Strong coupling constant from moments of quarkonium correlators revisited

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**A Virtual Tribute to
Quark Confinement and the Hadron Spectrum 2021**
08/05/2021

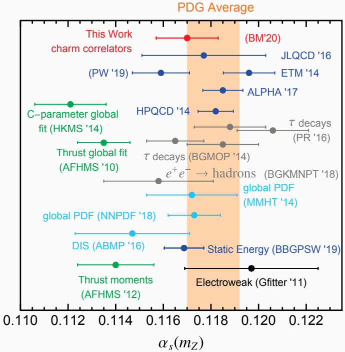
Y. Maezawa, P. Petreczky: PR D94 (2016)

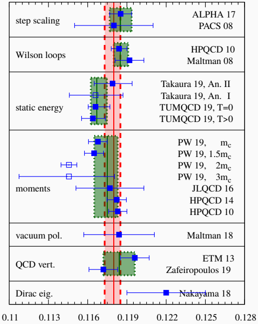
P. Petreczky, JHW: PR D100 (2019) no.3 & arXiv:2012.06193

J. Komijani, P. Petreczky, JHW: PPNP 113 (2020)

Outline

Lattice determinations of α_s in context





source: Boito, Mateu, PL B806 (2020) 135482 source: *JK, PP, JHW, PPNP 113 (2020)*

- Spread hints at **underestimated systematic uncertainties**
- I.e., **moments subaverage** of $\alpha_s(M_Z)$ in PPNP 113 (2020):

  $\alpha_s(M_Z) = 0.1175^{+08}_{-10}$ @ $\chi^2/\text{d.o.f.} = {}^{5.24}/_{4}$ via PW 19 $m_h < 2m_c$.
- Lattice QCD dominates the global average and error

⇒ urgent need for **reexamining errors of lattice calculations**

Conceptual idea of lattice determinations of α_s

- We sacrifice a few hadronic observables on the lattice to determine the quark masses and set the lattice scale (ultimately through f_π)
- We compute hadronic observables $O(\nu)$ on the lattice **at sufficiently high scales ν** for the weak-coupling approach to be applicable

Window problem: $^1/_L \ll \Lambda_{\mathrm{QCD}} \ll \nu \ll {}^1/_a$ in practice difficult

- We separate any non-perturbative physics through an **operator product expansion** (OPE) – suppressed by powers of $^\Lambda/_\nu$
- Applicability of OPE does not depend on continuum limit
- We compare continuum extrapolated lattice results for $O(\nu)$ to perturbative continuum results in $\overline{\mathbf{MS}}$ scheme to determine parameters

Conceptual idea of lattice determinations of α_s

The time moments of (pseudoscalar) quarkonium correlators (2008-2021)

- The scale is set by the quark mass, $\nu = m_h$ where $m_h \gtrsim m_c$
- Conceptually similar to non-lattice methods: *sum-rules*
- Large quark masses cause large discretization errors $\sim (am_h)^n$
    - need calculations at $am_h \sim 1$
    - continuum limit: challenging!

Bibliography α_s from quarkonium correlators

- Karlsruhe group[1] using experimental data on $R(s)$ (2001-2008)
- Boito et al.[2] using experimental data on $R(s)$ (2019-2020)

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[1] Kühn, Steinhauser, Nucl. Phys. B619 (2001) 588
Kühn, Steinhauser, JHEP 10 (2002) 018
Kühn et al., Nucl. Phys. B778 (2007) 192
Sturm, JHEP 0809 (2008) 075

[2] Boito, Mateu, arXiv:2001.11041

[3] Allison et al., Phys. Rev. D78 (2008) 054513
McNeile et al., Phys. Rev. D82 (2010) 034512
Chakraborty et al., Phys. Rev. D91 (2015) 5, 054508

[4] Nakayama et al., Phys. Rev. D94 (2016) 054507

[5] Maezawa, Petreczky, Phys. Rev. D94 (2016) 3, 034507
Petreczky, JHW: Phys. Rev. D100 (2019) 3, 034519; arXiv:2012.06193

[6] Bazavov et al., *in preparation*

Bibliography α_s from quarkonium correlators

- Karlsruhe group[1] using experimental data on $R(s)$ (2001-2008)
- Boito et al.[2] using experimental data on $R(s)$ (2019-2020)
- HPQCD collaboration[3] using 3 or 4 sea quark flavors (2008-2015)
- JLQCD collaboration[4] using 3 sea quark flavors (2016)
- Petreczky et al.[5] using 3 sea quark flavors (2016-2021)
- MILC/FNAL collaboration[6] using 4 sea quark flavors (ongoing)

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[1] Kühn, Steinhauser, Nucl. Phys. B619 (2001) 588
Kühn, Steinhauser, JHEP 10 (2002) 018
Kühn et al., Nucl. Phys. B778 (2007) 192
Sturm, JHEP 0809 (2008) 075

[2] Boito, Mateu, arXiv:2001.11041

[3] Allison et al., Phys. Rev. D78 (2008) 054513
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[4] Nakayama et al., Phys. Rev. D94 (2016) 054507

[5] Maezawa, Petreczky, Phys. Rev. D94 (2016) 3, 034507
Petreczky, JHW: Phys. Rev. D100 (2019) 3, 034519; arXiv:2012.06193

[6] Bazavov et al., *in preparation*

Lattice setup and heavy quark parameters

$\beta = 10/g_0^2$	$\frac{m_\ell}{m_s}$	$N_\sigma^3 \times N_\tau$	a^{-1} GeV	L_σ fm	am_{c0}	am_{b0}
6.740	0.05	48^4	1.81	5.2	0.5633(10)	
6.880	0.05	48^4	2.07	4.6	0.4800(10)	
7.030	0.05	48^4	2.39	4.0	0.4047(9)	
7.150	0.05	$48^3 \times 64$	2.67	3.5	0.3547(9)	
7.280	0.05	$48^3 \times 64$	3.01	3.1	0.3086(13)	
7.373	0.05	$48^3 \times 64$	3.28	2.9	0.2793(5)	
7.596	0.05	64^4	4.00	3.2	0.2220(2)	1.019(8)
7.825	0.05	64^4	4.89	2.6	0.1775(3)	0.7985(5)
7.030	0.20	48^4	2.39	4.0	0.4047(9)	
7.825	0.20	64^4	4.89	2.6	0.1775(3)	0.7985(5)
8.000	0.20	64^4	5.58	2.3	0.1495(6)	0.6710(6)
8.200	0.20	64^4	6.62	1.9	0.1227(3)	0.5519(6)
8.400	0.20	64^4	7.85	1.6	0.1019(27)	0.4578(6)

[HotQCD] [PR D90 094503]

[Bazavov et al.] [PR D97 014510]

- Valence HISQ, $m_{c,b}$ tuned via $\eta_{c,b}$, at $m_{h0} = 1, 1.5, 2, 3, 4\ m_{c0}$ (or m_{b0})
- Pseudoscalar meson operator $j_5(x) = \bar{\psi}(x)\gamma_5\psi(x)$ and RGI correlator

$$G(\tau) = a^8 m_{h0}^2 \sum_{\boldsymbol{x}} \langle j_5(\boldsymbol{x},\tau) j_5(0,0)\rangle_U \quad \lim_{\tau\to 0} \quad \left(\frac{a}{\tau}\right)^4$$

- Time moments finite for $n \geq 4$, defined on periodic lattice as

$$G_n = \sum_{\tau/a=1}^{N_\tau/2} \left(\frac{\tau}{a}\right)^n [G(\tau) + G(aN_\tau - \tau)]$$

- Stat. errors, sea quark effects $\lesssim$ finite volume or tuning errors

Quarkonium moments with HISQ action

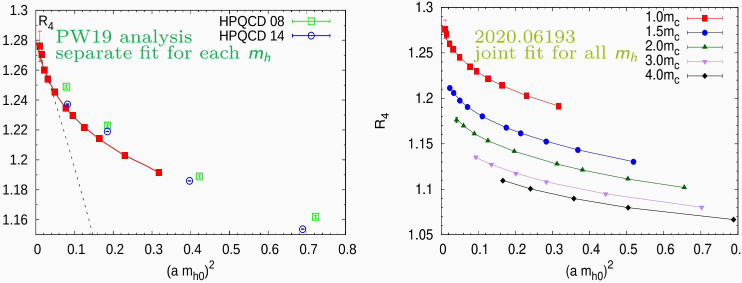
- Fit correlators, then extrapolate to larger N_τ, then compute moments
- Use random color wall sources – statistical errors become irrelevant
- Fluctuations and mass dependence reduced in ratios $G_n^{\frac{1}{n-4}} / G_{n+2}^{\frac{1}{n-2}}$
- Artifacts $\sim \alpha_s^0(a)^n$ cancel in reduced moments $R_n = \left(\frac{G_n^{QCD}}{G_n^0}\right)^{\frac{1}{n-4}}$
- Artifacts are worse in lower moments ($\tau \sim a$) and for larger masses
- Finite size effects are worse in higher moments ($\tau \sim aN_\tau$) and for free theory moments G_n^0 ("quark-antiquark" scattering states, not hadrons)
- Split R_n via OPE: **high energy** $\sim m_h$ or **low energy** $\sim \Lambda_{\mathrm{QCD}}$ parts

$$R_n(a, m_h, \alpha_s, \Lambda_{\mathrm{QCD}}) = \sum_{i=1}^{\infty} \alpha_s^i R_{n,i}^{\mathrm{pert}}((am_h)^2) \quad \to \text{QCD coupling}$$

$$+ \sum_{j=1}^{\infty} \underbrace{\left(\frac{\Lambda_{\mathrm{QCD}}}{m_h}\right)^{2j}}_{\ll 1} R_{n,j}^{\mathrm{np}}((am_h)^2, (a\Lambda_{\mathrm{QCD}})^2)$$

- So far, **no sensitivity** to artifacts of **low energy contribution**

Approach to the continuum limit



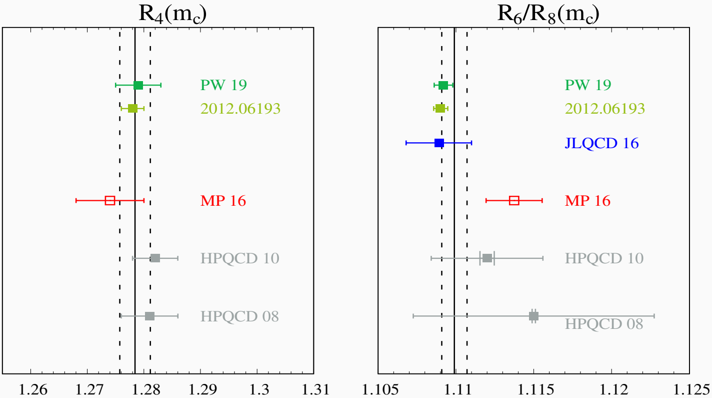
- Extrapolate R_4 (or ${}^{R_n}/m_{h0}$, $n \geq 6$) to continuum via **truncated Ansatz** (R_4 @ $N = 2, M_1 = 8, M_2 = 6$ for $(am_{h0})^2 \leq 0.7$, ${}^{R_n}/m_{h0}$ @ $N = 1, M_1 = 2$)

$$R_4^{\text{lat}}(\alpha_s, a, m_h) = R_4^{\text{cont}}(m_h) + \sum_{i=1}^{N} \sum_{j=1}^{M_i} b_{ij} \; (\alpha_s^{\text{lat}})^i (am_h)^{2j},$$

$$\frac{R_n^{\text{lat}}(\alpha_s, a, m_h)}{m_{h0}} = \left(\frac{R_n(m_h)}{m_{h0}}\right)^{\text{cont}} + \sum_{i=1}^{N} \sum_{j=1}^{M_i} c_{ij}^{(n)} \; (\alpha_s^{\text{lat}})^i (am_h)^{2j}, \quad n \geq 6.$$

- **Boosted coupling** $\alpha_s^{\text{lat}} = 10/(4\pi\beta u_0^4) \sim \log(a\Lambda)$, with $u_0^4 = \langle \text{tr } U_\square \rangle / 3$

Nonperturbative continuum results at the charm scale



- Differences with MP16: due to oversimplied continuum extrapolation
- Continuum R_4 (and $^{R_6}/_{R_8}$) at $m_h \leq 1.5m_c$: unchanged, reduced errors
- Continuum $^{R_6}/_{R_8}(m_h)$ only reliable at $m_h \leq 1.5m_c$; $m_h = 2m_c$ consistent
- Continuum R_4 (and $^{R_8}/_{R_{10}}$) for $m_h \geq 2m_c$: significant changes
- Continuum $^{R_8}/_{R_{10}}(m_c)$ cannot be obtained reliably due to severe finite volume effects (need $m_{h0}L \geq 23$), but feasible for $m_h \geq 1.5m_c$

Reduced quarkonium moments in perturbation theory

- We compare to the known weak-coupling result[7] at order α_s^3

$$R_n = \begin{cases} r_4 & (n=4) \\ r_n \cdot \frac{m_{h0}}{m_h} & (n \geq 6) \end{cases} \quad , \qquad r_n = 1 + \sum_{j=1}^{3} r_{nj} \left(\log \frac{\nu}{m_h(\nu_m)} \right) \alpha_s^j(\nu)$$

- **Intuitive** choice $\nu = \nu_m = m_h$; $\nu_m = m_h$ but varying is ν necessary[8];
- We estimate the uncertainty due to the truncation of the perturbative series with an α_s^4 term, whose coefficient is varied in the range $\pm 5 r_{n3}$
- **Low-energy physics $\sim \Lambda_{\mathrm{QCD}}$ via local operators**

⇒ leading low-energy contribution due to the gluon condensate[9]

- We determine $\alpha_s(m_h)$ from the nonlinear equations

$$R_4(\alpha_s(m_h)) = 1 + \sum_{j=1}^{3} r_{4,j}\,(m_h, 1)\,\alpha_s^j(m_h) + \frac{1}{m_h^4}\frac{11}{4}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle, \quad \text{etc.} \ ,$$

using the gluon condensate $\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = -0.006(12)\,\mathrm{GeV}^4$ from τ decays[10]

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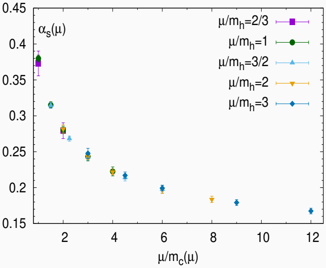
[7] Sturm, JHEP 0809 (2008) 075
Kiyo et al., Nucl. Phys. B 823, 269 (2009)
Maier et al., Nucl. Phys. B 824, 1 (2010)

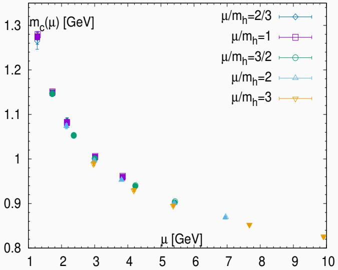
[8] Boito, Mateu, JHEP 03 (2020) 094

[9] Broadhurst et al., Phys. Lett. B **329**, 103 (1994)

[10] Geshkenbein et al., Phys. Rev. D **64**, 093009 (2001)

Running of the coupling and the charm mass

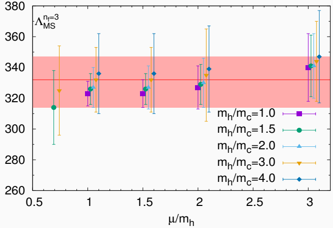




source:PP,JHW, 2012.06193

- We obtain the moment $R_4(m_h)$ at different values of the quark mass m_h
- We fit $R_4(m_h)$ as function of $\alpha_s(\nu)$ with different $n = {}^{\nu}/_{m_h}$
- Use $\alpha_s(\nu)$ in perturbative expr. to obtain $m_h(\nu)$ from ${}^{R_n(m_h)}/_{m_{h0}}, n \geq 6$
- $m_c(\nu)$ at ${}^{\nu}/_{m_h} = 3$ is 2σ lower – hint of underestimated truncation error?
- Combine $\alpha_s(\nu)$ and $m_c(\nu)$ for each set of data/analysis to obtain $\Lambda^{N_f=3}_{\overline{\mathrm{MS}}}$

The Lambda parameter in α_s at the Z mass from the moments



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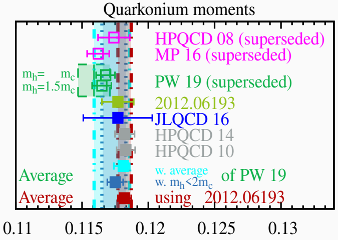


adapted from: JK, PP, JHW PPNP 113 (2020)

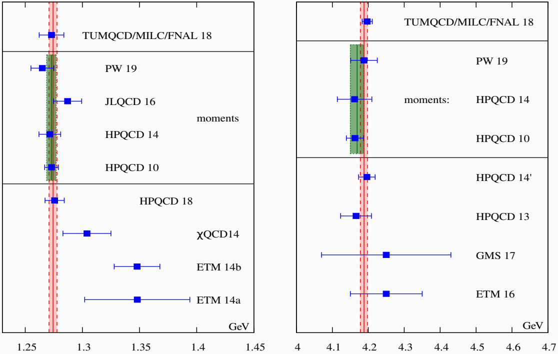
- Averaging methodology does not matter much; we obtain

$$\Lambda_{\overline{\rm MS}}^{N_f=3} = 332 \pm 17 \pm 2\,(\text{scale})\,\text{MeV}, \quad \alpha_s(M_Z) = 0.1177(12),$$

  matching to 4 or 5 flavors at 1.5 resp. 4.8 GeV in RunDec or at $\nu \in m_c(m_c), \ldots, 2\,\text{GeV}$ contributes 0.0003 to the error of $\alpha_s(M_Z)$
- Moments subaverage of $\alpha_s(M_Z)$ revised using 2012.06193:

$\alpha_s(M_Z) = 0.1182(5)$ @ $\chi^2/\text{d.o.f.} = {}^{0.23}/_{3}$ believing errors of HPQCD.

Heavy-quark masses from the moments on the lattice



source: JK, PP, JHW PPNP 113 (2020)

- Heavy-quark masses in good agreement (except m_c from ETM)

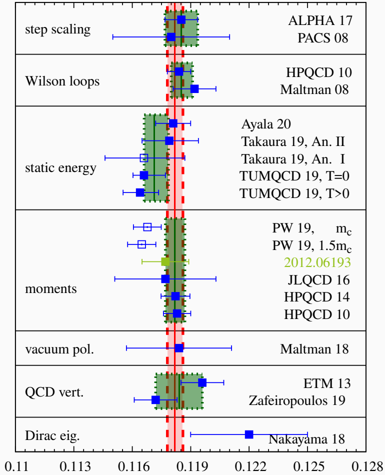
$$m_c(m_c, N_f = 4) = 1.2729(42) \quad @ \quad \chi^2/\text{d.o.f.} = {}^{1.98}/_{3} \quad \text{(moments)},$$

$$m_c(m_c, N_f = 4) = 1.2743(35) \quad @ \quad \chi^2/\text{d.o.f.} = {}^{2.16}/_{3} \quad \text{(lat. avg.)},$$

$$m_b(m_b, N_f = 5) = 4.169(19) \quad @ \quad \chi^2/\text{d.o.f.} = {}^{0.34}/_{2} \quad \text{(moments)},$$

$$m_b(m_b, N_f = 5) = 4.188(10) \quad @ \quad \chi^2/\text{d.o.f.} = {}^{2.30}/_{5} \quad \text{(lat. avg.)}$$

QCD coupling from the lattice



adapted from: JK, PP, JHW PPNP 113 (2020)

- Continuum limit for use in α_s determination is challenging
- Bottleneck for heavy-quark obs. is perturbative truncation error

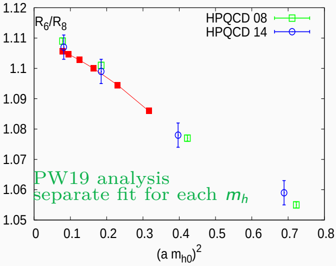
→ need $\mathrm{N}^{4,5}\mathrm{LO}$ results to improve

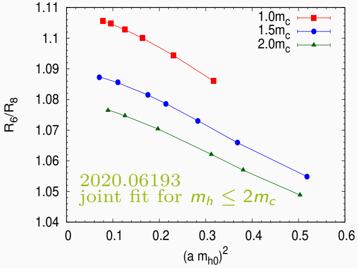
- α_s from lattice fully consistent
- If we believe errors of HPQCD:

$\alpha_s(M_Z) = 0.1182(4)$ @ $\chi^2/\text{d.o.f.} = {}^{4.92}/_{6}$

Thank you for your attention!

Continuum extrapolation of ${}^{R_6}/_{R_8}$



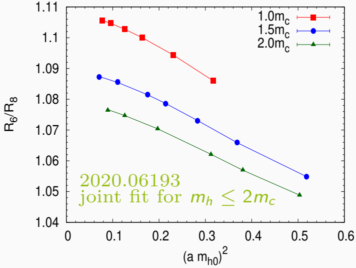


- Extrapolate ${}^{R_6}/_{R_8}$ using a **truncated Ansatz** $(N = 2, M_1 = M_2 = 3, 4)$

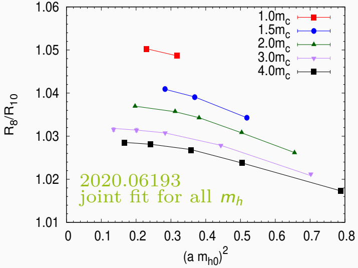
$$ {}^{R_n}/_{R_{n+2}}{}^{\mathrm{lat}}(\alpha_s, a, m_h) = {}^{R_n}/_{R_{n+2}}{}^{\mathrm{cont}}(m_h) + \sum_{i=1}^{N} \sum_{j=1}^{M_i} d_{ij}^{(n)} \; (\alpha_s^{\mathrm{lat}})^i (am_h)^{2j} $$

- Must exclude on fine lattices due to **severe finite volume effects**
- Need high order polynomials in $(am_{h0})^2$, slope decreases for larger m_h
- We **cannot constrain** ${}^{R_6}/_{R_8}{}^{\mathrm{cont}}(2m_c)$ – fixed parameter in fit function via 4-loop result using $\alpha_s(2m_c)$ from $R_4(2m_c)$ → **consistency check**!

Continuum extrapolation of R_8/R_{10}



2020.06193
joint fit for $m_h \leq 2m_c$



2020.06193
joint fit for all m_h

- Extrapolate R_8/R_{10} using a **truncated Ansatz** ($N = 2, M_1 = 4, M_2 = 3$ for $(am_{h0})^2 \leq 0.8$)

$$R_n/R_{n+2}{}^{\text{lat}}(\alpha_s, a, m_h) = R_n/R_{n+2}{}^{\text{cont}}(m_h) + \sum_{i=1}^{N} \sum_{j=1}^{M_i} d_{ij}^{(n)} \, (\alpha_s^{\text{lat}})^i (am_h)^{2j}$$

- Continuum $R_8/R_{10}(m_c)$ cannot be obtained reliably due to severe finite volume effects (need $m_{h0}L \geq 23$), but feasible for $m_h \geq 1.5m_c$
- We **cannot constrain** $R_8/R_{10}{}^{\text{cont}}(m_c)$ – fixed parameter in fit function via 4-loop result using $\alpha_s(m_c)$ from $R_4(m_c)$ → **consistency check**!

α_s from R_4

m_h	R_4	R_6/R_8	R_8/R_{10}	R_6/m_{h0}	R_8/m_{h0}	R_{10}/m_{h0}
$1.0m_c$	1.279(4)	1.1092(6)	1.0485(8)	1.0195(20)	0.9174(20)	0.8787(50)
$1.5m_c$	1.228(2)	1.0895(11)	1.0403(10)	0.7203(35)	0.6586(16)	0.6324(13)
$2.0m_c$	1.194(2)	1.0791(7)	1.0353(5)	0.5584(35)	0.5156(17)	0.4972(17)
$3.0m_c$	1.158(6)	1.0693(10)	1.0302(5)	0.3916(23)	0.3647(19)	0.3527(20)
$4.0m_c$				0.3055(23)	0.2859(12)	0.2771(23)
m_b				0.2733(17)	0.2567(17)	0.2499(16)

$\frac{m_h}{m_c}$	R_4	R_6/R_8	R_8/R_{10}	av.	$\Lambda^{N_f=3}_{\text{QCD}}$ MeV
1.0	0.3815(55)(30)(22)	0.3837(25)(180)(40)	0.3550(63)(140)(88)	0.3782(65)	314(10)
1.5	0.3119(28)(4)(4)	0.3073(42)(63)(7)	0.2954(75)(60)(17)	0.3099(48)	310(10)
2.0	0.2651(28)(7)(1)	0.2689(26)(35)(2)	0.2587(37)(34)(6)	0.2648(29)	284(8)
3.0	0.2155(83)(3)(1)	0.2338(35)(19)(1)	0.2215(367)(17)(1)	0.2303(150)	284(48)

m_h	R_4	R_6/m_{c0}	R_8/m_{c0}	R_{10}/m_{c0}	$\alpha_s(m_h)$
$1.0m_c$	1.2778(20)	1.0200(16)	0.9166(17)	0.8719(21)	0.3798(28)(31)(22)
$1.5m_c$	1.2303(30)	1.0792(20)	0.9860(20)	0.9462(23)	0.3151(43)(14)(4)
$2.0m_c$	1.2051(37)	1.1182(23)	1.0317(23)	0.9944(26)	0.2804(51)(9)(1)
$3.0m_c$	1.1782(44)	1.1729(27)	1.0923(26)	1.0574(31)	0.2434(61)(5)(0)
$4.0m_c$	1.1631(45)	1.2098(31)	1.1321(30)	1.0985(31)	0.2226(62)(4)(0)

m_h/m_c	$\mu/m_h = 2/3$	$\mu/m_h = 1$	$\mu/m_h = 3/2$	$\mu/m_h = 2$	$\mu/m_h = 3$
1.0		323(4)(6)(3)	323(4)(7)(3)	327(4)(13)(3)	340(4)(21)(3)
1.5	314(8)(23)(1)	326(9)(4)(1)	326(8)(5)(1)	329(8)(10)(1)	341(9)(18)(1)
2.0		327(13)(3)(0)	327(13)(4)(0)	330(13)(9)(0)	341(14)(16)(0)
3.0	325(20)(20)(0)	332(21)(2)(0)	332(21)(4)(0)	335(22)(22)(0)	344(22)(14)(0)
4.0		336(26)(2)(0)	336(26)(3)(0)	339(27)(7)(0)	347(28)(17)(0)

α_s from ratios of moments

m_h	R_4	$R_6\,/\,R_8$	$R_8\,/\,R_{10}$	$R_6\,/\,m_{h0}$	$R_8\,/\,m_{h0}$	$R_{10}\,/\,m_{h0}$
$1.0m_c$	1.279(4)	1.1092(6)	1.0485(8)	1.0195(20)	0.9174(20)	0.8787(50)
$1.5m_c$	1.228(2)	1.0895(11)	1.0403(10)	0.7203(35)	0.6586(16)	0.6324(13)
$2.0m_c$	1.194(2)	1.0791(7)	1.0353(5)	0.5584(35)	0.5156(17)	0.4972(17)
$3.0m_c$	1.158(6)	1.0693(10)	1.0302(5)	0.3916(23)	0.3647(19)	0.3527(20)
$4.0m_c$				0.3055(23)	0.2859(12)	0.2771(23)
m_b				0.2733(17)	0.2567(17)	0.2499(16)

$\frac{m_h}{m_c}$	R_4	$R_6\,/\,R_8$	$R_8\,/\,R_{10}$	av.	$\Lambda_{\text{QCD}}^{N_f=3}$ MeV
1.0	0.3815(55)(30)(22)	0.3837(25)(180)(40)	0.3550(63)(140)(88)	0.3782(65)	314(10)
1.5	0.3119(28)(4)(4)	0.3073(42)(63)(7)	0.2954(75)(60)(17)	0.3099(48)	310(10)
2.0	0.2651(28)(7)(1)	0.2689(26)(35)(2)	0.2587(37)(34)(6)	0.2648(29)	284(8)
3.0	0.2155(83)(3)(1)	0.2338(35)(19)(1)	0.2215(367)(17)(1)	0.2303(150)	284(48)

m_h	R_4	$R_6\,/\,m_{c0}$	$R_8\,/\,m_{c0}$	$R_{10}\,/\,m_{c0}$	$\alpha_s(m_h)$
$1.0m_c$	1.2778(20)	1.0200(16)	0.9166(17)	0.8719(21)	0.3798(28)(31)(22)
$1.5m_c$	1.2303(30)	1.0792(20)	0.9860(20)	0.9462(23)	0.3151(43)(14)(4)
$2.0m_c$	1.2051(37)	1.1182(23)	1.0317(23)	0.9944(26)	0.2804(51)(9)(1)
$3.0m_c$	1.1782(44)	1.1729(27)	1.0923(26)	1.0574(31)	0.2434(61)(5)(0)
$4.0m_c$	1.1631(45)	1.2098(31)	1.1321(30)	1.0985(31)	0.2226(62)(4)(0)

$R_6\,/\,R_8$			$R_8\,/\,R_{10}$		
$m_h\,/\,m_c$	continuum	$\alpha_s(m_h)$	$m_h\,/\,m_c$	continuum	$\alpha_s(m_h)$
1.0	1.10895(32)	0.3826(14)(178)(39)	1.0	-	-
1.5	1.09100(25)	0.3137(10)(76)(8)	1.5	1.04310(45)	0.3166(34)(82)(17)
2.0	-	-	2.0	1.03830(68)	0.2808(51)(50)(4)
3.0	-	-	3.0	1.03249(94)	0.2382(69)(24)(1)
4.0	-	-	4.0	1.02987(106)	0.2191(293)(17)(0)

Heavy quark masses m_h from higher moments

$\frac{m_h}{m_c}$	R_6/m_{h0} [GeV]	R_8/m_{h0} [GeV]	R_{10}/m_{h0} [GeV]
1.0	1.2740(25)(17)(11)(61)	1.2783(28)(23)(00)(43)	1.2700(72)(46)(13)(33)
1.5	1.7147(83)(11)(03)(60)	1.7204(42)(14)(00)(40)	1.7192(35)(29)(04)(30)
2.0	2.1412(134)(07)(01)(44)	2.1512(71)(10)(00)(29)	2.1531(74)(19)(02)(21)
3.0	2.9788(175)(06)(00)(319)	2.9940(156)(08)(00)(201)	3.0016(170)(16)(00)(143)
4.0	3.7770(284)(06)(00)(109)	3.7934(159)(08)(00)(68)	3.8025(152)(15)(00)(47)
$\frac{m_b}{m_c}$	4.1888(260)(05)(00)(111)	4.2045(280)(07)(00)(69)	4.2023(270)(14)(00)(47)

- Four errors of m_h due to the continuum-extrapolated lattice data, truncation of the perturbative series, the gluon condensate, and $\alpha_s(m_h)$
- The error due to the lattice scale r_1 is not included in the table
- Continuum limit of $R_{6,8,10}/m_{h0}$ unproblematic for all m_h
- For $m_h > 3m_c$: unweighted average of Λ_{QCD}, then use 4-loop running to obtain $\alpha_s(4m_c)$ and $\alpha_s(m_b)$, matching to 4 or 5 flavors at 1.5 or 4.7 GeV

$$m_c(m_c, N_f = 4) = 1.265(10)\,\mathrm{GeV}, \quad m_b(m_b, N_f = 5) = 4.188(37)\,\mathrm{GeV}$$