exploring hadron dynamics in the large $N_c$ limit
based on:

- J.J. Baeza Ballesteros, P. Hernández, F. Romero-López, in preparation [see JBB’s talk @ Latt21]


- E. Endress, CP, Exploring the role of the charm quark in the Δl=1/2 rule, PRD 90 (2014) 094504.


motivation & plan

- lattice simulations can be used as a lab to explore quark mass, $N_f$, volume dependence of hadron dynamics

- ditto: number of colours

$\Rightarrow$ explore $N_c$ dependence in problems where qualitative/quantitative insight can be expected
motivation & plan

- lattice simulations can be used as a lab to explore quark mass, $N_f$, volume dependence of hadron dynamics

- ditto: number of colours

⇒ explore $N_c$ dependence in problems where qualitative/quantitative insight can be expected

- original motivation: non-leptonic kaon decay, especially $\Delta I=1/2$ rule

- spinoffs (current focus): meson properties (mass, decay constants) and interactions
QCD @ large $N_c$

varying-$N_c$ lattice simulations have become a mature tool

- **Yang-Mills**
  - string tension, glueball masses
  - topological susceptibility
  - Wilson-flow coupling and factorisation

- **quenched QCD**
  - meson masses, decay constants
  - kaon weak decay matrix elements

- **dynamical**
  - $N_f=2$: meson+baryon masses, topological susceptibility, finite $T$
  - $N_f=4$: kaon weak decay matrix elements, ChiPT LECs, meson interactions

- also: reduced models

[cf. review by Hernández, Romero-López 2012.03331]
[Bali et al. 2013]
[Cè, García Vera, Giusti, Schaefer 2016]
[García Vera, Sommer 2019]
[García Vera, Sommer 2019]
[DeGrand, Liu 2017; DeGrand 2021]
[this talk]
[this talk]
[cf. review by García Pérez, Lattice 2019]
QCD @ large $N_c$

't Hooft's large $N_c$ limit of QCD: $N_c \to \infty \big|_{\lambda = g^2 N_c = \text{fixed}}$

à la 't Hooft proper: $\frac{N_f}{N_c} \to 0 \Rightarrow \ m_{\eta'}^2 = m_\pi^2$, $U(N_f)$ chiral symmetry

à la Veneziano: $\frac{N_f}{N_c} = \text{const} \Rightarrow \ m_{\eta'}^2 \gg m_\pi^2$, $SU(N_f)$ chiral symmetry

[‘t Hooft 1974]

[Veneziano 1979] [also: Witten 1979]
QCD @ large $N_c$

't Hooft’s large $N_c$ limit of QCD: $N_c \to \infty |_{\lambda = g^2 N_c = \text{fixed}}$

à la 't Hooft proper: $\frac{N_f}{N_c} \to 0 \Rightarrow m_{\eta'}^2 = m_{\pi}^2$, $U(N_f)$ chiral symmetry

à la Veneziano: $\frac{N_f}{N_c} = \text{const} \Rightarrow m_{\eta'}^2 \gg m_{\pi}^2$, $SU(N_f)$ chiral symmetry

- preserves asymptotic freedom
- captures most non-perturbative properties (confinement, chiral SSB, ...)
- simplifies the theory by suppressing dynamical quark effects

\[
\frac{\mu}{d\mu} = - \left( \frac{11}{3} - \frac{2 N_f}{3 N_c} \right) \frac{\lambda^2}{8 \pi^2} \to \Lambda_{\infty}
\]

\[
\sim g^2 N_c^2 \sim N_c
\]

\[
\sim g^2 N_c \sim 1
\]
**lattice setup**

- simulate for $N_c=3,...,6$ (+7,8,17 quenched) at fixed lattice spacing (+1 at $N_c=3$), change quark mass along $m_u=m_d=m_s=m_c$
  
  - **quenched**: use line of constant physics provided by quenched study of meson physics
  
  - **dynamical**: use gradient flow scale $t_0$ to set constant physics

- use Wilson fermions for sea (HiRep code), **twisted-mass QCD** for valence
  
  - twisted valence à la Frezzotti-Rossi allows to avoid mixing with wrong-chirality operators
  
  - mixed-action approach requires matching of valence and sea, performed with meson mass
  
  - check for residual cutoff effects by changing value of $c_{sw}$ + simulation on finer lattice
  
  - use perturbative renormalization and running (non-perturbative results unavailable)

- develop/check necessary bits of SU(4) $\chi$PT

---

[Bursa et al. 2013] [Frezzotti, Rossi 2004] [Constantinou et al. 2011; Alexandrou et al. 2012] [Ciuchini et al. 1998; Buras et al. 2000]
Goldstone boson physics is well-parametrized by Chiral Perturbation Theory

\[ F_\pi = F \left\{ 1 + \frac{M_\pi^2}{F_\pi^2} [4L_5(\mu) + 4N_fL_4(\mu)] + \frac{N_f}{2} \frac{M_\pi^2}{(4\pi F_\pi)^2} \log \left( \frac{M_\pi^2}{\mu} \right) \right\} \]

\[ F_\pi^2 = \mathcal{O}(N_c) \]
\[ L_5 = \mathcal{O}(N_c) \]
\[ L_4 = \mathcal{O}(1) \]

\[ F_\pi \xrightarrow{N_c \to \infty} F \left\{ 1 + 4 \frac{M_\pi^2}{F_\pi^2} L_5 + \text{logs} \right\} \]

[Gasser, Leutwyler 1985]
$N_c$ scaling of $\chi$PT LECs

\[ F_\pi = F \left\{ 1 + \frac{M_\pi^2}{F_\pi^2} 4L_F + \frac{N_f}{2} \frac{M_\pi^2}{(4\pi F_\pi)^2} \log \left( \frac{M_\pi^2}{\mu^2} \right) \right\} \]

\[ F = \sqrt{N_c} \left( F^{(0)} + \frac{F^{(1)}}{N_c} \right) \quad \quad L_F = N_c L_F^{(0)} + L_F^{(1)} \]
Motivation

Ensembles

M \overset{\text{c}}{\longrightarrow} \mathcal{F} \overset{\text{Scattering}}{\longrightarrow} \mathcal{K} \overset{\text{Scattering}}{\longrightarrow} \mathcal{F} \overset{\text{c}}{\longrightarrow} \mathcal{M} \overset{\text{c}}{\longrightarrow} \mathcal{F} \overset{\text{c}}{\longrightarrow} \mathcal{M}

Summary

$N_c$ scaling of \chiPT LECs

\[
M^2 = 2Bm \left\{ 1 + \frac{M^2}{F^2} 8L_M + \frac{1}{N_f} \frac{M^2}{(4\pi F)^2} \log \left( \frac{M^2}{\mu^2} \right) \right\}
\]

\[
B = B^{(0)} + \frac{B^{(1)}}{N_c}
\]

\[
L_M = N_cL_M^{(0)} + L_M^{(1)}
\]
$N_c$ scaling of ChPT LECs

- **LO LECs:**

$$\frac{F}{\sqrt{N_c}} = \left[ 67(3) - 26(4) \frac{N_f}{N_c} \right] \text{MeV} \Rightarrow F_{N_f=2} = 86(3) \text{ MeV} \quad F_{N_f=3} = 71(3) \text{ MeV}$$

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>Ref.</th>
<th>Value (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ETM 15A</td>
<td>[386]</td>
<td>86.3(2.8)</td>
</tr>
<tr>
<td>Engel 14</td>
<td>[50]</td>
<td>85.8(0.7)(2.0)</td>
</tr>
<tr>
<td>Brandt 13</td>
<td>[49]</td>
<td>84(8)(2)</td>
</tr>
<tr>
<td>QCDSF 13</td>
<td>[402]</td>
<td>86(1)</td>
</tr>
<tr>
<td>TWQCD 11</td>
<td>[394]</td>
<td>83.39(35)(38)</td>
</tr>
<tr>
<td>ETM 09C</td>
<td>[48]</td>
<td>85.91(07)(^{+78}_{-107})</td>
</tr>
<tr>
<td>ETM 08</td>
<td>[53]</td>
<td>86.6(7)(7)</td>
</tr>
<tr>
<td>Hasenfratz 08</td>
<td>[397]</td>
<td>90(4)</td>
</tr>
<tr>
<td>JLQCD/TWQCD 08A</td>
<td>[376]</td>
<td>79.0(2.5)(0.7)(^{+4.2}_{-0.0})</td>
</tr>
<tr>
<td>JLQCD/TWQCD 07</td>
<td>[398]</td>
<td>87.3(5.6)</td>
</tr>
<tr>
<td>Colangelo 03</td>
<td>[403]</td>
<td>86.2(5)</td>
</tr>
</tbody>
</table>

[FLAG 2019]

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>Ref.</th>
<th>Value (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MILC 10</td>
<td>[36]</td>
<td>80.3(2.5)(5.4)</td>
</tr>
<tr>
<td>MILC 09A</td>
<td>[17]</td>
<td>78.3(1.4)(2.9)</td>
</tr>
<tr>
<td>MILC 09</td>
<td>[129]</td>
<td></td>
</tr>
<tr>
<td>PACS-CS 08</td>
<td>[162]</td>
<td>83.8(6.4)</td>
</tr>
<tr>
<td>RBC/UKQCD 08</td>
<td>[163]</td>
<td>66.1(5.2)</td>
</tr>
</tbody>
</table>
$N_c$ scaling of $\chi$PT LECs

- **LO LECs:**

  \[
  \frac{F}{\sqrt{N_c}} = \left[ 67(3) - 26(4) \frac{N_f}{N_c} \right] \text{ MeV} \quad \Rightarrow \quad F_{N_f=2} = 86(3) \text{ MeV} \quad F_{N_f=3} = 71(3) \text{ MeV}
  \]

  - $\Sigma_{N_f=3}^{1/3} = 223(9) \text{ MeV}$ vs $\Sigma_{N_f=3}^{1/3} = 214(6)(24) \text{ MeV}$ [Fukaya et al. 2010]

  - $\frac{\Sigma_{N_f=3}}{\Sigma_{N_f=2}} = 1.49(10) \quad \text{vs} \quad \frac{\Sigma_{N_f=3}}{\Sigma_{N_f=2}} = 1.51(11)$ [Bernard, Descotes-Genon, Toucas 2012]

- **NLO LECs:**

  - $\bar{\ell}_4 = 5.1(3) \quad \text{vs} \quad \bar{\ell}_4 = 4.40(28)$ [FLAG 2019]

  - n.b. subleading corrections to LECs are sizable: $L_{N_f=4}^{N_f=4} = 10^3 \times -0.2(2) + \frac{2.9(6)}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right)$
weak decay and $\Delta I=1/2$

**$K^0_s$**

\[ I(J^P) = \frac{1}{2}(0^-) \]

Mean life $\tau = (0.8954 \pm 0.0004) \times 10^{-10} \text{ s}$ ($S = 1.1$) Assuming CPT

Mean life $\tau = (0.89564 \pm 0.00033) \times 10^{-10} \text{ s}$ Not assuming CPT

$K^0_s$ DECAY MODES

<table>
<thead>
<tr>
<th>Decay Mode</th>
<th>Fraction ($\Gamma_I/\Gamma$)</th>
<th>Scale factor/ρ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0\pi^0$</td>
<td>(30.69 ± 0.05) %</td>
<td>209</td>
</tr>
<tr>
<td>$\pi^+\pi^-$</td>
<td>(69.20 ± 0.05) %</td>
<td>206</td>
</tr>
</tbody>
</table>

**$K^0_L$**

\[ I(J^P) = \frac{1}{2}(0^-) \]

\[ m_{K_L} - m_{K_S} = (0.5293 \pm 0.0009) \times 10^{10} \text{ s}^{-1} \] ($S = 1.3$) Assuming CPT

\[ = (3.484 \pm 0.006) \times 10^{-12} \text{ MeV} \] Assuming CPT

\[ = (0.5289 \pm 0.0010) \times 10^{10} \text{ s}^{-1} \] Not assuming CPT

Mean life $\tau = (5.116 \pm 0.021) \times 10^{-8} \text{ s}$ ($S = 1.1$)

$K^0_L$ DECAY MODES

<table>
<thead>
<tr>
<th>Decay Mode</th>
<th>Fraction ($\Gamma_I/\Gamma$)</th>
<th>Scale factor/ρ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0\nu_e$</td>
<td>(40.55 ± 0.11 ) %</td>
<td>229</td>
</tr>
<tr>
<td>$\pi^+\mu^+\nu_\mu$</td>
<td>(27.04 ± 0.07 ) %</td>
<td>216</td>
</tr>
</tbody>
</table>

Semileptonic modes, including Charge conjugation × Parity Violating (CPV) modes

<table>
<thead>
<tr>
<th>Decay Mode</th>
<th>Fraction ($\Gamma_I/\Gamma$)</th>
<th>Scale factor/ρ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3\pi^0$</td>
<td>(19.52 ± 0.12 ) %</td>
<td>139</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>(12.54 ± 0.05 ) %</td>
<td>133</td>
</tr>
<tr>
<td>$\pi^+\pi^-$</td>
<td>CPV [a]</td>
<td>( 1.967 ± 0.010) × $10^{-3}$</td>
</tr>
<tr>
<td>$\pi^0\pi^0$</td>
<td>CPV</td>
<td>( 8.64 ± 0.06 ) × $10^{-4}$</td>
</tr>
</tbody>
</table>

\[-iT[K^0 \rightarrow (\pi\pi)_I] = A_I e^{i\delta_I}\]

\[ T[(\pi\pi)_I \rightarrow (\pi\pi)_I]_{l=0} = 2e^{i\delta_I} \sin \delta_I \]

\[ \left| \frac{A_0}{A_2} \right| = 22.45(6) \]

(similar observations in baryon sector — e.g., $\Lambda/\Sigma \rightarrow N\pi$, heavy meson decay, ...)

[fully?] satisfactory understanding of result within SM lacking for almost 50 years
weak decay and $\Delta l=1/2$

CP-violation effects neglected ($\frac{V_{td}V_{ts}^*}{V_{ud}V_{us}^*} \sim 10^{-3}$), keep active charm quark:

$$\mathcal{H}_w^{\text{eff}} = \frac{g_w^2}{2M_W^2} V_{us}^* V_{ud} \sum_{\sigma=\pm} \left\{ k_1^\sigma Q_1^\sigma + k_2^\sigma Q_2^\sigma \right\}$$

$$Q_1^{\pm} = (\bar{s}_L \gamma_\mu u_L)(\bar{u}_L \gamma_\mu d_L) \pm (\bar{s}_L \gamma_\mu u_L)(\bar{u}_L \gamma_\mu d_L) - [u \leftrightarrow c]$$

$$Q_2^{\pm} = (m_u^2 - m_c^2) \left\{ m_d (\bar{s}_L d_R) + m_s (\bar{s}_R d_L) \right\}$$

(do not contribute to physical $K \to \pi\pi$ transitions)

(penguin contributions cancel in GIM limit $m_c = m_u$)
weak decay and $\Delta l = 1/2$

CP-violation effects neglected ($\frac{V_{td}V_{ts}^*}{V_{ud}V_{us}^*} \sim 10^{-3}$), keep active charm quark:

$$\mathcal{H}_{\text{eff}}^w = \frac{g_w^2}{2M_W^2} V_{us}^* V_{ud} \sum_{\sigma = \pm}\left\{ k_1^\sigma Q_1^\sigma + k_2^\sigma Q_2^\sigma \right\}$$

$$Q_1^\pm = (\bar{s}_L\gamma_\mu u_L)(\bar{u}_L\gamma_\mu d_L) \pm (\bar{s}_L\gamma_\mu d_L)(\bar{u}_L\gamma_\mu u_L) - [u \leftrightarrow c]$$

$$Q_2^\pm = (m_u^2 - m_c^2) \left\{ m_d(\bar{s}_L d_R) + m_s(\bar{s}_R d_L) \right\}$$

$$\left| A_0 \right| = \frac{k_1^- (M_W)}{k_1^+ (M_W)} \frac{\langle (\pi\pi)_{I=0} | Q_1^- | K \rangle}{\langle (\pi\pi)_{I=2} | Q_1^+ | K \rangle} \quad \frac{k_1^- (M_W)}{k_1^+ (M_W)} \approx 2.8$$

- bulk of effect should come from long-distance QCD contribution
  
  - reliable non-perturbative determination mandatory

[Gaillard, Lee; Altarelli, Maiani 1974]

[Cabibbo, Martinelli, Petronzio; Brower, Maturana, Gavela, Gupta 1984]
weak decay and $\Delta l=1/2$

approximate methods/effective theory

- spectacular failure of naive $1/N_c$ expansion

\[ T[K^0 \to \pi^0\pi^0] \sim 0 \quad \Rightarrow \quad \frac{|A_0|}{A_2} \mid_{N \to \infty} \sim \sqrt{2} \]

[Fukugita et al. 1977]
[Chivukula, Flynn, Georgi 1986]

- elaborate approaches that combine $1/N_c$, chiral perturbation theory + vector dominance, and quark-hadron duality claim (non-universal) success

[Aebischer, Bobeth, Buras 2020 (⊃ earlier)]
[Gisbert, Pich 2018 (⊃ earlier)]
weak decay and $\Delta l=1/2$

approximate methods/effective theory

- spectacular failure of naive $1/N_c$ expansion

\[
\begin{array}{ccc}
K & W & \pi \\
O(N_c^2) & O(N_c) & O(1)
\end{array}
\]

\[T[K^0 \rightarrow \pi^0\pi^0] \sim 0 \Rightarrow \left| \frac{A_0}{A_2} \right|_{N \rightarrow \infty} \sim \sqrt{2}\]

[Fukugita et al. 1977]
[Chivukula, Flynn, Georgi 1986]

- elaborate approaches that combine $1/N_c$, chiral perturbation theory + vector dominance, and quark-hadron duality claim (non-universal) success

[Aebischer, Bobeth, Buras 2020 (© earlier)]
[Gisbert, Pich 2018 (© earlier)]

lattice QCD

far-reaching effort by RBC/UKQCD collaboration

- Naive factorisation approach: ② $\sim 1/3$ ①
- Our computation: ② $\sim -0.7$ ①

“emerging understanding of the $\Delta l=1/2$ rule”

[Boyle et al., PRL 110 (2013) 152001]
several possible sources for $\Delta I=1/2$ enhancement:

- physics at “intrinsic” QCD scale $\sim \Lambda_{\text{QCD}}$
- physics at charm scale (penguins)
- final state interactions
- all of the above (no dominating “mechanism”)

separate low-energy QCD and charm-scale physics: consider amplitudes as a function of charm mass for fixed u,d,s masses

$$m_c = m_u = m_d = m_s \quad \rightarrow \quad m_c \gg m_u = m_d \leq m_s$$

implementation:

- active charm
- use fermions with good chiral properties (good renormalisation, arbitrarily low masses with GW)
- give up (too expensive) direct computation, use ChiPT ($\Rightarrow$ FSI captured at weak pion coupling only)
anatomy of $\Delta l = 1/2$ (GIM limit)

$g^\pm \propto A^\pm \propto \begin{array}{c}
\begin{array}{c}
\text{Color-disconnected } O(N_c^2) \\
\text{Color-connected } O(N_c)
\end{array}
\end{array}$

\[
\frac{A_0}{A_2} = \frac{1}{2\sqrt{2}} \left(1 + 3 \frac{g^-}{g^+}\right) \text{ Large } N_c \xrightarrow[g^+ = g^-]{\text{g+ = g-}} \sqrt{2}
\]

$m_u = m_d = m_s$ limit: $B_K \propto A_2$
anatomy of $\Delta l=1/2$ (GIM limit)

\[
g^{\pm} \propto A^{\pm} \propto \begin{array}{c}
\text{Color-disconnected } O(N_c^2) \\
\text{Color-connected } O(N_c)
\end{array}
\]

\[
\frac{A_0}{A_2} = \frac{1}{2\sqrt{2}} \left(1 + 3 \frac{g^-}{g^+} \right)
\]

for large $N_c$ and $g^+/g^- \to \sqrt{2}$

\[
m_u = m_d = m_s \text{ limit: } B_K \propto A_2
\]

\[
A^{\pm} = 1 \pm \tilde{a} \frac{1}{N_c} \pm \tilde{b} \frac{N_f}{N_c^2} + \tilde{c} \frac{1}{N_c^2} + \tilde{d} \frac{N_f}{N_c^3} + \cdots
\]
anatomy of $\Delta l=1/2$ (GIM limit)

$$g^\pm \propto A^\pm \propto \begin{array}{c}
\text{Color-disconnected } O(N_c^2) \\
\text{Color-connected } O(N_c)
\end{array} \mp \begin{array}{c}
\text{Color-connected } O(N_c) \\
\text{Color-disconnected } O(N_c^2)
\end{array}$$

$$\frac{A_0}{A_2} = \frac{1}{2\sqrt{2}} \left(1 + 3 \frac{g^-}{g^+}\right) \xrightarrow{\text{Large } N_c} \sqrt{2}$$

$$m_u=m_d=m_s \text{ limit: } B_K \propto A_2$$

$$A^\pm = 1 \pm \tilde{a} \frac{1}{N_c} \pm \tilde{b} \frac{N_f}{N_c^2} + \tilde{c} \frac{1}{N_c^2} + \tilde{d} \frac{N_f}{N_c^3} + \cdots$$
anatomy of $\Delta l=1/2$ (GIM limit)

$$g^\pm \propto A^\pm \propto \begin{array}{c}
\text{Color-connected } O(N_c^2) \\
\text{Color-disconnected } O(N_c)
\end{array} \implies \begin{array}{c}
\frac{A_0}{A_2} = \frac{1}{2\sqrt{2}} \left(1 + 3 \frac{g^-}{g^+}\right) \text{ Large } N_c \rightarrow \sqrt{2}
\end{array}$$

$m_u=m_d=m_s$ limit: $B_K \propto A_2$

\[ A^\pm = g^\pm \left[1 \mp 3 \left(\frac{M_\pi}{4\pi F_\pi}\right)^2 \left(\log \frac{M_\pi}{\mu^2} + L_\pm(\mu)\right)\right] \]
anatomy of $\Delta l=1/2$ (GIM limit)

\[
\frac{A^{-} + A^{+}}{2} = 1 + \tilde{c} \frac{1}{N_c^2} + \tilde{d} \frac{N_f}{N_c^3} + \ldots
\]

\[
\frac{A^{-} - A^{+}}{2} = -\tilde{a} \frac{1}{N_c} - \tilde{b} \frac{N_f}{N_c^2} + \ldots
\]

- scaling with $N_c$ confirms expectations, with natural $O(1)$ coefficients
- dynamical quarks enhance effect
- lighter quark masses enhance effect
anatomy of $\Delta l = 1/2$: $B_K$

![Graph showing $\hat{B}_K$ as a function of $1/N_c$ for different cases.

- BBG $N_f = 0$
- BBG $N_f = 4$
- this work $N_f = 0$
- this work $N_f = 4$

Errors are only statistical, but correlations are taken into account. From these results, we obtain for the ratio of couplings $\lambda_{\Delta K}$ at its physical value: }

$$\lambda_{\Delta K} = \frac{\Delta_{\pi^+}}{\Delta_{\pi^-}} \approx 1.1 \pm 0.05$$

$$\lambda_{\Delta K} \approx 1.1 \pm 0.05$$

The complete results of these fits are shown in Tables VI, and the product $\lambda_{\Delta K}$ is obtained as in Ref. [43]. We extract the effective couplings for this theory. This way, all data points with $M > 360$ MeV have been taken into account. On the other hand, to the quasi-physical (QP) situation with $M > 360$ MeV. We have not found results in the literature for the de-excitation of the unknown LECs. For $\Delta M = 0.50$, we define $\hat{B}_K = \frac{\Delta_{\pi^+}}{\Delta_{\pi^-}}$.

The main goal of this work is to compute the ratio $\lambda_{\Delta K}$ using only the data with $M > 360$ MeV. We fit using Eqs. (23) and (27), incorporating the required chiral extrapolation, we follow the same strategy as in Ref. [43]. We extract the effective couplings for this theory. This way, all data points with $M > 360$ MeV have been taken into account.
anatomy of $\Delta l=1/2$: conclusions

\[
\hat{B}_K \bigg|_{\substack{N_f=3 \nn M_K=M_\pi}} = 0.67(2)_{\text{stat}}(6)_{\text{Z}} + (3)_{\text{fit}} \\
\text{Re} \frac{A_0}{A_2} \bigg|_{N_f=4} = 24(5)_{\text{stat}}(4)_{\text{fit}}(5)_{Z\pm}(3)_{\text{NLO}} \\
\text{Re}(A_0)/\text{Re}(A_2) = 22.45(6) \quad [\text{expt}] \\
\text{Re}(A_0)/\text{Re}(A_2) = 19.9(2.3)(4.4) \\
[\text{RBC/UKQCD, PRD 102 (2020) 054509}]
anatomy of $\Delta l=1/2$: conclusions

- $B_K$ displays a large $N_f$ (in GIM limit), chiral dependence

- ratio of decay amplitudes in GIM limit comes very close to the physical value (!)

- handle on $N_f$ dependence in principle allows us to make connection with other physical kinematics, but we are still missing a direct analysis of the $m_c$ dependence

- "mechanism budget":
  - short-distance
  - physics at "intrinsc" QCD scale $\sim \Lambda_{QCD}$
  - physics at charm scale (penguins)
  - final state interactions

\[
\hat{B}_K \bigg|_{N_f=3, M_K=M_\pi} = 0.67(2)_{\text{stat}}(6)_{Z+3}^{(3)_{\text{fit}}}
\]
\[
\text{Re} \left( \frac{A_0}{A_2} \right) \bigg|_{N_f=4} = 24(5)_{\text{stat}}(4)_{\text{fit}}(5)_{Z}^{(3)_{\text{NLO}}}
\]

$\text{Re}(A_0)/\text{Re}(A_2) = 22.45(6)$ [expt]

$\text{Re}(A_0)/\text{Re}(A_2) = 19.9(2.3)(4.4)$

[RBC/UKQCD, PRD 102 (2020) 054509]
conclusions & outlook

- non-leptonic kaon decay remains an open problem... and a fertile ground to learn about strong interaction physics
  - indirect CP violation well under control
  - direct CP violation, isospin enhancement still witness claims of new physics

- lattice toolbox making steady progress
  - controlled quantitative predictions for amplitudes are at hand
  - the anatomy of the effect is ever better understood, pure “low-energy” dynamics seems to play major role in enhancement

- interesting spinoffs: qualitative understanding of meson interactions at low energies

- a theorist’s paradise: field-theory, phenomenology, and computational physics all simultaneously at play!
backup
QCD @ large $N_c$

't Hooft's large $N_c$ limit of QCD: $N_c \to \infty | \lambda = g^2 N_c = \text{fixed}$

à la 't Hooft proper: $\frac{N_f}{N_c} \to 0 \Rightarrow m_{\eta'}^2 = m_{\pi}^2$, $U(N_f)$ chiral symmetry

à la Veneziano: $\frac{N_f}{N_c} = \text{const} \Rightarrow m_{\eta'}^2 \gg m_{\pi}^2$, $SU(N_f)$ chiral symmetry

- preserves asymptotic freedom
- captures most non-perturbative properties (confinement, chiral SSB, ...)
- leads to some \textit{quantitative} non-perturbative predictions!

\[ \langle 0 | A^{a}_\mu (x) A^{b}_\mu (y) |0 \rangle \propto F^2_\pi \sim N_c \]

\[ g_s^2 N_c^2 \propto \mathcal{O}(N_c) \]
QCD @ large $N_c$

't Hooft’s large $N_c$ limit of QCD: $N_c \rightarrow \infty | \lambda = g^2 N_c = \text{fixed}$

à la ’t Hooft proper: $\frac{N_f}{N_c} \rightarrow 0 \Rightarrow m_{\eta'}^2 = m_\pi^2$, $U(N_f)$ chiral symmetry

à la Veneziano: $\frac{N_f}{N_c} = \text{const} \Rightarrow m_{\eta'}^2 \gg m_\pi^2$, $SU(N_f)$ chiral symmetry

- preserves asymptotic freedom
- captures most non-perturbative properties (confinement, chiral SSB, ...)
- leads to some quantitave non-perturbative predictions!

$$\langle 0 \mid \bar{q}q \mid 0 \rangle = -\Sigma \sim N_c$$

$$g_s^2 N_c^2 \propto \mathcal{O}(N_c)$$
lattice setup

**quenched** simulations in $16^3$ lattices at (roughly) constant PS mass [Wilson+Wilson]

<table>
<thead>
<tr>
<th>$N_c$</th>
<th>$T/a$</th>
<th>$\beta$</th>
<th>$am_{\text{PCAC}}$</th>
<th>$am_{\text{PS}}$</th>
<th>$R^+_{\text{bare}}$</th>
<th>$R^-_{\text{bare}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>48</td>
<td>6.0175</td>
<td>-0.002(14)</td>
<td>0.2718(61)</td>
<td>0.774(21)</td>
<td>1.218(31)</td>
</tr>
<tr>
<td>4</td>
<td>48</td>
<td>11.028</td>
<td>-0.0015(11)</td>
<td>0.2637(39)</td>
<td>0.783(15)</td>
<td>1.198(19)</td>
</tr>
<tr>
<td>5</td>
<td>48</td>
<td>17.535</td>
<td>0.0028(9)</td>
<td>0.2655(31)</td>
<td>0.839(8)</td>
<td>1.145(12)</td>
</tr>
<tr>
<td>6</td>
<td>32</td>
<td>25.452</td>
<td>0.0013(7)</td>
<td>0.2676(28)</td>
<td>0.871(6)</td>
<td>1.125(7)</td>
</tr>
<tr>
<td>7</td>
<td>32</td>
<td>34.8343</td>
<td>-0.0034(6)</td>
<td>0.2819(19)</td>
<td>0.880(5)</td>
<td>1.122(5)</td>
</tr>
</tbody>
</table>

renormalisation (RI scheme) at scale around 2 GeV performed using one-loop P.T.

[Constantinou et al. 2011]
[Alexandrou et al. 2012]

**dynamical** simulations at varying PS mass and constant $t_0$ [Iwasaki+Clover]

<table>
<thead>
<tr>
<th>Ensemble</th>
<th>$N_c$</th>
<th>$\beta$</th>
<th>$c_{sw}$</th>
<th>$T \times L$</th>
<th>$am^g_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3A10</td>
<td>3</td>
<td>1.778</td>
<td>1.69</td>
<td>36 $\times$ 20</td>
<td>-0.4040</td>
</tr>
<tr>
<td>3A11</td>
<td></td>
<td></td>
<td></td>
<td>48 $\times$ 24</td>
<td>-0.4040</td>
</tr>
<tr>
<td>3A20</td>
<td>3</td>
<td>1.820</td>
<td>1.66</td>
<td>48 $\times$ 24</td>
<td>-0.4060</td>
</tr>
<tr>
<td>3A30</td>
<td></td>
<td></td>
<td></td>
<td>48 $\times$ 24</td>
<td>-0.4070</td>
</tr>
<tr>
<td>3A40</td>
<td></td>
<td></td>
<td></td>
<td>60 $\times$ 32</td>
<td>-0.4080</td>
</tr>
<tr>
<td>3B10</td>
<td>3</td>
<td>1.820</td>
<td>1.66</td>
<td>48 $\times$ 24</td>
<td>-0.3915</td>
</tr>
<tr>
<td>3B20</td>
<td></td>
<td></td>
<td></td>
<td>60 $\times$ 32</td>
<td>-0.3946</td>
</tr>
<tr>
<td>4A10</td>
<td>4</td>
<td>3.570</td>
<td>1.69</td>
<td>36 $\times$ 20</td>
<td>-0.3725</td>
</tr>
<tr>
<td>4A30</td>
<td></td>
<td></td>
<td></td>
<td>48 $\times$ 24</td>
<td>-0.3760</td>
</tr>
<tr>
<td>4A40</td>
<td></td>
<td></td>
<td></td>
<td>60 $\times$ 32</td>
<td>-0.3780</td>
</tr>
<tr>
<td>5A10</td>
<td>5</td>
<td>5.969</td>
<td>1.69</td>
<td>36 $\times$ 20</td>
<td>-0.3458</td>
</tr>
<tr>
<td>5A30</td>
<td></td>
<td></td>
<td></td>
<td>48 $\times$ 24</td>
<td>-0.3500</td>
</tr>
<tr>
<td>5A40</td>
<td></td>
<td></td>
<td></td>
<td>60 $\times$ 32</td>
<td>-0.3530</td>
</tr>
<tr>
<td>6A10</td>
<td>6</td>
<td>8.974</td>
<td>1.69</td>
<td>36 $\times$ 20</td>
<td>-0.3260</td>
</tr>
<tr>
<td>6A30</td>
<td></td>
<td></td>
<td></td>
<td>48 $\times$ 24</td>
<td>-0.3311</td>
</tr>
<tr>
<td>6A40</td>
<td></td>
<td></td>
<td></td>
<td>60 $\times$ 32</td>
<td>-0.3340</td>
</tr>
</tbody>
</table>

perturbative two-loop RG running in RI to connect to RGIs

[Ciuchini et al. 1998]
[Buras et al. 2000]

+ extra quenched points ($N_c=8,17$)
chiral and finite volume corrections

\[
\text{Re } \left| \frac{A_0}{A_2} \right|_{M_\pi, M_D \to 0, M_K^{\text{phys}}} = \frac{1}{2\sqrt{2}} \left( 1 + 3 \frac{g^-}{g^+} \right) + \frac{17}{12\sqrt{2}} \left( 1 + \frac{1}{17} \frac{g^-}{g^+} \right) \frac{M_K^2}{(4\pi F_K)^2} \log \frac{\Lambda_{\text{eff}}^2}{M_K^2}
\]

[extension to \(N_f=4\) framework of Golterman, Leung PRD 56 (1997) 2950]
chiral and finite volume corrections

\[
\text{Re} \left( \frac{A_0}{A_2} \right)_{M_D \to 0, M_K^{\text{phys}}} = \frac{1}{2\sqrt{2}} \left( 1 + 3 \frac{g^-}{g^+} \right) + \frac{17}{12\sqrt{2}} \left( 1 + \frac{1}{17} \frac{g^-}{g^+} \right) \frac{M_K^2}{(4\pi F_K)^2} \log \frac{\Lambda_{\text{eff}}^2}{M_K^2}
\]

\[
\hat{B}_K^{QP} = \hat{B}_K \bigg|_{M_K = M_\pi} \left[ 1 + \frac{2}{3} \left( \frac{M_K}{4\pi F_K} \right)^2 \log \frac{\Lambda_{\text{eff}}^B}{M_K} \right]
\]

\[
\tilde{R}^\pm (L) = \tilde{R}^\pm \left[ 1 \pm 6\sqrt{2}\pi \xi \frac{e^{-M_\pi L}}{(M_\pi L)^{3/2}} (M_\pi L - 4) \right]
\]

[extension to $N_f=4$ framework of Golterman, Leung PRD 56 (1997) 2950]

[Hernández, Laine 2006]
[Colangelo, Dürr, Haefeli 2005]
Meson observables at large $N_c$

**Long-term goal:** understand QCD at large $N_c$
- Resonances $\rightarrow$ Stable ($\Gamma \sim 1/N_c$)
- Exotic states (tetraquarks?)
- $K \rightarrow (\pi\pi)_{l=0,2}$ \{ Intrinsic QCD effects [Donini, et al. 2020] 
  \begin{align*}
  &\text{Final state interactions} \\
  C_{l=2} & = D - C \\
  C_{AA} & = D + C
\end{align*}

**This work:** $\pi\pi$ scattering at large $N_c$ from lattice simulations
- $N_f = 4$ (u, d, s c) $\rightarrow$ 7 channels (4 with s-wave)
  \[15 \otimes 15 = 84 \oplus 45 \oplus 45 \oplus 20 \oplus 15 \oplus 15 \oplus 1\]
  \[\begin{array}{c}
  \pi^+ \pi^+ \\
  \pi^+ D_s^+ - K^+ D^+
  \end{array}\]
- Match to Chiral Perturbation Theory (ChPT) to constrain Low Energy Coupling (LECs)

[slides from Jorge Baeza-Ballesteros’ talk @ Lattice 2001]
meson scattering

\( \pi \pi \) scattering amplitudes for \( N_f \) flavours are known to NNLO

- [Weinberg 1979]
- [Gasser, Leutwyler 1985]
- [Bijnens, Lu 2011]

\[
M_\pi a_0^{I=2} = - \frac{M_\pi^2}{16\pi F_\pi^2} \left[ 1 - \frac{16M_\pi^2}{F_\pi^2} L_{I=2} + \frac{M_\pi^2}{32\pi^2 F_\pi^2} \left( \frac{13}{4} \ln \frac{M_\pi^2}{\mu^2} - \frac{3}{4} \right) \right]
\]

- \( L_{I=2} = L^{(0)} N_c + L^{(1)}_{I=2} + \ldots \)

\[
M_\pi a_0^{AA} = \frac{M_\pi^2}{16\pi F_\pi^2} \left[ 1 - \frac{16M_\pi^2}{F_\pi^2} L_{AA} - \frac{M_\pi^2}{32\pi^2 F_\pi^2} \left( \frac{21}{4} \ln \frac{M_\pi^2}{\mu^2} + \frac{5}{4} \right) \right]
\]

\[
L_{AA} = L^{(0)} N_c + L^{(1)}_{AA} + \ldots
\]

We have computed \( M_{I=2} \) and \( M_{AA} \) to NNLO in \( U(N_f) \) ChPT

\[
M_\pi a_0^{I=2} = - \frac{M_\pi^2}{16\pi F_\pi^2} \left\{ 1 - \frac{16M_\pi^2}{F_\pi^2} L_{I=2} + N_c^2 K^{(0)}_{I=2} \left( \frac{M_\pi^2}{F_\pi^2} \right)^2 \right. \\
+ \frac{M_\pi^2}{32\pi^2 F_\pi^2} \left[ \frac{15M_\pi^2 - 13M_{\eta'}^2}{4(M_\pi^2 - M_{\eta'}^2)} \ln \frac{M_\pi^2}{\mu^2} + \frac{M_\pi^2 - 3M_{\eta'}^2}{4(M_\pi^2 - M_{\eta'}^2)} \ln \frac{M_{\eta'}^2}{\mu^2} - \frac{1}{2} \right] \left. \right\}
\]

\[
M_\pi a_0^{AA} = \frac{M_\pi^2}{16\pi F_\pi^2} \left\{ 1 - \frac{16M_\pi^2}{F_\pi^2} L_{AA} + N_c^2 K^{(0)}_{AA} \left( \frac{M_\pi^2}{F_\pi^2} \right)^2 \right. \\
- \frac{M_\pi^2}{32\pi^2 F_\pi^2} \left[ \frac{15M_\pi^2 - 21M_{\eta'}^2}{4(M_\pi^2 - M_{\eta'}^2)} \ln \frac{M_\pi^2}{\mu^2} + \frac{M_\pi^2 + 5M_{\eta'}^2}{4(M_\pi^2 - M_{\eta'}^2)} \ln \frac{M_{\eta'}^2}{\mu^2} + \frac{3}{2} \right] \left. \right\}
\]

Large \( N_c \) or \( U(N_f) \) ChPT [Kaiser, Leutwyler 2000]:

- Leutwyler counting scheme

\[
\delta \sim O(m_q) \sim O(M_\pi^2) \sim O(k^2) \sim O(N_c^{-1})
\]

- \( F_\pi \sim \sqrt{N_c} \rightarrow \) Loop diagrams are NNLO

[slides from Jorge Baeza-Ballesteros’ talk @ Lattice 2001]
meson scattering

Finite-volume spectrum:
\[ \delta E_{\pi\pi} = E_{\pi\pi} - 2M_{\pi} \]

Scattering properties:
\[ \mathcal{M}_{\pi\pi}, \ k \cot \delta_0, \ a_0 \ldots \]

Lüscher’s formalism [1986] \[ k \cot \delta_0 = \frac{1}{\pi L} \mathcal{Z} \left( \frac{Lk}{2\pi} \right) \]

Threshold expansion
\[ \delta E_{\pi\pi} = -\frac{4\pi a_0}{M_{\pi} L^3} \left[ 1 + c_1 \left( \frac{a_0}{L} \right) + c_2 \left( \frac{a_0}{L} \right)^2 + c_3 \left( \frac{a_0}{L} \right)^3 + \frac{2\pi r_0 a_0}{L^3} + \frac{\pi a_0}{M_{\pi}^2 L^3} + \ldots \right] \]
\[ \mathcal{O}(L^{-6}) \text{ [Hansen, Sharpe 2017]} \]

\[ \text{Fit range} \]

\[ \text{SU}(4): \ \chi^2/\text{dof} = 1.00 \]
\[ \text{U}(4): \ \chi^2/\text{dof} = 0.94 \]

\[ \text{SU}(4): \ \chi^2/\text{dof} = 2.00 \]
\[ \text{U}(4): \ \chi^2/\text{dof} = 1.42 \]

[slides from Jorge Baeza-Ballesteros’ talk @ Lattice 2001]
meson scattering

Match to ChPT to constrain LECs

SU(4) \[ \frac{L_{l=2}}{N_c} \times 10^3 = -0.11(4) - \frac{1.43(16)}{N_c} \]

U(4) \[ \frac{L_{l=2}}{N_c} \times 10^3 = -0.10(7) - \frac{1.29(16)}{N_c} \]

SU(4): \[ \frac{L_{AA}}{N_c} \times 10^3 = -1.08(13) + \frac{2.2(3)}{N_c} \]

U(4): \[ \frac{L_{AA}}{N_c} \times 10^3 = -0.6(4) + \frac{2.4(3)}{N_c} \]

[slides from Jorge Baeza-Ballesteros’ talk @ Lattice 2001]