Determination of $\alpha(M_z)$ from a hyperasymptotic approximation to the energy of a static quark-antiquark pair

Xabier Lobregat

IFAE-BIST

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Work in collaboration with César Ayala and Antonio Pineda

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Outline of the talk

- Introduction
  - Static energy of a quark antiquark pair
  - Terminants
- The fits
- Conclusions

The static energy of a quark-antiquark pair

- $E^{\mathrm{latt}}(r) - E^{\mathrm{latt}}(r_{\mathrm{ref}}) = E^{\mathrm{th}}(r) - E^{\mathrm{th}}(r_{\mathrm{ref}})$
- In pNRQCD the singlet static energy admits the expression

$$E^{\mathrm{th}}(r) = V(r, \nu_{\mathrm{us}} = \nu_{\mathrm{s}}) + \delta V_{\mathrm{RG}}(r, \nu_{\mathrm{s}}, \nu_{\mathrm{us}}) + \delta E_{\mathrm{us}}(r, \nu_{\mathrm{us}})$$

- V has a $u = 1/2$ renormalon
- This renormalon is r independent
- We use the derivative of the static energy ($\frac{d}{dr}V \equiv F$)

$$\begin{aligned}\mathcal{F} \equiv& \frac{d}{dr}E^{\mathrm{th}}(r)\\ =& F(r, \nu_{\mathrm{us}} = \nu_{\mathrm{s}}) + \frac{d}{dr}\delta V_{\mathrm{RG}}(r, \nu_{\mathrm{s}}, \nu_{\mathrm{us}}) + \frac{d}{dr}\delta E_{\mathrm{us}}(r, \nu_{\mathrm{us}})\end{aligned}$$

- We fit

$$E^{\mathrm{latt}}(r) - E^{\mathrm{latt}}(r_{\mathrm{ref}}) = \int_{r_{\mathrm{ref}}}^{r} dr' \, \mathcal{F}(r')$$

The principal value Borel sum

- Not a new idea
- Based on Phys.Rev.D99,074019 we use the principal value Borel sums

$$\mathcal{F} = F(r, \nu_{\rm us} = \nu_{\rm s})\Big|_{\rm PV} + \frac{d}{dr}\delta V_{\rm RG}(r, \nu_{\rm s}, \nu_{\rm us})\Big|_{\rm N^3LL} + \frac{d}{dr}E_{\rm us}(r, \nu_{\rm us})\Big|_{\rm PV}$$

- We define the principal value Borel sum

$$R = \sum_{n=0}^{\infty} r_n \alpha^{n+1}(\mu)\,, \qquad R_{\rm PV} = {\rm PV}\int_0^{\infty} dt\, e^{-t/\alpha(\mu)} \sum_{n=0}^{\infty} \frac{r_n}{n!} t^n$$

Terminants

- Leading singularity in $t = \frac{2\pi d}{\beta_0}$; $N_{\rm P}(d) = \frac{2\pi|d|}{\beta_0\alpha(\mu)}\{1 - c\alpha(\mu)\}$

$$R_{\rm PV} \approx \sum_{n=0}^{N_{\rm P}(d)} r_n\alpha^{n+1}(\mu) + T(d, N_{\rm P}, \mu) + \mathcal{O}\left(e^{-\frac{2\pi|d|}{\beta_0\alpha(\mu)}(1+\log\frac{|d_{\rm next}|}{|d|})}\right)$$

- In our case $d = 3$ and the terminant takes the form

$$T = \sqrt{\alpha(\mu)}K^{(\rm P)} r\mu^3 e^{-\frac{6\pi}{\beta_0\alpha(\mu)}}\left(\frac{\beta_0\alpha(\mu)}{4\pi}\right)^{-3b}\left(1 + \bar{K}_1^{(\rm P)}\alpha(\mu) + \mathcal{O}\left(\alpha^2(\mu)\right)\right)$$

where $\eta_c = -3b + \frac{6\pi c}{\beta_0} - 1$; $b = \frac{\beta_1}{2\beta_0^2}$ and

$$K^{(\rm P)} = -\frac{Z_3^F 2^{1-3b}\pi 3^{3b+1/2}}{\Gamma(1+3b)}\beta_0^{-1/2}\left[-\eta_c + \frac{1}{3}\right]$$

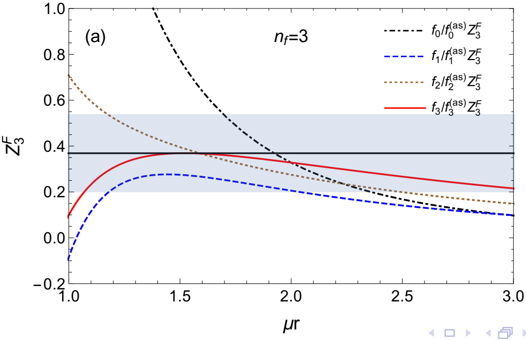
Normalization of the $u = 3/2$ renormalon

- The PV Borel sum of the force requires knowledge of Z_3^F (recall $Z_3^F = 2Z_3^V$)

$$r^2 f_n^{\rm as} = Z_3^F (r\mu)^3 \left(\frac{\beta_0}{6\pi}\right)^n \frac{\Gamma(n+1+3b)}{\Gamma(1+3b)} \left\{1 + \frac{3b}{n+3b} b_1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}$$

- We estimate it dividing by the exact value ($x = 1.52$)

$$Z_3^F\big|_{n_f=3} = 0.37^{-0.06}_{-0.16}(\Delta x) + 0.02(\mathrm{N^2LO}) - 0.05(\mathcal{O}(1/n)) + 0.005(\mathrm{us}) = 0.37(17)$$



$$F = \frac{d}{dr} V$$

$$V = \sum_{n=0}^{\infty} V_n(r, \nu_s, \nu_{us}) \alpha^{n+1}(\nu_s)$$

$$V_n(r, \nu_s, \nu_{us}) = -\frac{C_F}{r} \frac{1}{(4\pi)^n} a_n(r, \nu_s, \nu_{us})$$

$$a_0 = 1$$

$$a_1(r, \nu_s) = a_1 + 2\beta_0 \log\left(\nu_s e^{\gamma_E} r\right)$$

$$a_2(r, \nu_s) = a_2 + \frac{\pi^2}{3}\beta_0^2 + (4a_1\beta_0 + 2\beta_1)\log\left(\nu_s e^{\gamma_E} r\right) + 4\beta_0^2 \log^2\left(\nu_s e^{\gamma_E} r\right)$$

$$a_3(r, \nu_s, \nu_{us}) = a_3 + a_1\beta_0^2\pi^2 + \frac{5\pi^2}{6}\beta_0\beta_1 + 16\zeta_3\beta_0^3 + \left(2\pi^2\beta_0^3 + 6a_2\beta_0 + 4a_1\beta_1 + 2\beta_2\right)\log\left(\nu_s e^{\gamma_E} r\right) + \frac{16}{3}C_A^3\pi^2 \log\left(\nu_{us} e^{\gamma_E} r\right) + \left(12a_1\beta_0^2 + 10\beta_0\beta_1\right)\log^2\left(\nu_s e^{\gamma_E} r\right) + 8\beta_0^3 \log^3\left(\nu_s e^{\gamma_E} r\right)$$

Details on the fits I

- Data from Phys.Rev.D100,114511
- $\beta = 8.4$ lattice spacing $a = 0.025\,\mathrm{fm} = 0.125\,\mathrm{GeV}^{-1}$
- We consider the ranges
  - Set I: 0.353 $\mathrm{GeV}^{-1} \leq r \leq$ 0.499 GeV^{-1}; 8 points
  - Set II: 0.353 $\mathrm{GeV}^{-1} \leq r \leq$ 0.612 GeV^{-1}; 17 points
  - Set III: 0.353 $\mathrm{GeV}^{-1} \leq r \leq$ 0.8002 GeV^{-1}; 31 points
  - Set IV: 0.353 $\mathrm{GeV}^{-1} \leq r \leq$ 1 GeV^{-1}; 50 points
- $r_{\mathrm{ref}} = 0.353\mathrm{GeV}^{-1}$
- The central values for the soft and ultrasoft scales are
  - $\nu_{\mathrm{s}} = 1/r$
  - $\nu_{\mathrm{us}} = \frac{C_A \alpha(\nu_{\mathrm{s}})}{2r}$

Details on the fits II

We will explore the following orders

- LL/LO

$$\mathcal{F}_{\rm LO}(r) = F(r, \nu_{\rm us} = \nu_{\rm s})\Big|_{\rm LO\,in\,\alpha(\nu_s)}$$

- NLL/NLO

$$\mathcal{F}_{\rm NLO}(r) = F(r, \nu_{\rm us} = \nu_{\rm s})\Big|_{\rm NLO\,in\,\alpha(\nu_s)}$$

- $\mathrm{N^2LL}$

$$\mathcal{F}_{\rm N^2LL}(r) = F(r, \nu_{\rm us} = \nu_{\rm s})\Big|_{\rm N^2LO\,in\,\alpha(\nu_s)} + \frac{d}{dr}\delta V_{\rm RG}(r, \nu_{\rm s}, \nu_{\rm us})\Big|_{\rm N^2LL}$$

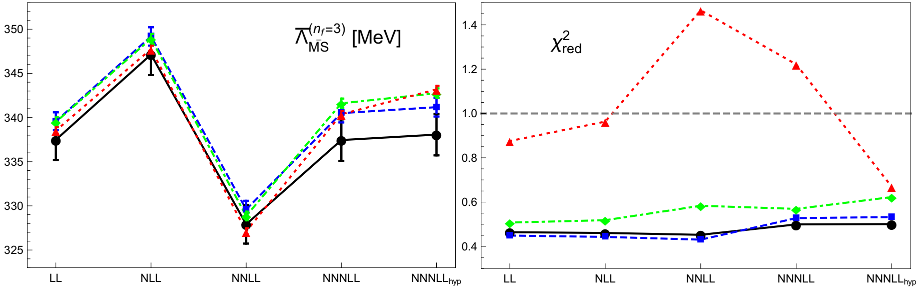
- $\mathrm{N^3LL}$

$$\mathcal{F}_{\rm N^3LL}(r) = F(r, \nu_{\rm us} = \nu_{\rm s})\Big|_{\rm N^3LO\,in\,\alpha(\nu_s)} + \frac{d}{dr}\delta V_{\rm RG}(r, \nu_{\rm s}, \nu_{\rm us})\Big|_{\rm N^3LL} + \frac{d}{dr}\delta E_{\rm us}(r, \nu_{\rm us})\Big|_{\rm LO\,in\,\alpha(\nu_{us})}$$

- $\mathrm{N^3LL_{hyp}}$

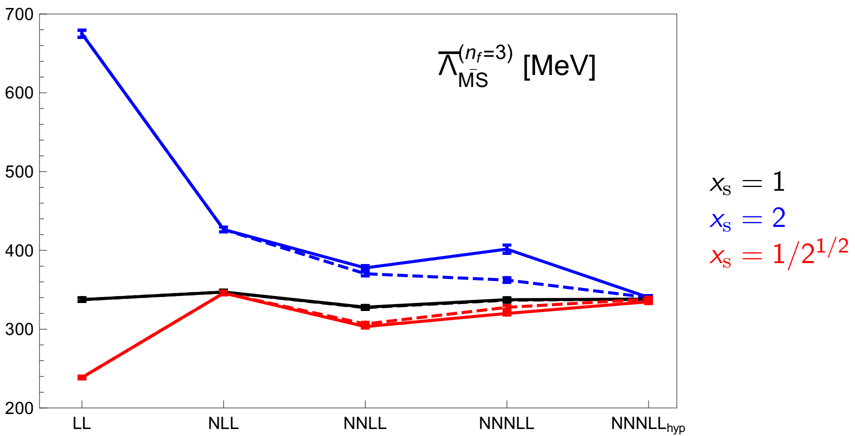
$$\begin{aligned}\mathcal{F}_{\rm N^3LL}(r) =& F(r, \nu_{\rm us} = \nu_{\rm s})\Big|_{\rm N^3LO\,in\,\alpha(\nu_s)} + T(d=3, N_{\rm P}=3, \nu_{\rm s}) + \frac{d}{dr}\delta V_{\rm RG}(r, \nu_{\rm s}, \nu_{\rm us})\Big|_{\rm N^3LL} \\ &+ \frac{d}{dr}\delta E_{\rm us}(r, \nu_{\rm us})\Big|_{\rm LO\,in\,\alpha(\nu_{us})} - T(d=3, N_{\rm P}=0, \nu_{\rm us})\end{aligned}$$

Results



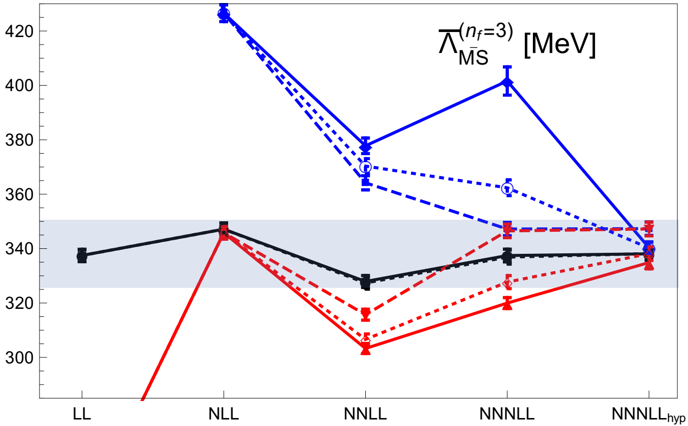
- $r \in [0.353, 0.499] \times \mathrm{GeV}^{-1}$
- $r \in [0.353, 0.612] \times \mathrm{GeV}^{-1}$
- $r \in [0.353, 0.8002] \times \mathrm{GeV}^{-1}$
- $r \in [0.353, 1] \times \mathrm{GeV}^{-1}$

Variations on $\nu_{\rm s} \equiv x_{\rm s}/r$ and $\nu_{\rm us} \equiv x_{\rm us}\frac{C_A\alpha(\nu_{\rm s})}{2r}$



Continuous $x_{\rm us} = 1$; Dashed $\nu_{\rm us} = 1$ GeV

Variations on $\nu_s \equiv x_s/r$ and $\nu_{us} \equiv x_{us}\frac{C_A\alpha(\nu_s)}{2r}$



$x_s = 1$
$x_s = 2$
$x_s = 1/2^{1/2}$

Continuous $x_{us} = 1$; Dashed $x_{us} = x_s$; Dotted $\nu_{us} = 1$ GeV

Final numbers

$$\text{Set I} \quad \Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 338(2)_{\text{stat}}(10)_{\text{h.o.}}(8)_{r_{\text{ref}}}\ \text{MeV} = 338(12)\ \text{MeV}$$

$$\text{Set II} \quad \Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 341(1)_{\text{stat}}(11)_{\text{h.o.}}(6)_{r_{\text{ref}}}\ \text{MeV} = 341(14)\ \text{MeV}$$

$$\text{Set III} \quad \Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 343(1)_{\text{stat}}(13)_{\text{h.o.}}(7)_{r_{\text{ref}}}\ \text{MeV} = 343(14)\ \text{MeV}$$

$$\text{Set IV} \quad \Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 343(0)_{\text{stat}}(13)_{\text{h.o.}}(9)_{r_{\text{ref}}}\ \text{MeV} = 343(16)\ \text{MeV}$$

Therefore our central value result for the strong coupling

- $\alpha^{(n_f=3)}(M_\tau) = 0.3151(65)$
- $\alpha^{(n_f=5)}(M_z) = 0.1181(8)_{\Lambda_{\overline{\text{MS}}}}(4)_{M_\tau \to M_z} = 0.1181(9)$

Conclusions

- We have obtained the normalization of the $d = 3$ renormalon of V

$$Z_3^F\Big|_{n_f=3} = 0.37(17)$$

- Making use of the hyperasymptotic expansion of principal value Borel sums and $\mathrm{N^3LL}$ resummation we have obtained an estimate of the QCD strong coupling

$$\alpha^{(n_f=5)}(M_z) = 0.1181(9)$$

Formulas I

$$F \equiv \frac{d}{dr}V\,, \qquad F = \sum_{n=0}^{\infty} f_n(r,\nu_s,\nu_{us})\alpha^{n+1}(\nu_s)$$

$$\begin{aligned}
f_0 &= \frac{C_F}{r^2}\,, \qquad f_1 = \frac{C_F}{4\pi r^2}\left\{a_1(r,\nu_s) - 2\beta_0\right\} \\
f_2 &= \frac{C_F}{(4\pi)^2 r^2}\left\{a_2(r,\nu_s) - 4a_1(r,\nu_s)\beta_0 - 2\beta_1\right\} \\
f_3 &= \frac{C_F}{(4\pi)^3 r^2}\left\{a_3(r,\nu_s,\nu_{us}) - 6a_2(r,\nu_s)\beta_0 - 4a_1(r,\nu_s)\beta_1 - 2\beta_2 - \frac{16}{3}C_A^3\pi^2\right\}
\end{aligned}$$

$$\begin{aligned}
a_1(r,\nu_s) &= a_1 + 2\beta_0\log\left(\nu_s e^{\gamma_E} r\right) \\
a_2(r,\nu_s) &= a_2 + \frac{\pi^2}{3}\beta_0^2 + (4a_1\beta_0 + 2\beta_1)\log\left(\nu_s e^{\gamma_E} r\right) + 4\beta_0^2\,\log^2\left(\nu_s e^{\gamma_E} r\right) \\
a_3(r,\nu_s,\nu_{us}) &= a_3 + a_1\beta_0^2\pi^2 + \frac{5\pi^2}{6}\beta_0\beta_1 + 16\zeta_3\beta_0^3 \\
&\quad + \left(2\pi^2\beta_0^3 + 6a_2\beta_0 + 4a_1\beta_1 + 2\beta_2\right)\log\left(\nu_s e^{\gamma_E} r\right) + \frac{16}{3}C_A^3\pi^2\log\left(\nu_{us} e^{\gamma_E} r\right) \\
&\quad + \left(12a_1\beta_0^2 + 10\beta_0\beta_1\right)\log^2\left(\nu_s e^{\gamma_E} r\right) + 8\beta_0^3\log^3\left(\nu_s e^{\gamma_E} r\right)
\end{aligned}$$

Formulas II

$$a_1 = \frac{31 C_A - 20 T_F n_f}{9}$$

$$a_2 = \frac{400 n_f^2 T_F^2}{81} - C_F n_f T_F \left(\frac{55}{3} - 16\zeta(3)\right) + C_A^2 \left(\frac{4343}{162} + \frac{16\pi^2 - \pi^4}{4} + \frac{22\zeta(3)}{3}\right) - C_A n_f T_F \left(\frac{1798}{81} + \frac{56\zeta(3)}{3}\right)$$

$$a_3 = a_3^{(3)} n_f^3 + a_3^{(2)} n_f^2 + a_3^{(1)} n_f + a_3^{(0)}$$

Formulas III

$$a_3^{(3)} = -\left(\frac{20}{9}\right)^3 T_F^3$$

$$a_3^{(2)} = \left(\frac{12541}{243} + \frac{368\zeta(3)}{3} + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} - \frac{416\zeta(3)}{3}\right) C_F T_F^2$$

$$a_3^{(1)} = \frac{d_F^{abcd} d_F^{abcd}}{N_A}\left\{\pi^2\left(\frac{1264}{9} - \frac{976}{3}\zeta(3) + \log(2)[64 + 672\zeta(3)]\right) + \pi^4\left(-\frac{184}{3} + \frac{32}{3}\log(2) - 32\log^2(2)\right) + \frac{10\pi^6}{3}\right\}$$
$$+ T_F\left\{C_F^2\left(\frac{286}{9} + \frac{296}{3}\zeta(3) - 160\zeta(5)\right) + C_A C_F\left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5)\right)\right.$$
$$+ C_A^2\left(-\frac{58747}{486} + \pi^2\left[\frac{17}{27} - 32\theta_4 + \log(2)\left\{-\frac{4}{3} - 14\zeta(3)\right\} - \frac{19}{3}\zeta(3)\right] - 356\zeta(3)\right.$$
$$\left.\left. + \pi^4\left[-\frac{157}{54} - \frac{5}{9}\log(2) + \log^2(2)\right] + \frac{1091}{6}\zeta(5) + \frac{57}{2}\zeta^2(3) + \frac{761}{2520}\pi^6 - 48y_6\right)\right\}$$

$$a_3^{(0)} = \frac{d_F^{abcd} d_A^{abcd}}{N_A}\left\{\pi^2\left(\frac{7432}{9} - 4736\theta_4 + \log(2)\left[\frac{14752}{3} - 3472\zeta(3)\right] - \frac{6616}{3}\zeta(3)\right)\right.$$
$$\left. + \pi^4\left(-156 + \frac{560}{3}\log(2) + \frac{496}{3}\log^2(2)\right) + \frac{1511\pi^6}{45}\right\} + C_A^3\left\{\frac{385645}{2916} + \pi^2\left(-\frac{953}{54} + \frac{584}{3}\theta_4 + \frac{175}{2}\zeta(3)\right.\right.$$
$$\left. + \log(2)\left[-\frac{922}{9} + \frac{217}{3}\zeta(3)\right]\right) + \frac{584}{3}\zeta(3) + \pi^4\left(\frac{1349}{270} - \frac{20}{9}\log(2) - \frac{40}{9}\log^2(2)\right) - \frac{1927}{6}\zeta(5) - \frac{143}{2}\zeta^2(3)$$
$$\left. - \frac{4621\pi^6}{3024} + 144y_6\right\}$$

Formulas IV

$$\frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{18 - 6N_c^2 + N_c^4}{96 N_c^2}$$

$$\frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N_c(N_c^2 + 6)}{48}$$

$$\theta_n = \mathrm{Li}_n(1/2) + \frac{(-\log(2))^n}{n!}$$

$$y_6 = \zeta(-5, -1) + \zeta(6)$$

Where $\zeta(z)$ is the Riemann zeta function, $\zeta(z_1, z_2)$ is the multiple zeta function, and $\mathrm{Li}_n(z)$ is the polylogarithm.

Formulas V

$$\frac{d}{dr}\delta V_{\mathrm{RG}}(r,\nu_s,\nu_{us})\Big|_{\mathrm{N^2LL}} = -\frac{1}{6\beta_0}C_F C_A^3 \frac{1}{r^2}\alpha^3(\nu_s)\log\left(\frac{\alpha(\nu_{us})}{\alpha(\nu_s)}\right)$$

$$\frac{d}{dr}\delta V_{\mathrm{RG}}(r,\nu_s,\nu_{us})\Big|_{\mathrm{N^3LL}} = C_F C_A^3 \frac{1}{r^2}\alpha^3(\nu_s)\Big\{ -\frac{1}{6\beta_0}\log\left(\frac{\alpha(\nu_{us})}{\alpha(\nu_s)}\right)$$

$$+\frac{\pi}{4\beta_0}K\{\alpha(\nu_{us})-\alpha(\nu_s)\} + \frac{1}{8\pi}\left(2-\frac{1}{\beta_0}\left[a_1 + 2\beta_0\log(r\nu_s e^{\gamma_E})\right]\right)\alpha(\nu_s)\log\left(\frac{\alpha(\nu_{us})}{\alpha(\nu_s)}\right)\Big\}$$

$$K \equiv \frac{8}{3}\frac{\beta_1}{\beta_0}\frac{1}{(4\pi)^2} - \frac{1}{27\pi^2}\left(C_A\left(47+6\pi^2\right) - 10T_F n_f\right)$$

$$L \equiv \log\frac{\nu_{\mathrm{us}}}{\nu_{\mathrm{s}}}$$

$$\mathrm{N^2LL}: \alpha^{3+n}(\nu_{\mathrm{s}})L^n \sim \alpha^{3+n}(\nu_{\mathrm{s}})\log^n(\alpha(\nu_{\mathrm{s}})) \sim \alpha^3(\nu_{\mathrm{s}}) \text{ if } \alpha(\nu_{\mathrm{s}})L \sim 1$$

$$\mathrm{N^3LL}: \alpha^{4+n}(\nu_{\mathrm{s}})L^n \sim \alpha^{4+n}(\nu_{\mathrm{s}})\log^n(\alpha(\nu_{\mathrm{s}})) \sim \alpha^4(\nu_{\mathrm{s}}) \text{ if } \alpha(\nu_{\mathrm{s}})L \sim 1$$

$$\frac{d}{dr}\delta E_{us}\Big|_{\mathrm{LO\ in\ }\alpha(\nu_{\mathrm{us}})} = C_F C_A^3 \frac{1}{12\pi r^2}\alpha^3(\nu_s)\alpha(\nu_{us})\Big\{\log\left(\frac{C_A\alpha(\nu_s)}{2r\nu_{us}}\right) + \log(2) + 1/6\Big\}$$

Formulas VI

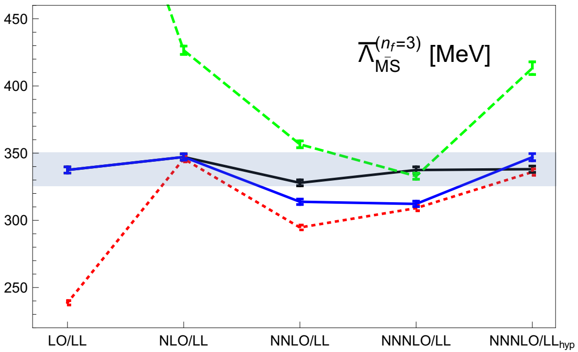
$$T = \sqrt{\alpha(\mu)} K^{(\mathrm{P})} r \mu^3 e^{-\frac{6\pi}{\beta_0 \alpha(\mu)}} \left(\frac{\beta_0 \alpha(\mu)}{4\pi} \right)^{-3b} \left(1 + \bar{K}_1^{(\mathrm{P})} \alpha(\mu) + \mathcal{O}\left(\alpha^2(\mu)\right) \right)$$

where $\eta_c = -3b + \frac{6\pi c}{\beta_0} - 1$; $b = \frac{\beta_1}{2\beta_0^2}$ and

$$K^{(\mathrm{P})} = -\frac{Z_3^F 2^{1-3b} \pi 3^{3b+1/2}}{\Gamma(1+3b)} \beta_0^{-1/2} \left[-\eta_c + \frac{1}{3} \right]$$

$$\bar{K}_1^{(\mathrm{P})} = \frac{\beta_0/(3\pi)}{-\eta_c + \frac{1}{3}} \left[-3b_1 b \left(\frac{1}{2}\eta_c + \frac{1}{3} \right) - \frac{1}{12}\eta_c^3 + \frac{1}{24}\eta_c - \frac{1}{1080} \right]$$

Fixed order fits $\nu_{\rm us} = \nu_{\rm s}$



- Central value fits with log resummation
- $\nu_{\rm s} = \nu_{\rm us} = 1/r$
- $\nu_{\rm s} = \nu_{\rm us} = 2/r$
- $\nu_{\rm s} = \nu_{\rm us} = 1/(\sqrt{2}r)$