically free QFT: IR renormalons in perturbation theory $\Rightarrow$ ambiguous imaginary non-perturbative terms
- Resurgence idea: cancelled by other (non-perturbative) path integral saddles
- "Bion" and Lefschetz thimble interpretation
- 2 dim. Sigma models: CP(N-1), O(N), PCM, …



Adiabatic Continuity in QFT

- Asymptotically free QFT: weakly-coupled & perturbatively calculable at short distances but strongly-coupled & non-perturbative at long distances

- Adiabatic continuity idea: deform the theory to preserve the interesting IR physics while staying continuously connected to the strong-coupling theory

- Deformations: compactify (twisted b.c.s); matter content; double-trace; SUSY…
- E.g.: spatial vs thermal compactification

- "Bion" and Lefschetz thimble interpretation

- "hidden topological angle" (HTA) & critical points at infinity

Some recent developments

- Lattice investigations (QCD_adj, …)
- 't Hooft anomaly interpretation
- Graded partition functions and distillation of Hilbert space
- Renormalons

Resurgence in QFT

Effective Field Theory: Euler-Heisenberg QED effective action

- Paradigm of effective field theory
- Effective Lagrangian: $\mathcal{L} = \mathcal{L}(F_{\mu\nu}, \nabla F_{\mu\nu}, \nabla^2 F_{\mu\nu}, \dots)$
- Expansion is divergent, but Borel-Ecalle summable (also derivatives)
- Manifestation of Dyson's instability

Matrix Models

- 2d Yang-Mills: Gross-Witten-Wadia, …
- "Localizable" SUSY QFT
- Path integral reduces to an NxN matrix integral
- Phase transition structure at large N: "transmutation of transseries"

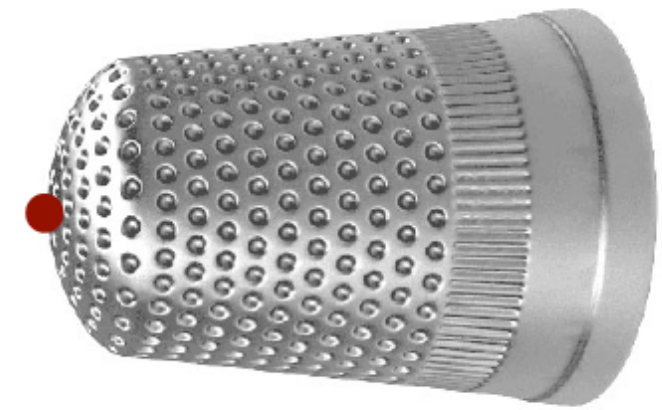
Chern-Simons Theory

- 2+1 dim. gauge theory
- Saddles: $F_{\mu\nu} = 0$ ("flat connections")
- Remarkable result: expansions about different saddles "see" one another
- Technicalities: large gauge transformations; geometry of spacetime

Analytic Continuation of Path Integrals: "Lefschetz Thimbles"

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \stackrel{?}{=} \sum_{\text{thimble}} \mathcal{N}_{\text{th}} \, e^{i\,\phi_{\text{th}}} \int_{\text{th}} \mathcal{D}A \, \times (\mathcal{J}_{\text{th}}) \times \exp\left(\mathcal{R}e\left[\frac{i}{\hbar} S[A]\right]\right)$$

Lefschetz thimble = "functional steepest descents contour"



on a thimble, the path integral becomes
well-defined and computable

complexified gradient flow:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

Conceptual and algorithmic challenges:

- Finding all thimbles ?
- Relative phases ?
- Intersection numbers ?

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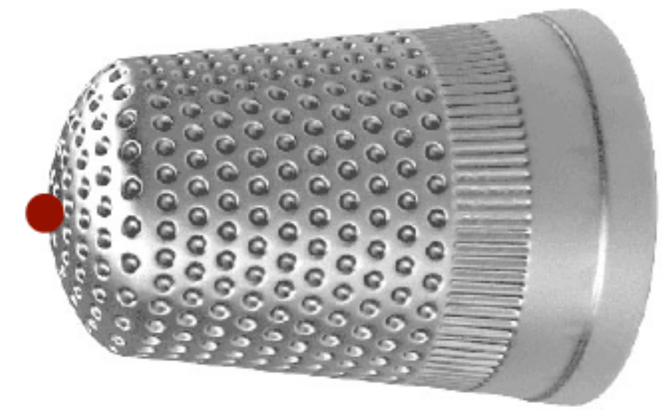

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Comment: despite great progress recently, lattice methods have not yet taken advantage of the full power of resurgence

Analytic Continuation of Path Integrals: “Lefschetz Thimbles”

CRISTOFORETTI *et al.* (2013)

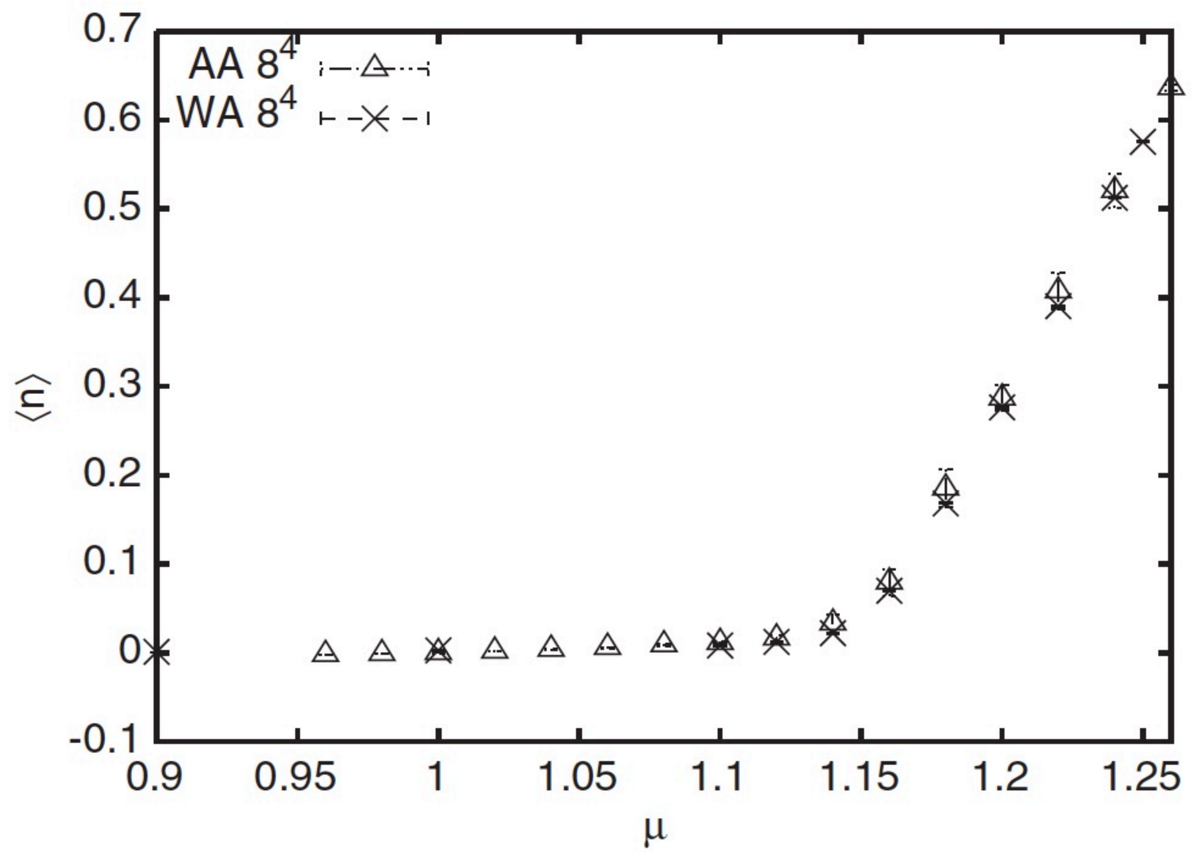
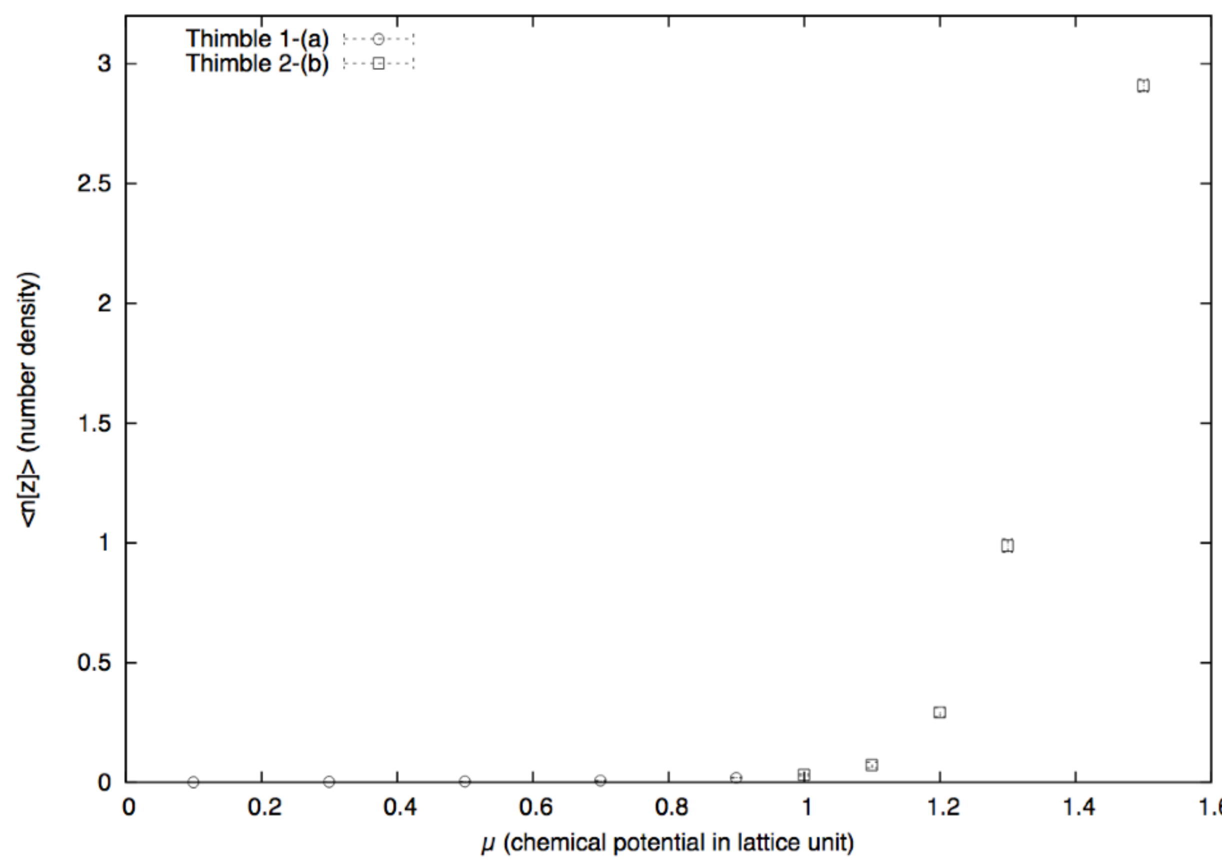


FIG. 3. Comparison of the average density $\langle n \rangle$ obtained with the worm algorithm (WA) [22] with the Aurora algorithm (AA)

Fujii et al (2013)



- 4d relativistic Bose gas: complex scalar field theory
- Monte Carlo on thimble softens the sign problem
- results comparable to “worm algorithm”

Phase Transitions in QFT: 2 dim. Thirring Model

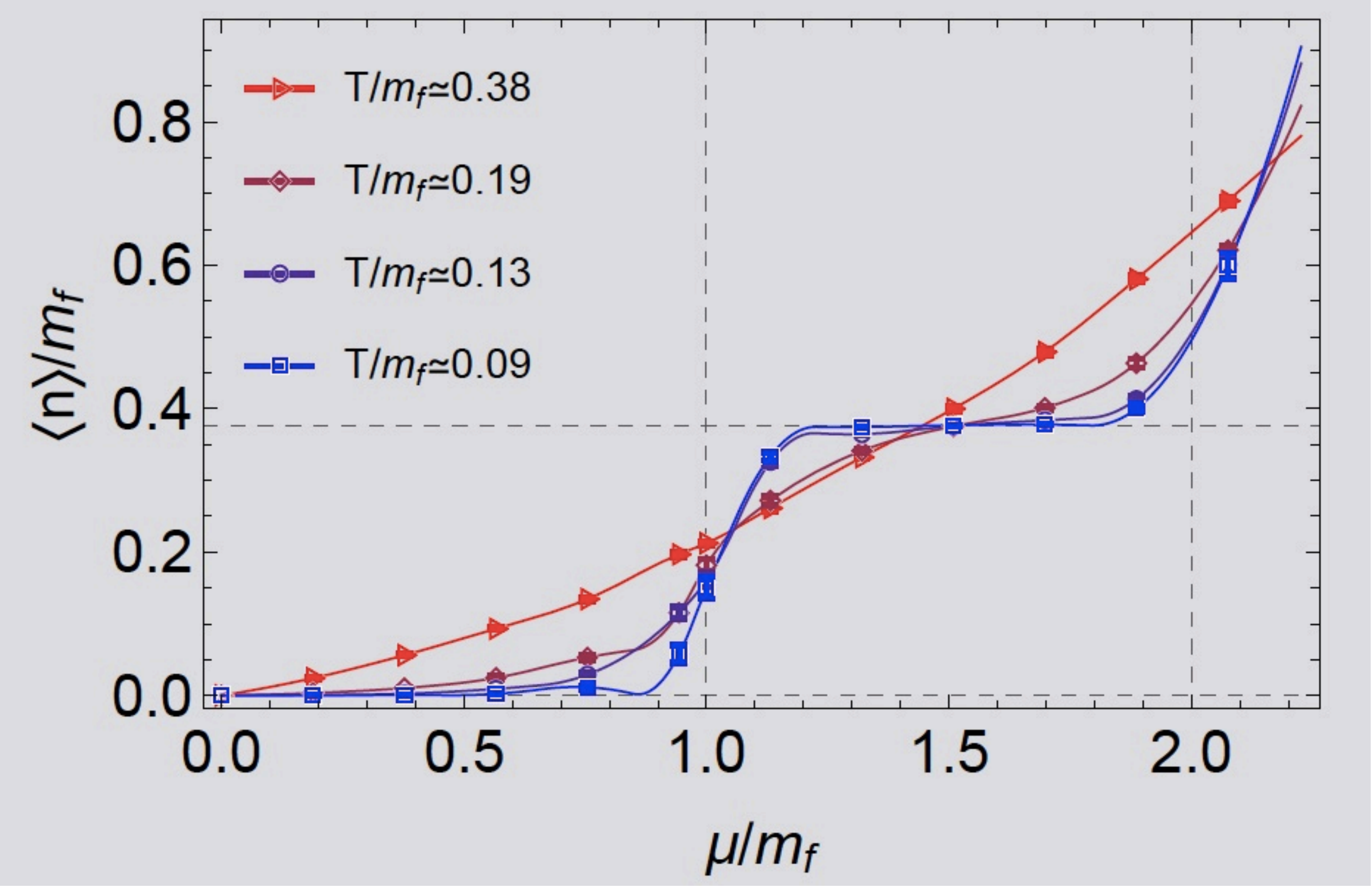
$$\mathcal{L} = \bar{\psi}^a \left(\gamma_\nu \partial_\nu + m + \mu\, \gamma_0\right) \psi^a + \frac{g^2}{2N_f} \left(\bar{\psi}^a \gamma_\nu \psi^a\right) \left(\bar{\psi}^b \gamma_\nu \psi^b\right)$$

- interacting fermions: asymptotically free
- prototype for dense quark matter
- sign problem at nonzero density

idea: flow to an approximate Lefschetz thimble(s)

(Alexandru et al, 2016)

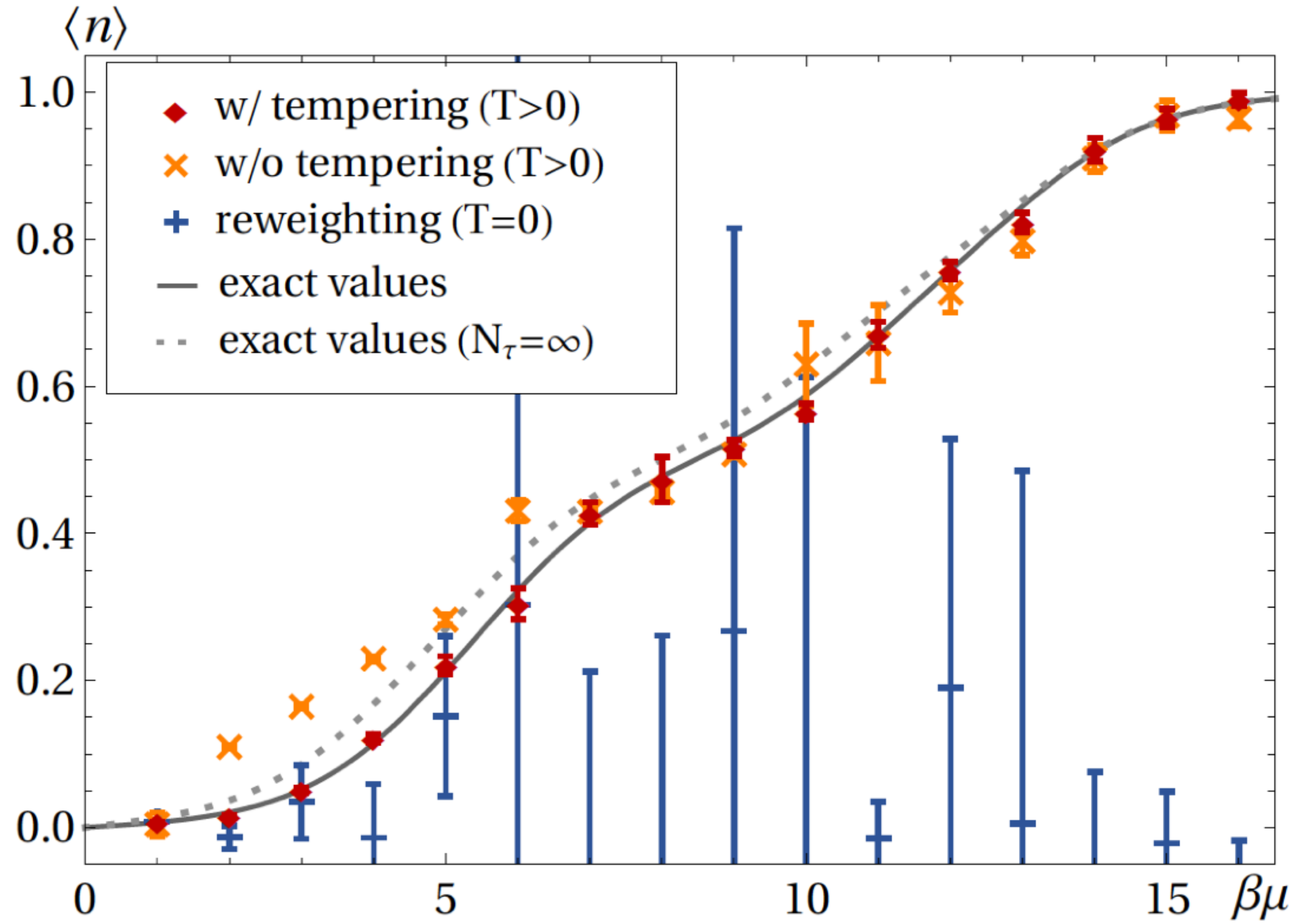
Monte Carlo thimble computation

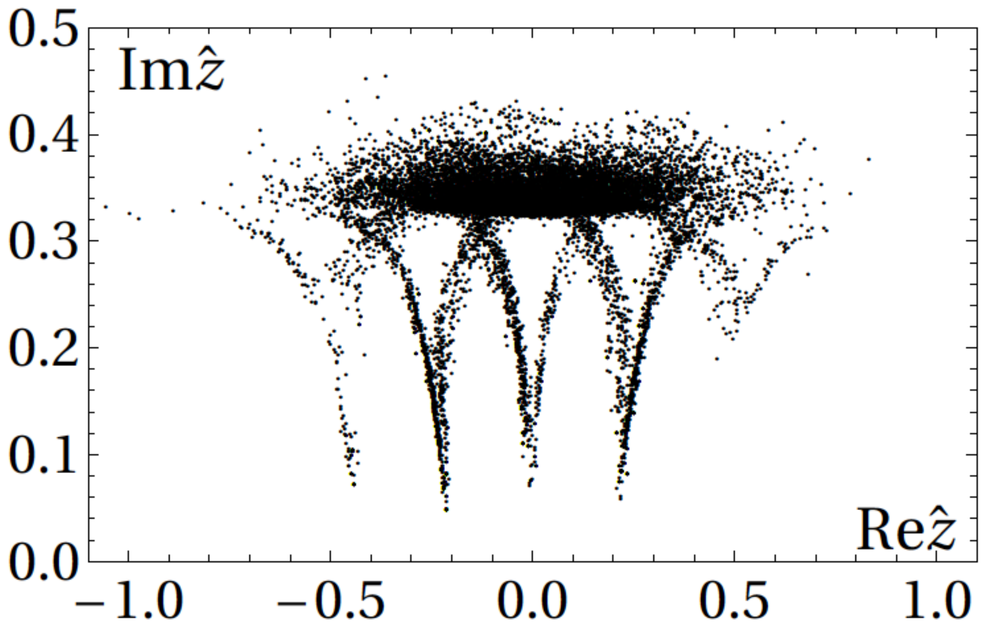


Tempered Lefschetz Thimble Method

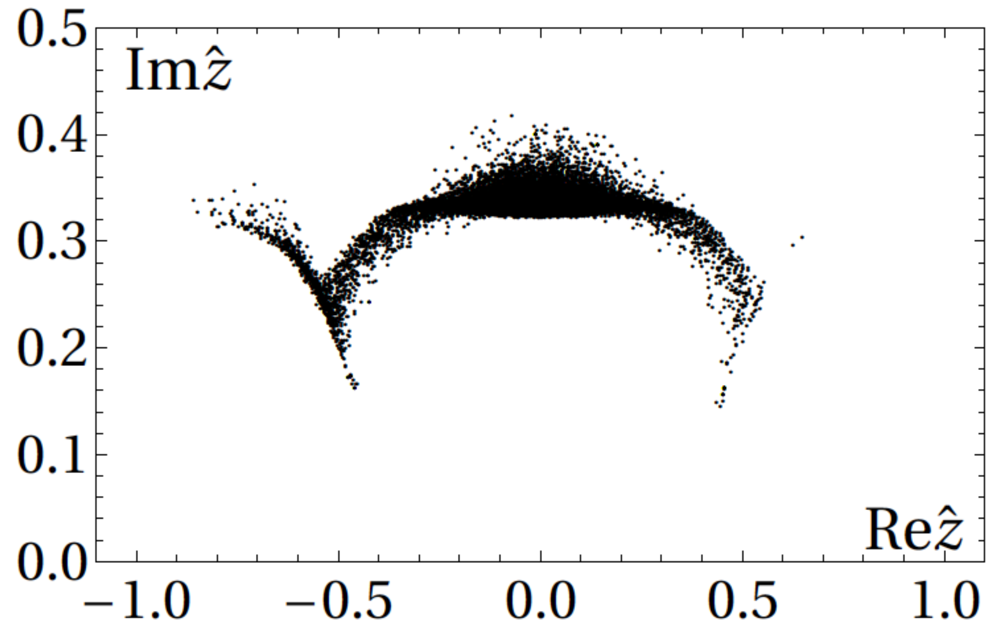
(Fukuma et al, 2017, 2019,…)

- probe <u>all</u> relevant thimbles ???
- sign problem vs. ergodicity
- coupling → dynamical variable
- parallelized tempering
- e.g. 2d Hubbard model
- probes multiple thimbles





with tempering

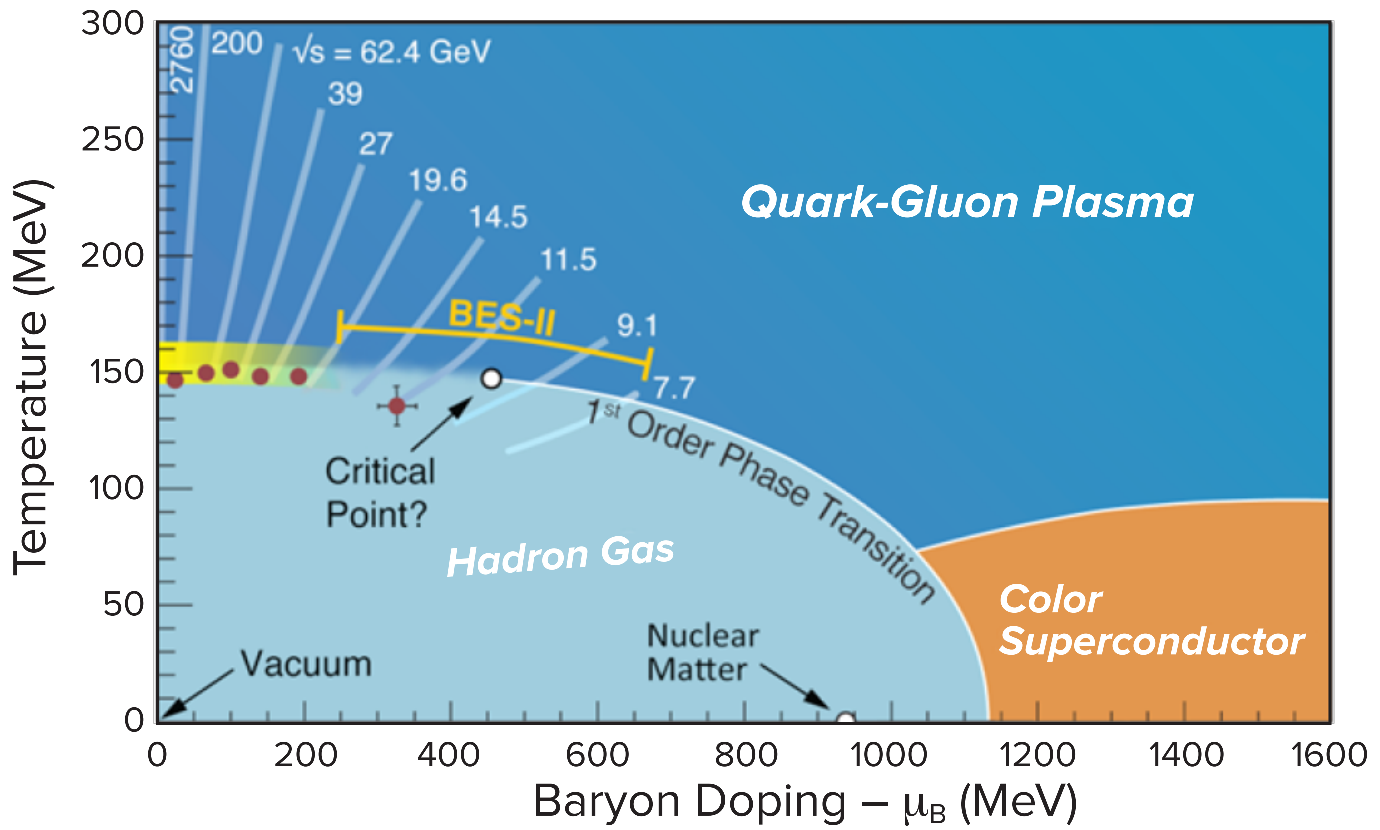


without tempering

Resurgent Extrapolation

Costin, GD: 2003.07451, 2009.01962 , 2108.01145

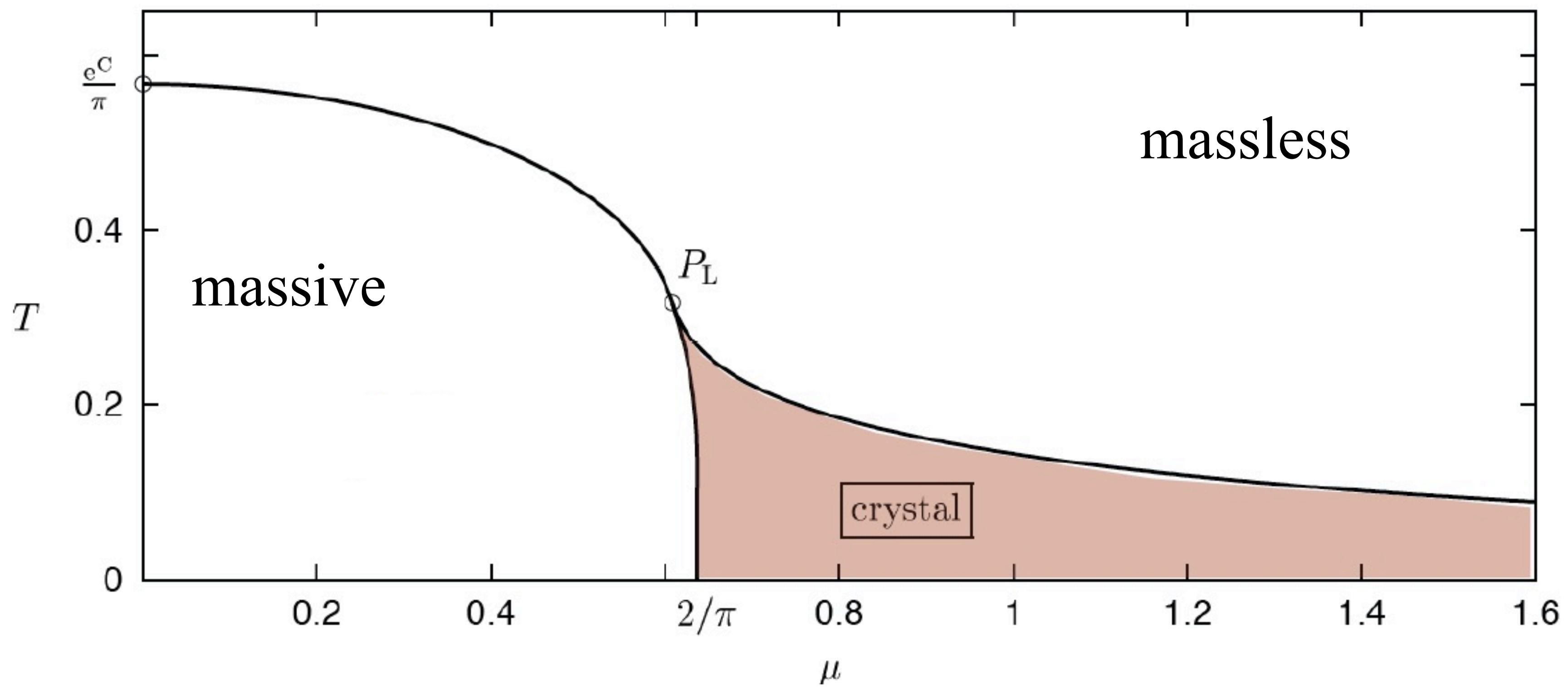
- Often, perturbation theory/asymptotics is the ONLY thing we can do
- E.g. imaginary chemical potential for the sign problem
- Dramatic recent progress in computing QFT (perturbative) amplitudes
- Question: how much global information can be decoded from a FINITE number of perturbative coefficients ?
- How much "perturbative" information is required to detect, and to probe the properties of, a phase transition, possibly at a distant point ?



Phase Transitions in 2d Gross-Neveu Model

$$\mathcal{L}_{\rm Gross-Neveu} = \bar{\psi}_a i \partial\!\!\!/ \psi_a + \frac{g^2}{2}\left(\bar{\psi}_a \psi_a\right)^2$$

- asymptotically free; dynamical mass; chiral symmetry; model for QCD
- large N_f chiral symmetry breaking phase transition



chiral symmetry breaking condensate

$$\sigma(x;T,\mu) \equiv \langle \bar{\psi}\psi \rangle (x;T,\mu)$$

develops crystalline phases

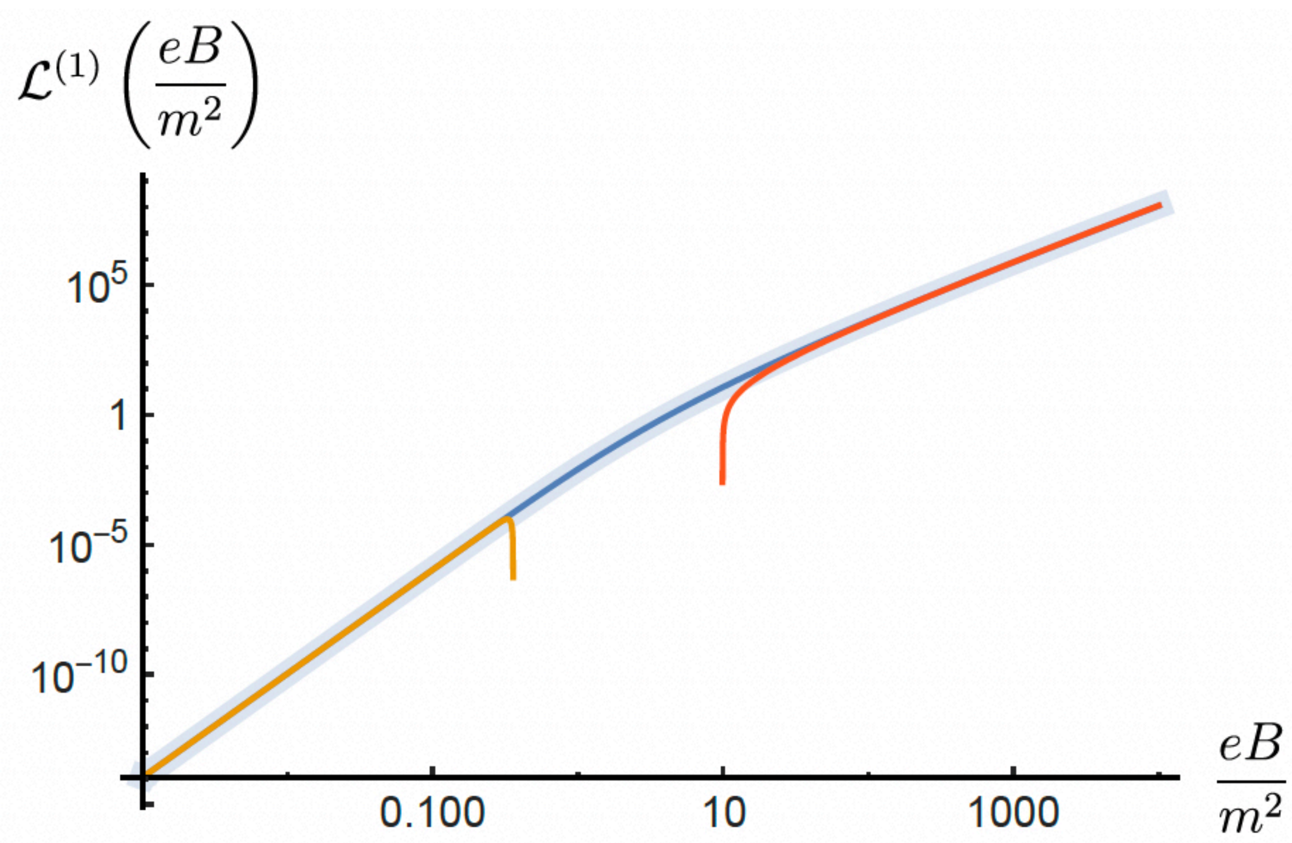
- Expansion about tricritical point = (inhomogeneous) Ginzburg-Landau
- T=0 High-density expansion is convergent: critical chemical potential
- T=0 Low-density expansion is divergent: non-perturbative trans-series

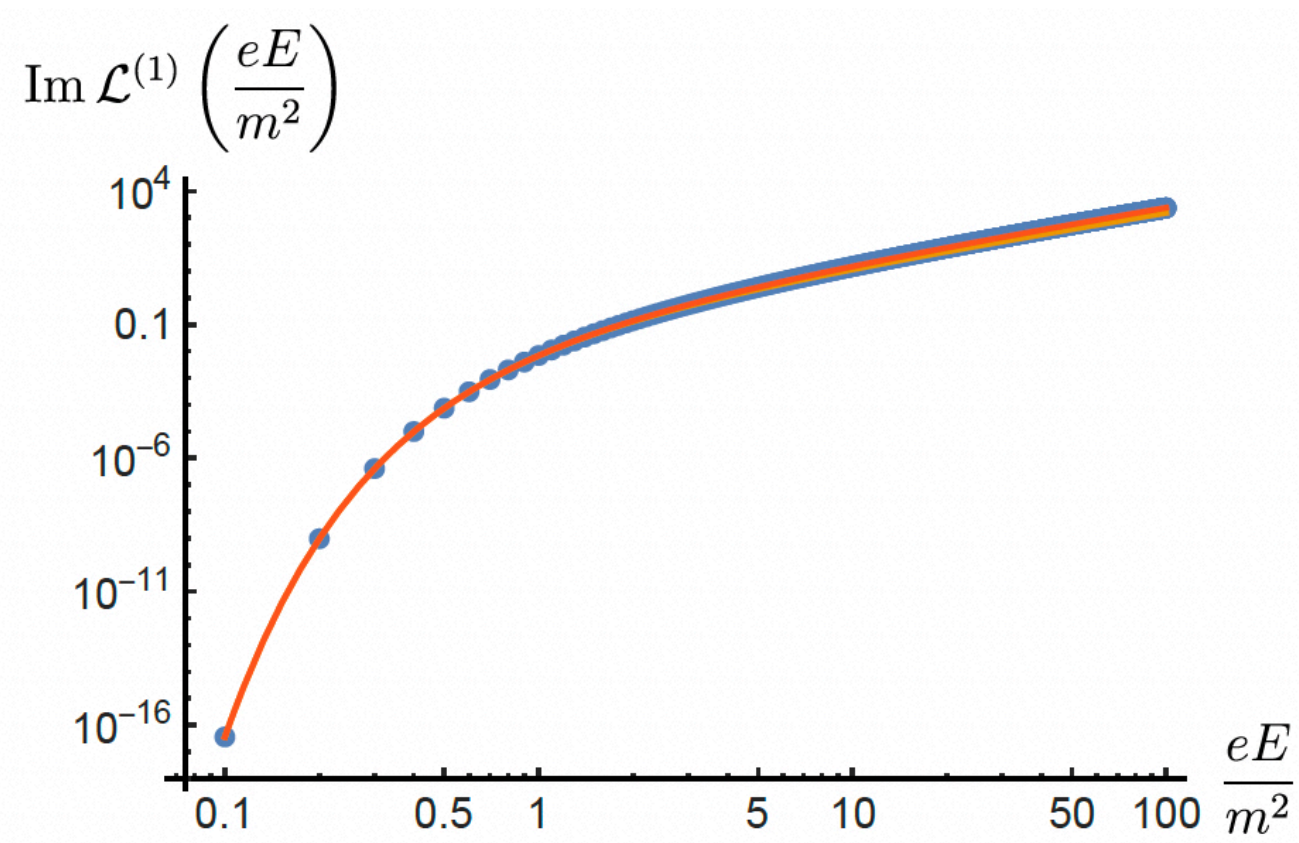
$$\mathcal{E}(\rho) \sim -\frac{1}{4\pi} + \frac{2\rho}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{e^{-k/\rho}}{\rho^{k-2}} \mathcal{F}_{k-1}(\rho) \qquad\qquad \mu_{\rm critical} = \frac{2}{\pi} \quad \leftrightarrow \quad \rho = 0$$

Resurgent Extrapolation: Euler-Heisenberg example

$$
\begin{aligned}
\mathcal{L}^{(1)}\left(\frac{eB}{m^2}\right) &= -\frac{B^2}{2}\int_0^\infty \frac{dt}{t^2}\left(\coth t - \frac{1}{t} - \frac{t}{3}\right) e^{-m^2 t/(eB)} \\
&\sim \frac{B^2}{\pi^2}\left(\frac{eB}{m^2}\right)^2 \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2n+2)}{\pi^{2n+2}} \zeta(2n+4) \left(\frac{eB}{m^2}\right)^{2n} \quad , \quad eB \ll m^2 \\
&\sim \frac{1}{3}\cdot\frac{B^2}{2}\left(\ln\left(\frac{eB}{\pi m^2}\right) - \gamma + \frac{6}{\pi^2}\zeta'(2)\right) + \dots \quad , \quad eB \gg m^2
\end{aligned}
$$

- Weak to strong B field extrapolation from just 10 terms of weak B expansion
- B field to E field analytic continuation from just 10 terms of weak B expansion





Conclusions

- "Resurgence" is based on a new and improved form of asymptotics
- Established for differential & difference equations
- Deep(er) connections between perturbative and non-perturbative physics
- Many examples in QM, matrix models, QFT
- Overall goal: computable access to strongly-coupled QFT, high density QFT, far-from-equilibrium physics, phase transitions, particle production, …

- Resurgence, Renormalons and the Operator Product Expansion
- Incorporating large N information systematically
- Lattice QFT developments: more examples, new algorithms and ideas
- Resurgence and hydrodynamics: (i) gradient expansion is generically divergent; (ii) relating initial conditions to late-time expansion; (iii) holographic methods
- Adiabatic Continuity and 't Hooft anomaly matching
- Resurgent extrapolation: high-precision extraction of physical information from finite order expansions. (e.g. 2d Gross-Neveu: Basar, 2105.08080)