A VIRTUAL TRIBUTE TO
QUARK CONFINEMENT
AND THE HADRON SPECTRUM
August 2nd - 6th, 2021 online

Quark free energies and electric fluxes

Milad Ghanbarpour
Lorenz von Smekal

3 August 2021

HFHF Helmholtz Forschungsakademie Hessen für FAIR

Bundesministerium für Bildung und Forschung

CRC-TR 211 Strong-interaction matter under extreme conditions

JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

Outline

- Quark free energies — the puzzle
- Static charges in box
- Heavy-dense QCD
- Flux tube models
- Full QCD — another puzzle
- Summary and Outlook

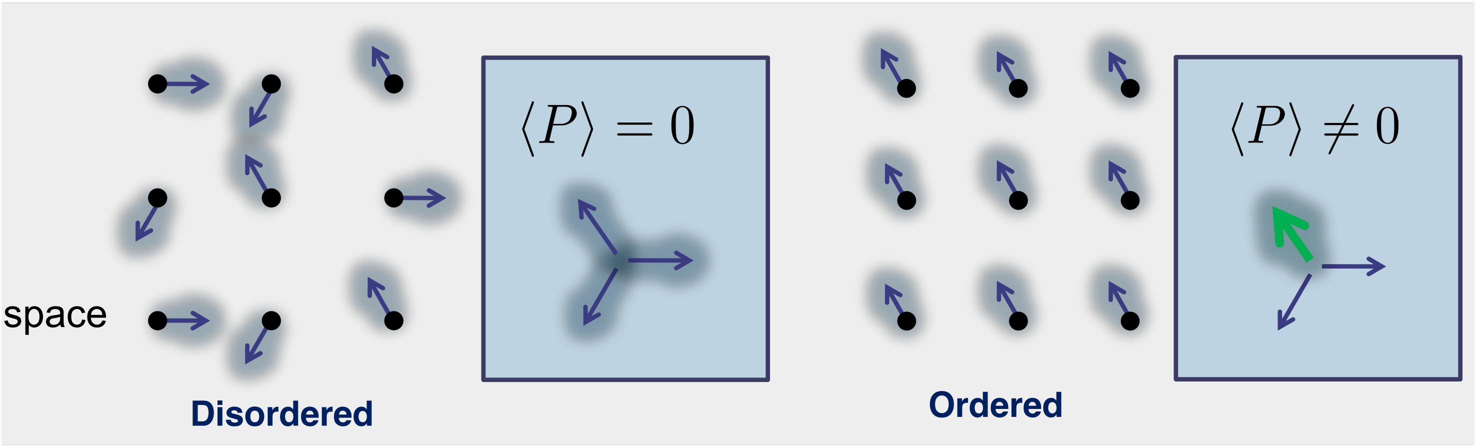
Quark Free Energies

- **Why bother?**

Polyakov loop:

$$P(\vec{x}) = \mathcal{P} \exp\left\{ ig \int_0^{1/T} dt A_0(t,\vec{x}) \right\} \in SU(N_c)$$

$$\langle \mathrm{tr} P(\vec{x}) \rangle_T \sim e^{-F_q/T}$$



Quark Free Energies

- **imaginary chemical potential:** $\mu/T = i\theta$

fugacity expansion

→ Fourier series

$$Z^I(\theta) \equiv Z(T,V,i\theta T) = \sum_{N_q} e^{iN_q\theta}\, Z_c(T,V,N_q)$$

grand canonical at imaginary *μ*

canonical ensembles

- **in QCD, periodicity:**

$$Z^I(\theta + 2\pi/3) = Z^I(\theta)$$

period 2*π*/3

Roberge & Weiss, NPB 275 (1986) 734

→ only every 3rd Fourier coefficient ≠ 0:

$$Z_c(T,V,N_q) = 0\,, \quad \text{for} \quad N_q \neq 0 \mod 3$$

- **Polyakov loop paradox**

Kratochvila & de Forcrand, PRD 73 (2006) 114512

Single Quark in Finite Volume

…we've just changed the temporal b.c.'s without changing the spatial ones!

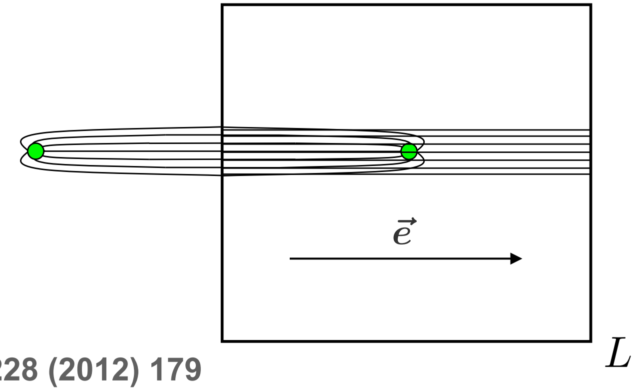
- **change b.c.'s to account for Gauss' law**

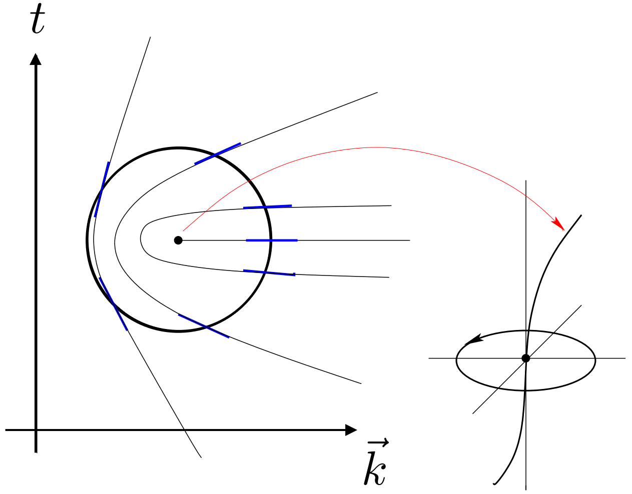
back up — pure *SU*(*N*) gauge theory: (*d*+1)-dim spacetime

$$Z_e(\vec{e}) = \frac{1}{N^d} \sum_{\vec{k}\in\mathbb{Z}_N^d} e^{2\pi i\, \vec{e}\cdot\vec{k}/N}\, Z_k(\vec{k})$$

't Hooft's electric flux ensembles (mirror charges) — dual — 't Hooft's twisted b.c.'s (no of center vortices mod *N*)

$$\frac{Z_e(\vec{e})}{Z_e(0)} = \frac{1}{N}\Big\langle \mathrm{tr}\big(P_\Omega(\vec{x})P_\Omega^\dagger(\vec{x}+\vec{e}L)\big)\Big\rangle_{\text{no-flux}}$$





LvS, NPB (PS) 228 (2012) 179

Heavy-Dense QCD

- **effective Polyakov-loop theory**

(1 flavor Wilson)

$$Z_{\text{eff}} = \int \Big(\prod_i \mathrm{d}L_i \, J(L_i) \, Q(L_i) \Big) \prod_{\langle i,j \rangle} \big(1 + 2\lambda \, \mathrm{Re} L_i L_j^* \big)$$

leading order hopping expansion

static fermion determinat → site factors

Fromm, Langelage, Lottini, Philipsen, JHEP 01 (2012) 042
Langelage, Neuman, Philipsen, JHEP 09 (2014) 131

$$Q(L) = \big(1 + hL + h^2 L^* + h^3\big)^2 \big(1 + \bar{h} L^* + \bar{h}^2 L + \bar{h}^3\big)^2$$

where

$$h(\mu) = e^{(\mu - m)/T}$$

$$\bar{h}(\mu) = h(-\mu)$$

Pietri, Feo, Seiler, Stamatescu, PRD 76 (2007) 114501

- **reduce to Potts model**

maintain leading moments of reduced Haar measure

tabulated in

$$T_{m,n} = \int \mathrm{d}L \, J(L) \, L^m L^{*n}$$

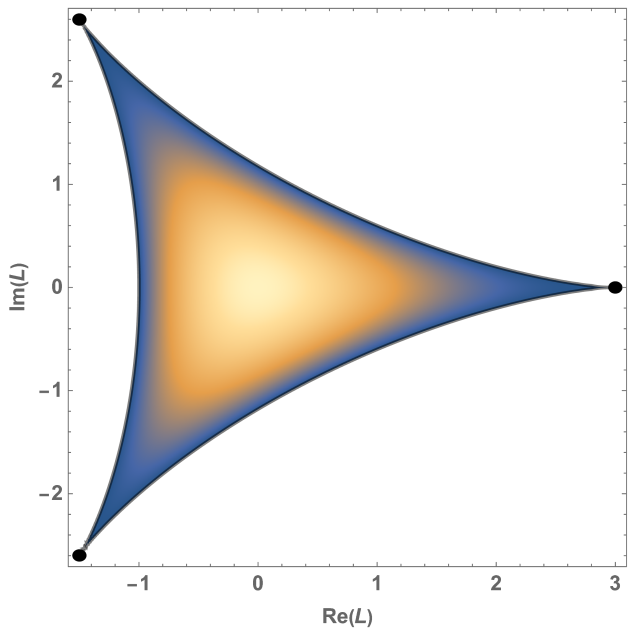
Uhlmann, Meinel, Wipf, J. Phys. A 40 (2007) 4367

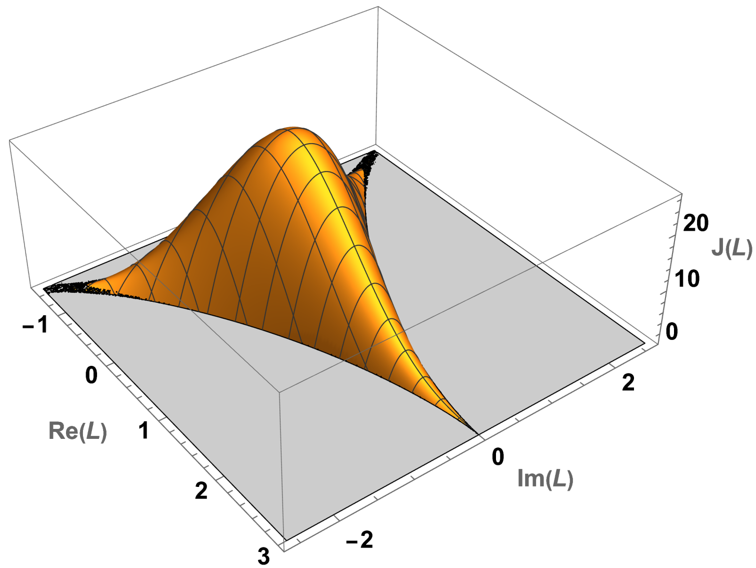
Effective Potts Model

- **reduced Haar measure of SU(3)**

describes Polyakov-loop distributions of pure gauge theory very well up to $T \lesssim T_c$

Smith, Dumitru, Pisarski, LvS, PRD 88 (2013) 054020
Endrodi, Gattringer, Schadler, PRD 89 (2014) 054509





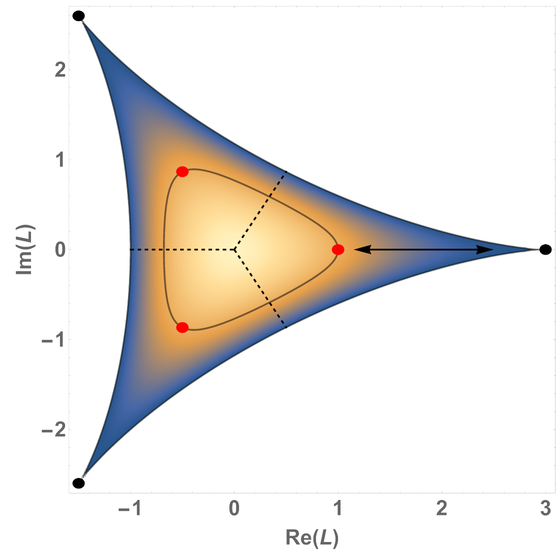
CRC-TR 211

Effective Potts Model

JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

replace group integrations

$$L \to z \in Z_3 \,, \quad \int \mathrm{d}L \, J(L) \, f(L) \to \frac{1}{3} \sum_{z \in Z_3} f(z)$$



reproduces $T_{m,n}$ with $m+n<4$

- **check strong coupling limit**

$\lambda \to 0$

$$\int \mathrm{d}L \, J(L) \, Q(L) = 1 + 4h\bar{h} + 4h^3 + 4\bar{h}^3 + \cancel{10}\,[9]\,h^2\bar{h}^2$$
$$+ 6h^4\bar{h} + 6h\bar{h}^4 + \bar{h}^6 + h^6 + \cancel{20}\,[16]\,h^3\bar{h}^3$$
$$+ 6h^5\bar{h}^2 + 6h^2\bar{h}^5 + \cancel{10}\,[9]\,h^4\bar{h}^4 + 4h^6\bar{h}^3 + 4h^3\bar{h}^6$$
$$+ 4h^5\bar{h}^5 + h^6\bar{h}^6$$

$\to 1 + 4h^3 + h^6$ for $\bar{h} \to 0$

only difference from $T_{2,2} = 2$ missing one of two $h^2\bar{h}^2$

scalar mesonic tetra-quark states in $3 \otimes 3 \otimes \bar{3} \otimes \bar{3}$

o.k. in heavy-dense limit
$m \sim \mu \gg T$

Effective Potts Model

- **for QCD at strong coupling**

  with static fermion determinant

$$Z_{\text{eff}} = \frac{1}{3^{N_s}} \sum_{\{z_i \in Z_3\}} \prod_{\langle i,j \rangle} \left(1 + 2\lambda \,\text{Re}\, z_i z_j^*\right) \times$$
$$\left(\prod_i \left(1 + h z_i + h^2 z_i^* + h^3\right)^2 \left(1 + \bar{h} z_i^* + \bar{h}^2 z_i + \bar{h}^3\right)^2 \right)$$

$$= \mathcal{N} \sum_{\{z_i \in Z_3\}} \exp\left\{ \sum_{\langle i,j \rangle} 2\gamma \,\text{Re}\, z_i z_j^* \right\} \times$$
$$\left(\prod_i \left(1 + h z_i + h^2 z_i^* + h^3\right)^2 \left(1 + \bar{h} z_i^* + \bar{h}^2 z_i + \bar{h}^3\right)^2 \right)$$

with $\gamma = \frac{1}{3} \ln\left(\frac{1+2\lambda}{1-\lambda}\right)$

- **Roberge-Weiss symmetric**

  from global Z_3 symmetry

$$Z_{\text{eff}}(T, \mu = i\theta T) \equiv Z^I_{\text{eff}}(\theta) = Z^I_{\text{eff}}(\theta + 2\pi/3)$$

Equivalent Flux-Tube Model

- **flux-tube model representation (dual)**

$$Z_{\text{eff}}(T,\mu) = \sum_{\{n,l\}_{\text{phys}}} \exp\Big\{ -\beta\Big(H(n,l) - \mu \sum_i q_i \Big) \Big\}$$

analogous to:

Patel, NPB 243 (1984) 411

Bernard, DeGrand, DeTar, Gottlieb, Krasnitz, Sugar, Toussain, PRD 49 (1994) 6051

Condella & DeTar, PRD 61(2000) 074023

here with:

string tension → σ

$$H(n,l) = \sum_{\langle i,j\rangle} \sigma |l_{\langle i,j\rangle}| + \sum_{i,s} m(n_{i,s} + \bar{n}_{i,s})$$

fluxes represented by link variables: $l_{\langle i,j\rangle} \in \{-1,0,1\}$

(anti-)quark occupation numbers: $n_{i,s} \in \{0,\dots,3\}$ and $\bar{n}_{i,s} \in \{0,\dots,3\}$ spin *s* = {↑,↓}

- ***Z*₃-Gauss' law:**

(Poisson equation)

$$\sum_{j\sim i} l_{\langle i,j\rangle} - \sum_s (n_{i,s} - \bar{n}_{i,s}) = 0 \mod 3$$

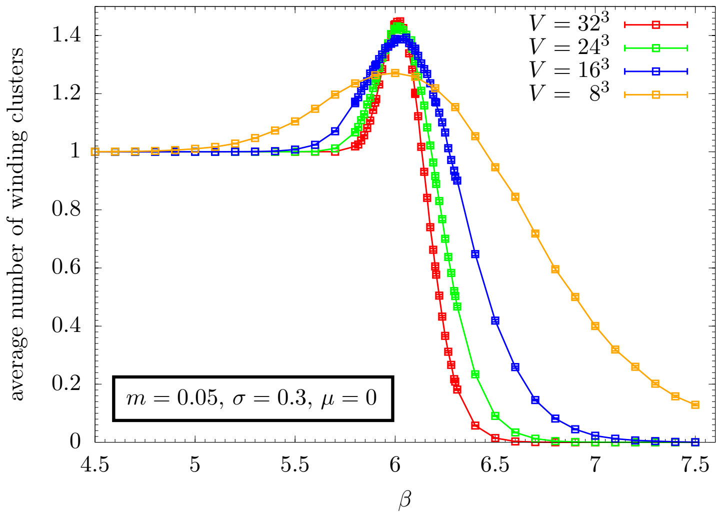
flux from volume around site *i*: $\phi_i \;=\; q_i \mod 3 \;\equiv\; q_i^z$:net-quark number modulo 3

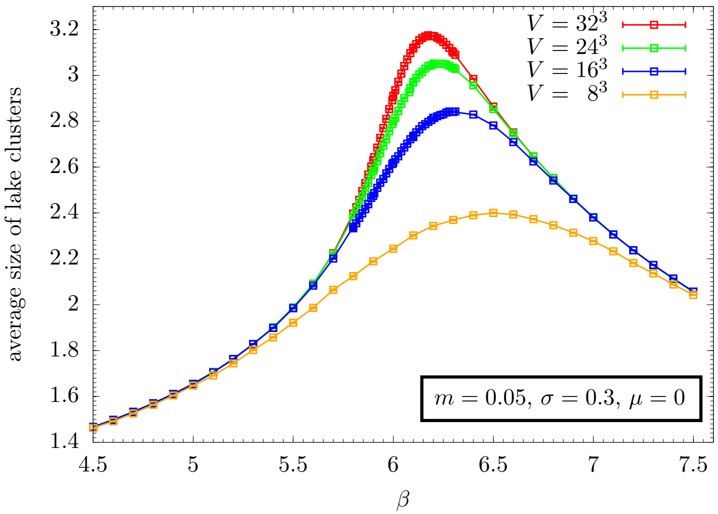
Flux-Tube Model

- **simulate with worm algorithm**

  example with light quarks / crossover

Prokof'ev & Svistunov, PRL 87 (2001) 160601
Korzec & Vierhaus, 2011, CPC 182 (2011) 1477
Delgado, Evertz, Gattringer, CPC 183 (2012) 1920
Rindlisbacher, Akerlund, de Forcrand, NPB (2016) 542





- **measure with fully-dynamic connectivity algorithm**

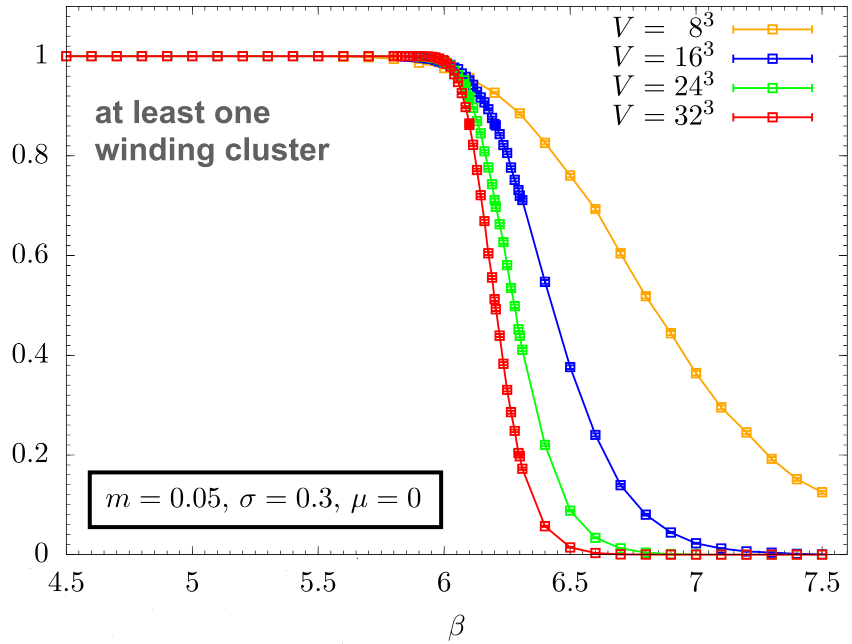
Holm, Lichtenberg, Thorup, J. ACM 48 (2001) 723
Alexandru, Bergner, Schaich, Wenger, PRD 97 (2018) 114503

Flux-Tube Model

- **deconfinement crossover**

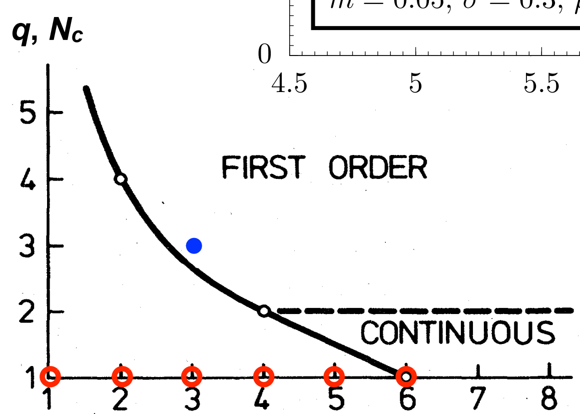
  percolation of Z_3-fluxes

Patel, NPB 243 (1984) 411



- **percolation??**

second order



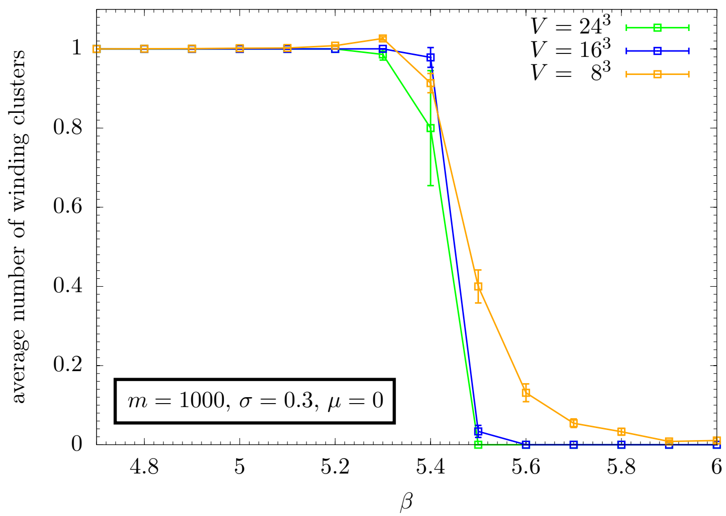
← percolation: q = 1

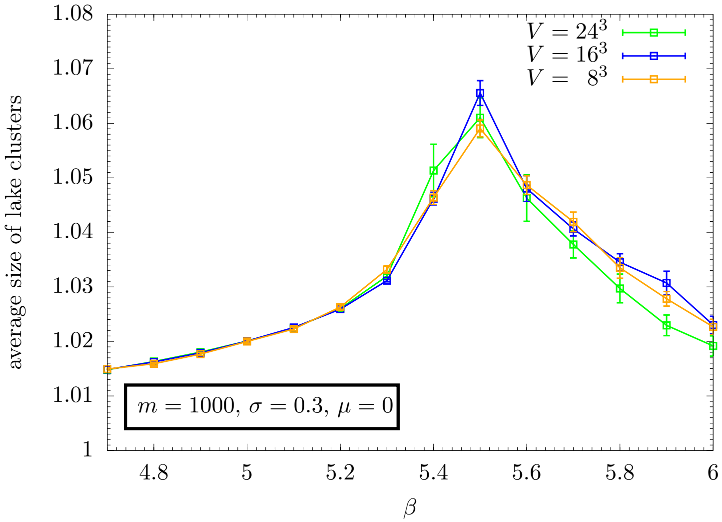
spatial dimensions d

Flux-Tube Model

- **heavy quarks**

  becomes 1st order





- **ok, still Potts!**

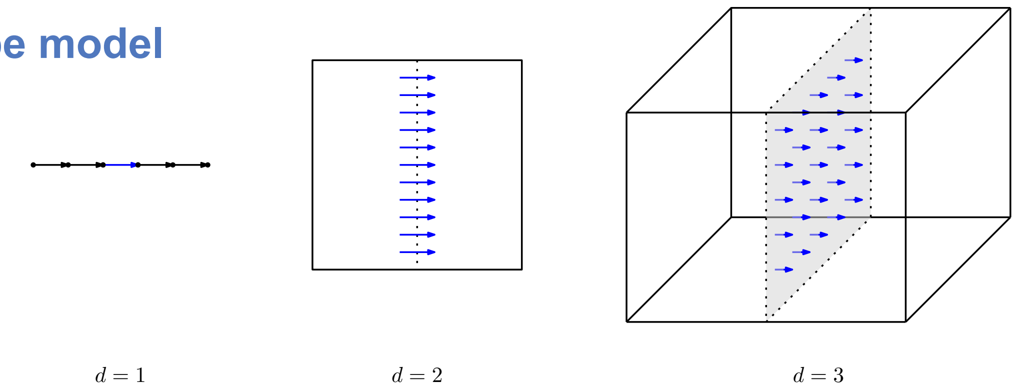
percolation with Z_N-Gauss' law = N-state Potts

Gauss' law makes the difference

Electric fluxes

- **interfaces in flux-tube model**

  dual stacks of links

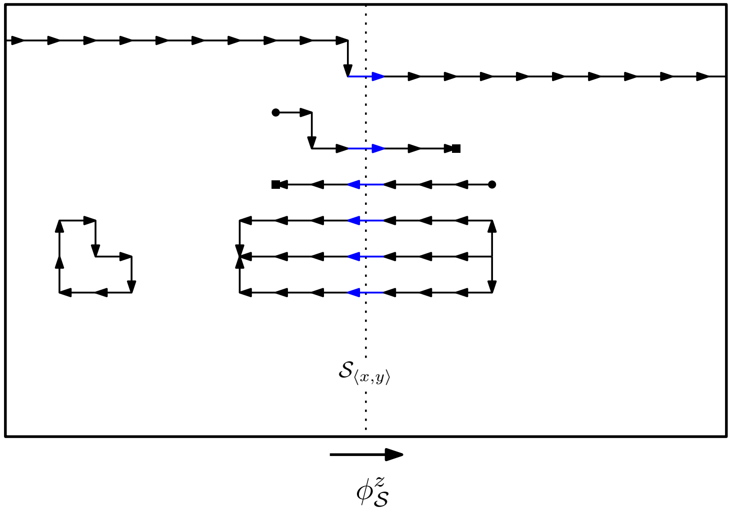


- **flux through interface**

$$\phi_{\mathcal{S}}^{z} = \sum_{\langle u,v\rangle \in \mathcal{S}^*} l^{z}_{\langle u,v\rangle}$$

- **add constraint to fix**

$$\phi_{\mathcal{S}}^{z} = e$$



Ensembles with Fixed Flux

- flux-tube model with interface

$$Z^e_{\mathcal{S}}(\beta,\mu) = \sum_{\{l,n\}} \exp\Big\{-\beta\Big(H(\{l,n\}) - \mu\sum_i q_i\Big)\Big\} \underbrace{\Big(\frac{1}{3}\sum_{z\in Z_3} z^{\phi_{\mathcal{S}}-e}\Big)}_{\text{flux constraint}} \underbrace{\prod_j \Big(\frac{1}{3}\sum_{z\in Z_3} z^{\phi_j - q_j}\Big)}_{\text{Gauss' law}}$$

$$= \frac{1}{3}\sum_{z\in Z_3} z^{-e} \sum_{\{l,n\}} \exp\Big\{-\beta\Big(H(\{l,n\}) - \mu\sum_i q_i\Big)\Big\} z^{\phi_{\mathcal{S}}} \prod_j \Big(\frac{1}{3}\sum_{z\in Z_3} z^{\phi_j - q_j}\Big)$$

- back to dual Potts model

$$\sum_{\{l\}} \Big(\prod_{\langle i,j\rangle} e^{-\beta\sigma|l_{\langle i,j\rangle}|}\Big) z^{\phi_{\mathcal{S}}} \Big(\prod_k z_k^{\phi_k}\Big)$$

$$= \prod_{\langle i,j\rangle} \Big(1 + 2\lambda\,\mathrm{Re}\Big(z^{\omega_{\mathcal{S}}(\langle i,j\rangle)} z_i z_j^*\Big)\Big)$$

with interface:

$$\omega_{\mathcal{S}}(\langle i,j\rangle) = \begin{cases} 1, & \langle i,j\rangle \in \mathcal{S}^* \\ -1, & \langle j,i\rangle \in \mathcal{S}^* \\ 0, & \text{otherwise} \end{cases}$$

Ensembles with Fixed Flux

- Z_3-Fourier transform of interfaces

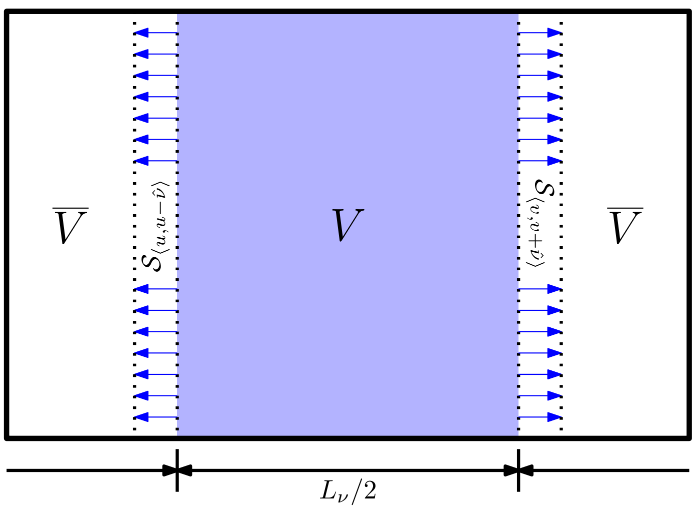
$$Z^e_{\mathcal{S}}(\beta,\mu) = \frac{1}{3}\sum_{z\in Z_3} z^{-e} Z^z_{\mathcal{S}}(\beta,\mu)$$

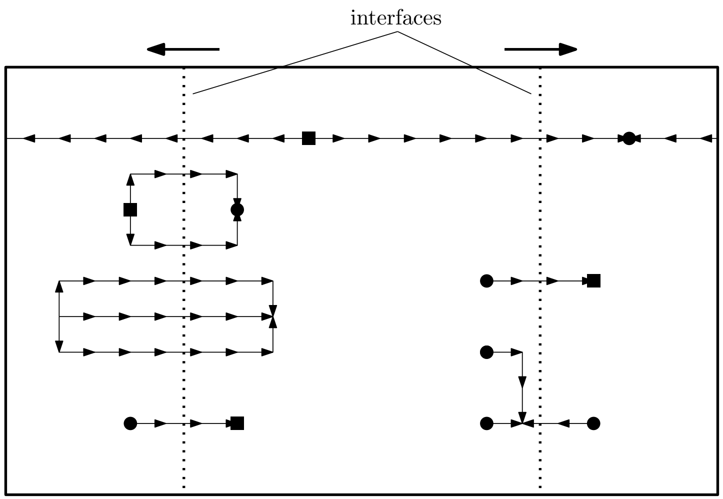
Z_3-flux ensembles

Z_3-interface ensembles

- generalize to combinations of interfaces

with total flux e





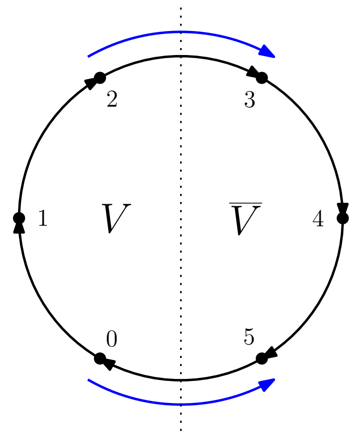
with total flux e = 2

Test in (1+1)d

- **Potts model representation**

total charge in sub-volume

$V: \; q = 1 \mod 3$



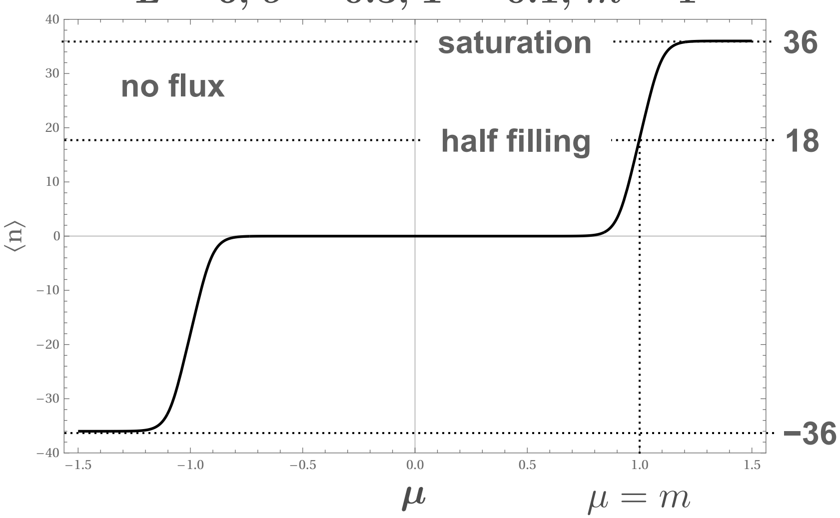
total charge in complement

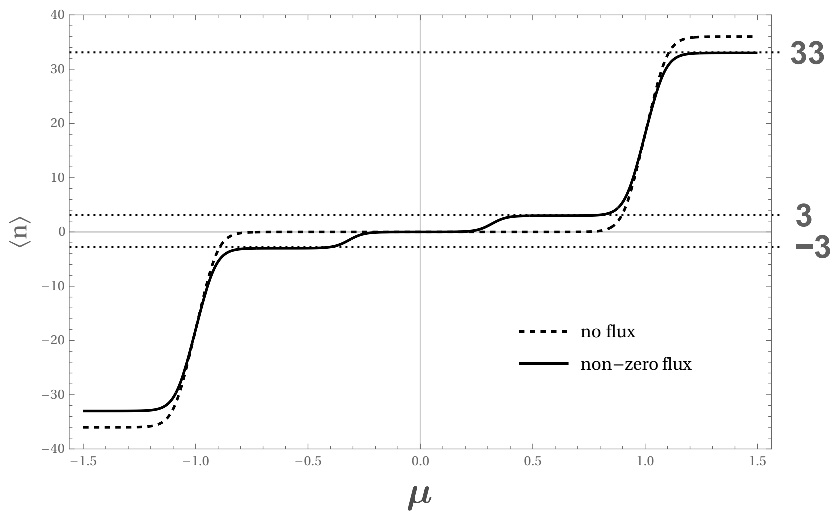
$\overline{V}: \; q = 2 \mod 3$

total flux: $\phi^z_{\cup\mathcal{S}} = 1 \mod 3$

- **compare:**

$L = 6, \sigma = 0.3, T = 0.1, m = 1$





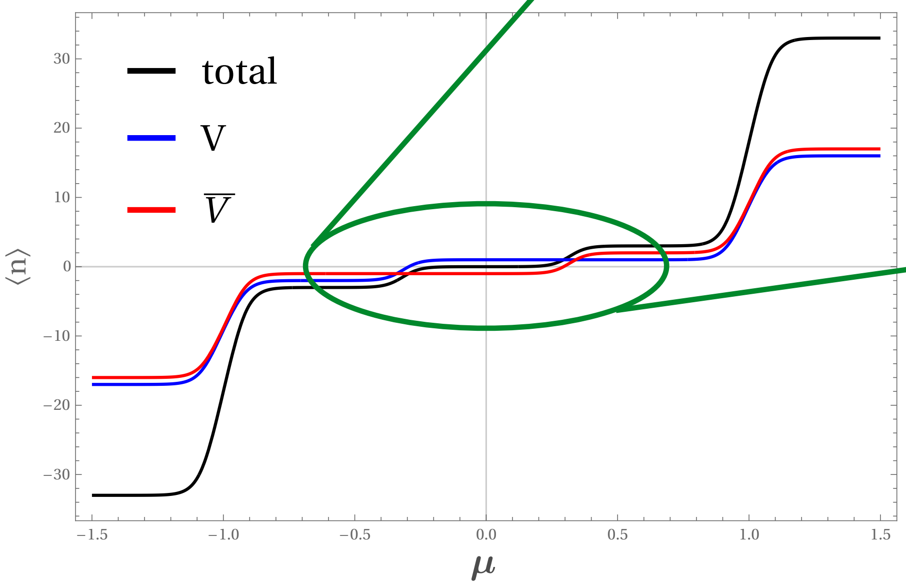
Bundesministerium für Bildung und Forschung

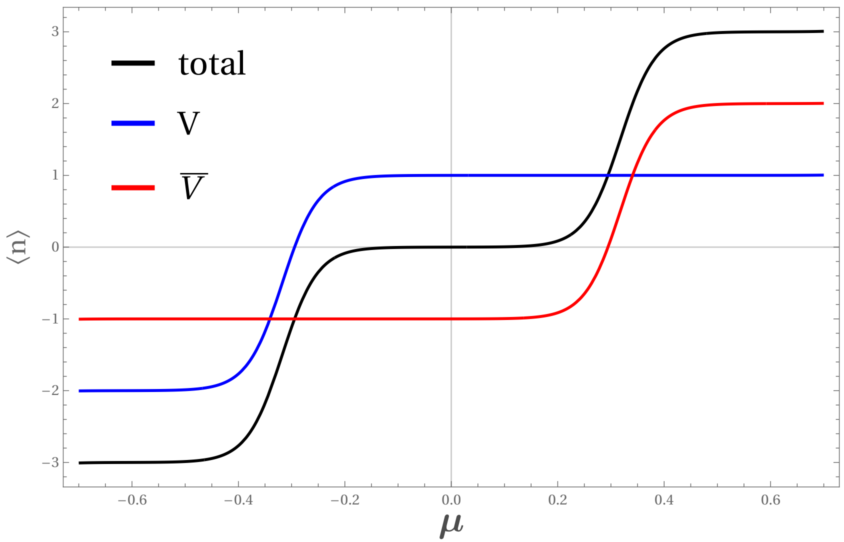
Test in (1+1)d

- **compute quark numbers in sub-volumes:**

  net-quark number

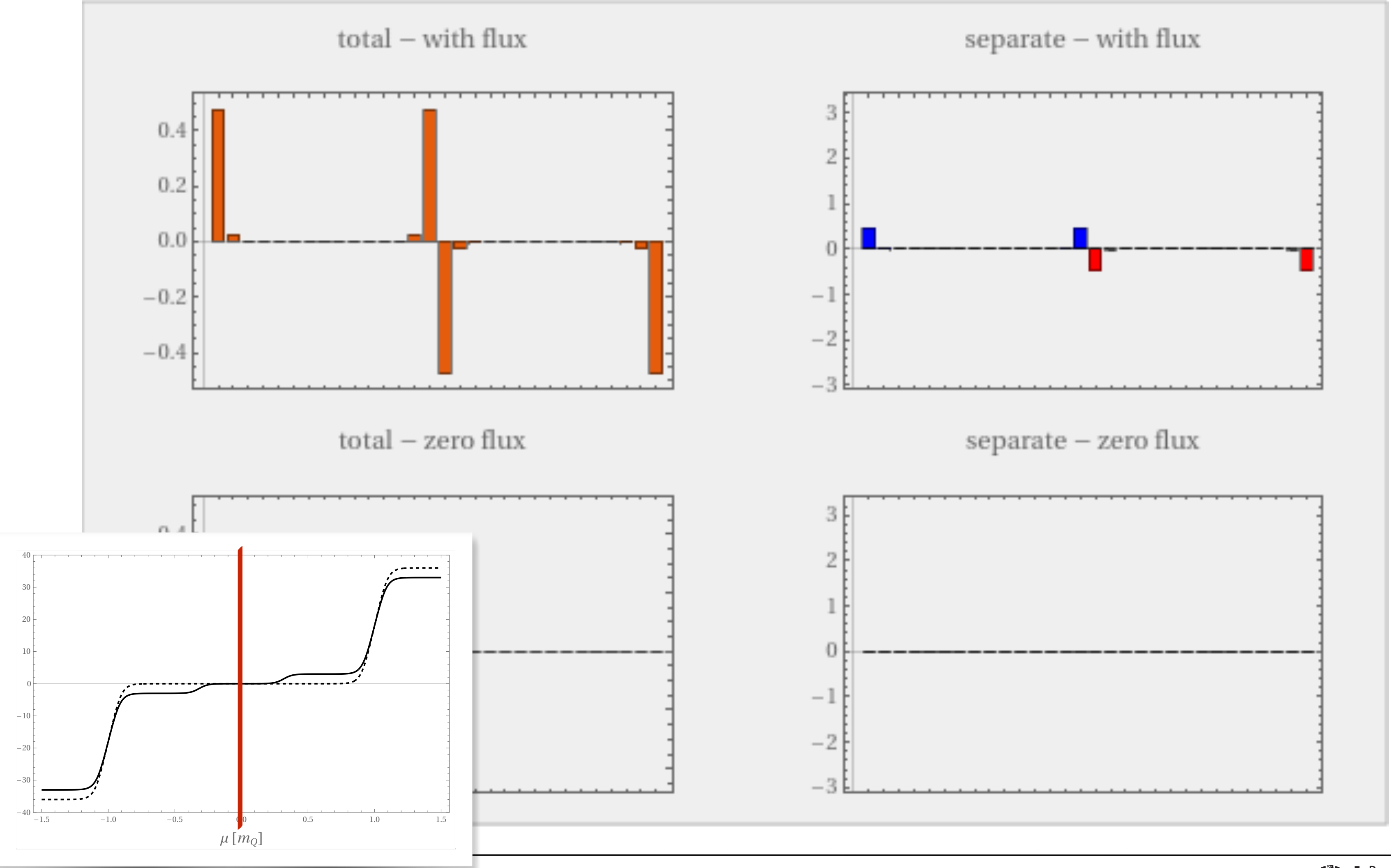
$L = 6,\ \sigma = 0.3,\ T = 0.1,\ m = 1$





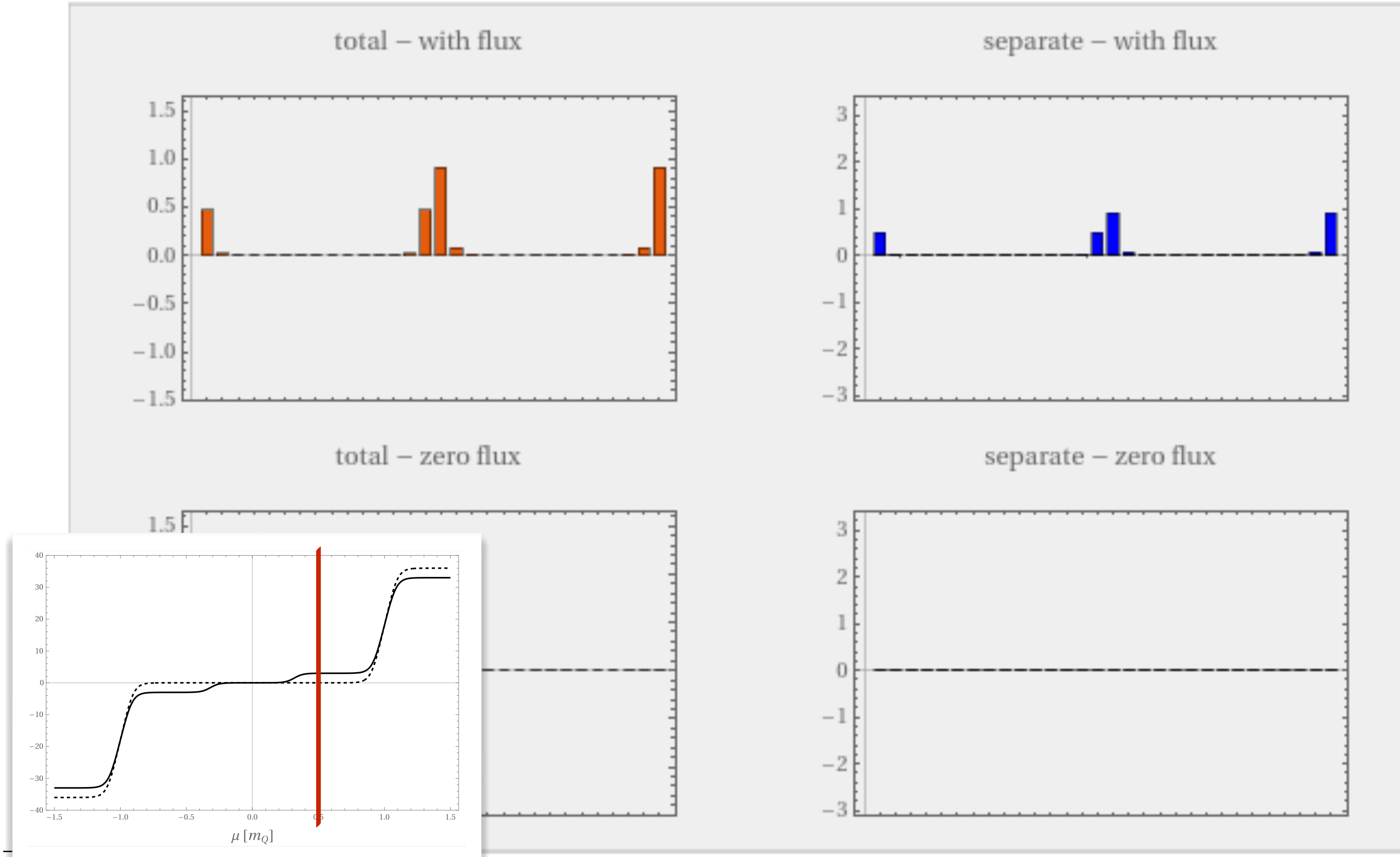
Quark-Number Densities

σ=0.3m$_Q$, μ = 0, T = 0.101m$_Q$



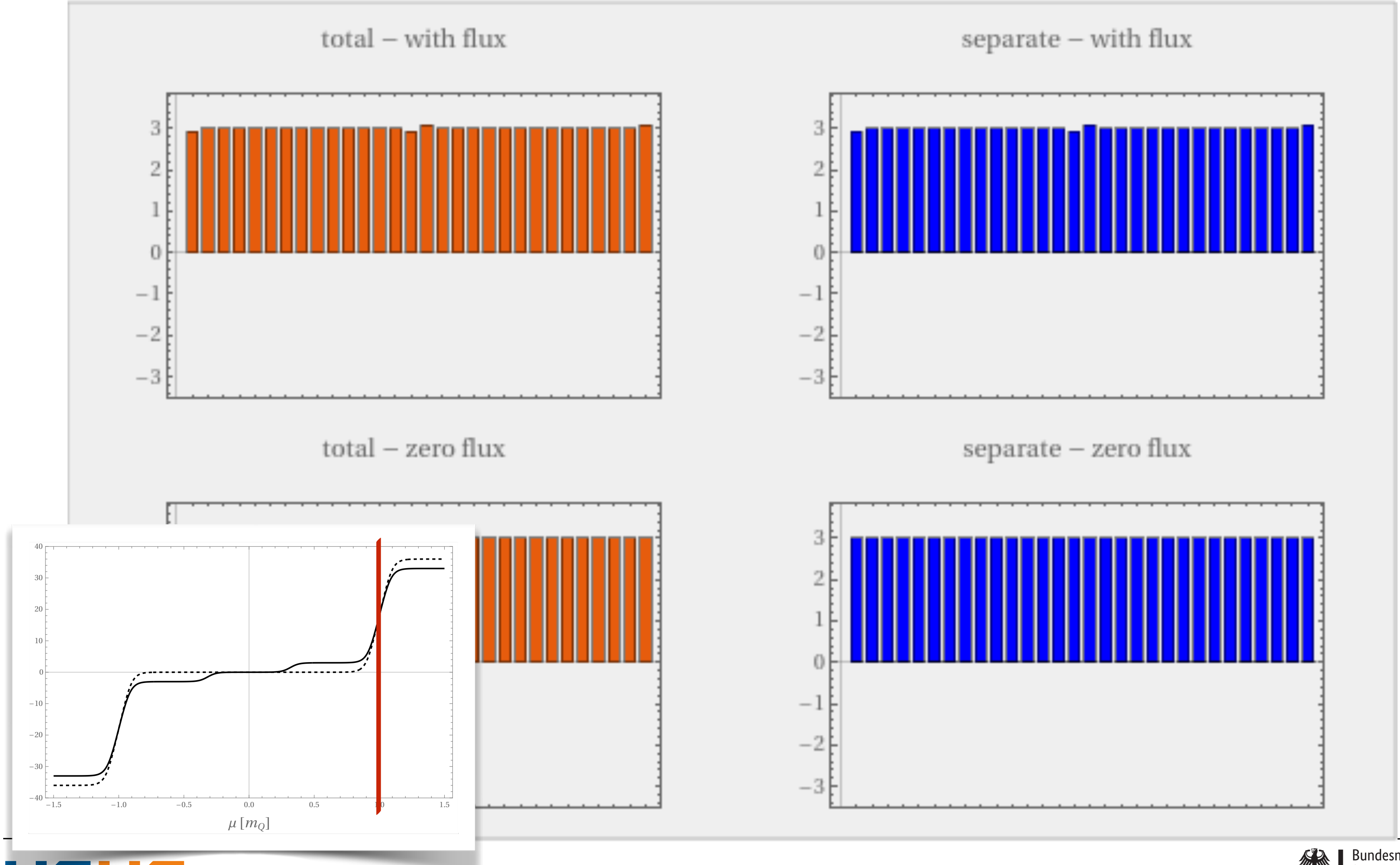
Quark-Number Densities

σ=0.3m$_Q$, μ = 0.5m$_Q$, T = 0.101m$_Q$



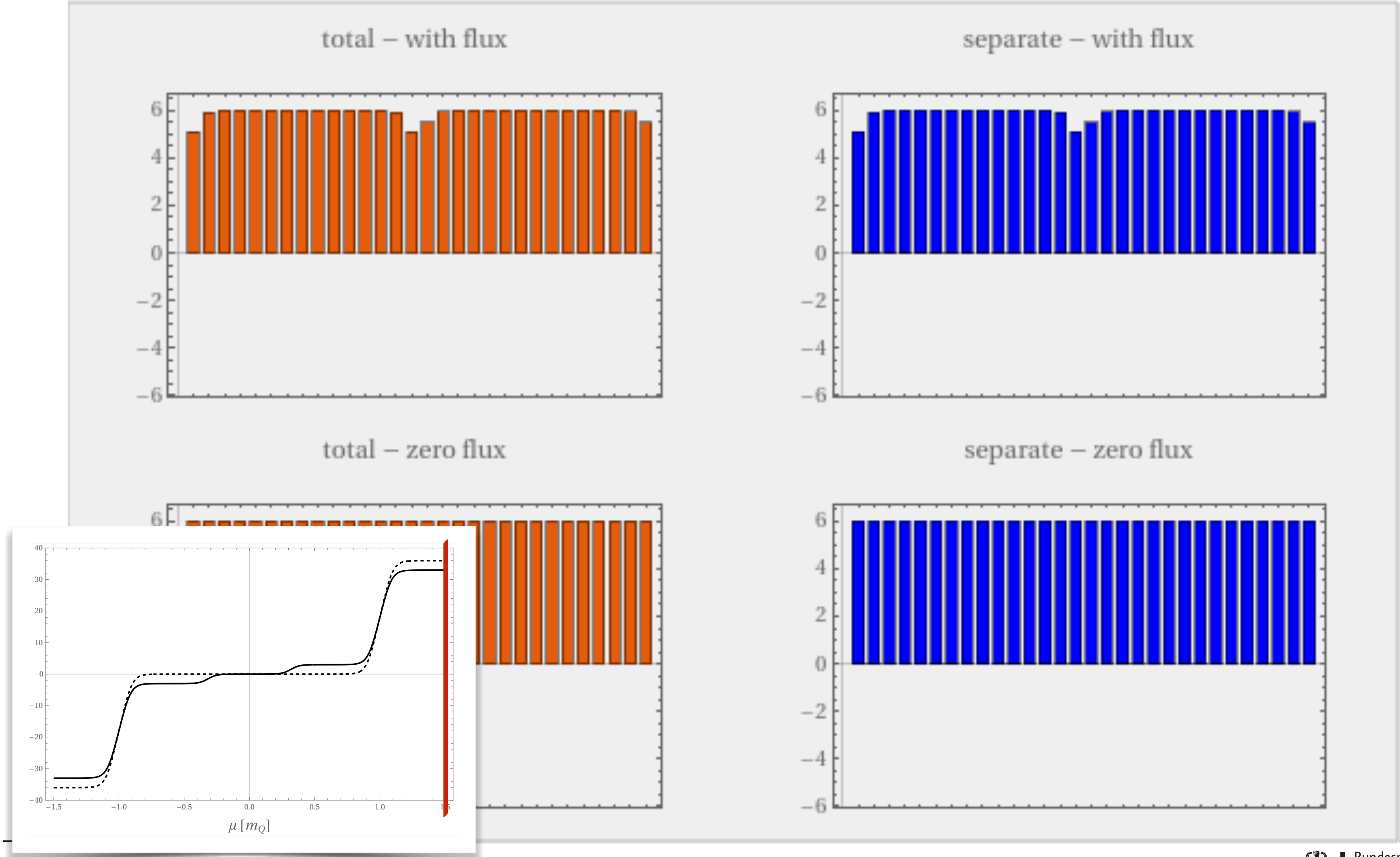
Quark-Number Densities

$\sigma=0.3m_Q$, $\mu = m_Q$, $T = 0.101m_Q$



Quark-Number Densities

$\sigma=0.3m_Q$, $\mu = 1.5m_Q$, $T = 0.101m_Q$



Quark Free Energies

- **define for quark in volume *V*:**

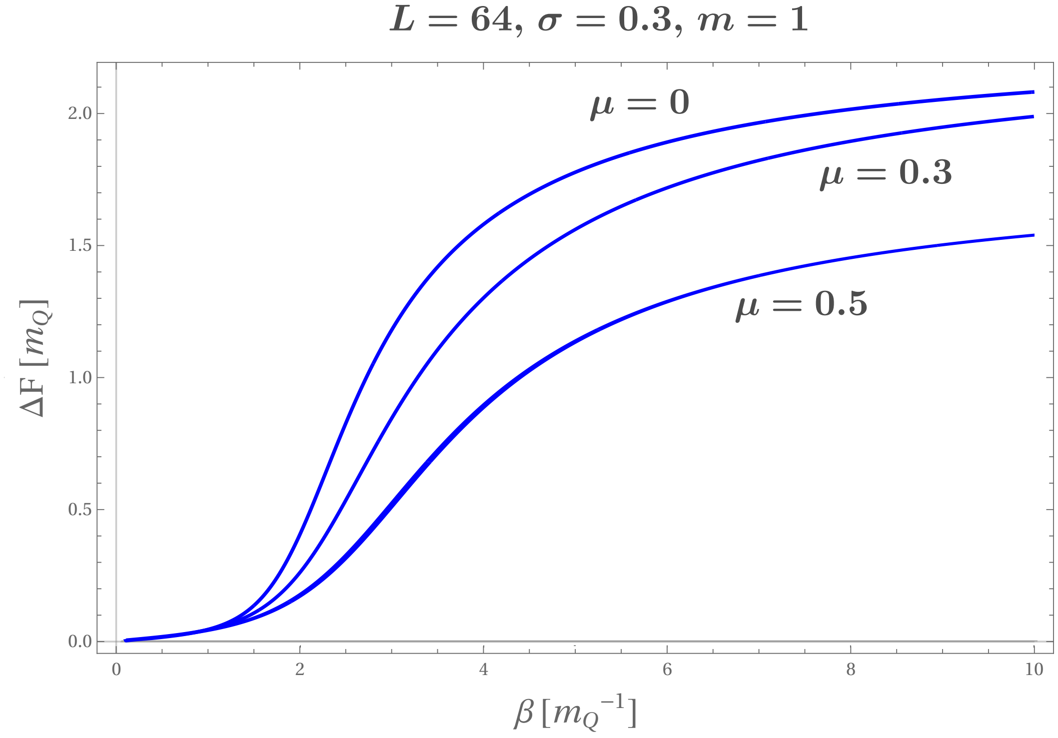
$$\Delta F_q(\beta, V, \mu) = -T \ln \left(\frac{Z_{\partial V}^{e=1}(\beta,\mu)}{Z_{\partial V}^{e=0}(\beta,\mu)} \right)$$

- **still (1+1)-dim**

  Potts model representation, flux-tube model parameters ($V = L/2$)

- **independent of *L*!**

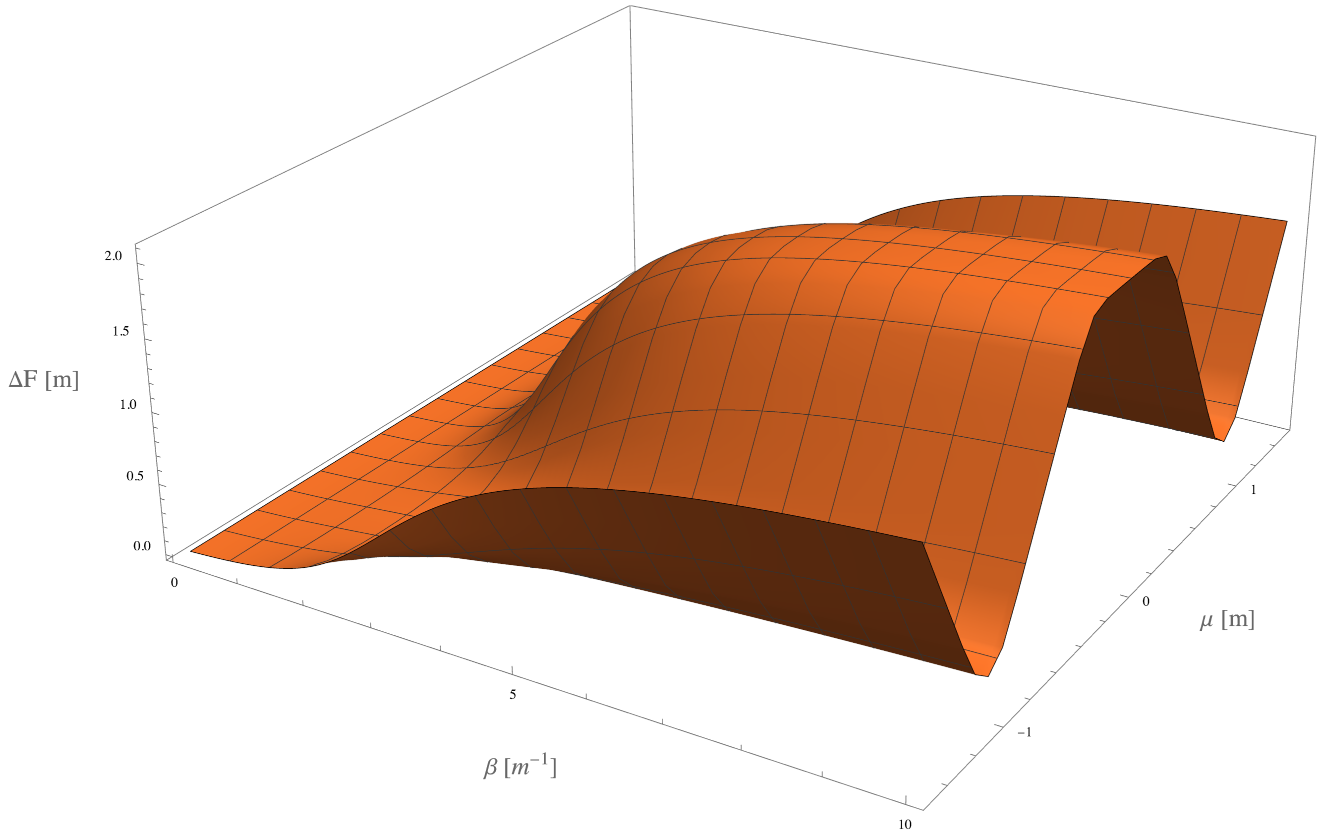
  no asymptotic string tension



Quark Free Energies

- still (1+1)-dim

$$\Delta F_q(\beta, V, \mu) = -T \ln\left(\frac{Z_{\partial V}^{e=1}(\beta,\mu)}{Z_{\partial V}^{e=0}(\beta,\mu)}\right)$$



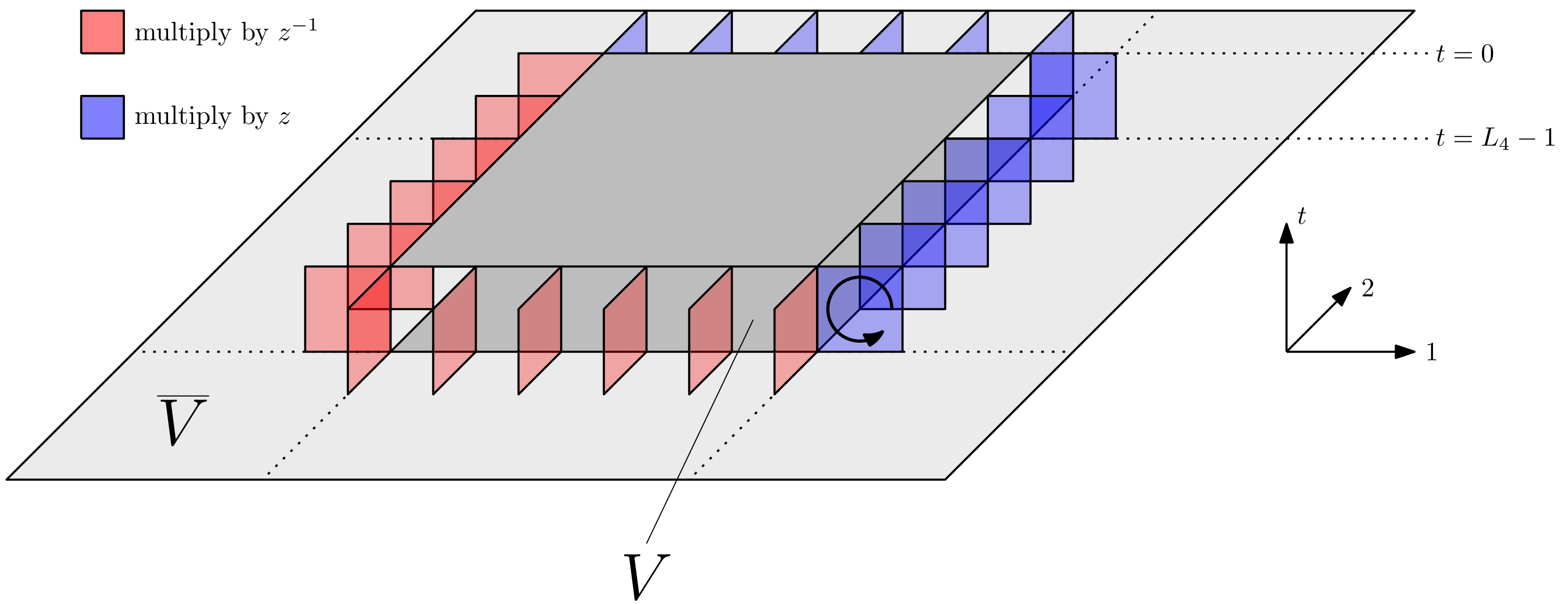
Gauge Theory

- **heavy-dense limit of QCD**

  static fermion determinant — unchanged

- **reintroduce plaquette action**

  with closed centre-vortex sheets



Full QCD

- **dualization of Wilson quark determinant**

  introduce site and link variables

  $s_x, k_{x,\mu}, \overline{k}_{x,\mu} \in \{0,1\}$

Gattringer & Marchis, NPB 916 (2017) 627

Marchis & Gattringer, PRD 97 (2018) 034508

- **integrate original *SU*(3) link variables *U***

complicated weights

$$Z(\beta,\mu) = \sum_{\{s,k,\overline{k}\}} \exp\Big(a\mu \sum_x \sum_{\omega,\omega'} \big(k_{x,4}^{\omega\omega'} - \overline{k}_{x,4}^{\omega\omega'}\big)\Big)\, W(k,\overline{k})$$

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi \prod_x \prod_\omega \left(-\overline{\psi}_x^\omega \psi_x^\omega\right)^{s_x^\omega} \prod_{\omega'} \prod_\mu \left(\overline{\psi}_x^\omega \psi_{x+\hat{\mu}}^{\omega'}\right)^{k_{x,\mu}^{\omega\omega'}} \left(\overline{\psi}_{x+\hat{\mu}}^\omega \psi_x^{\omega'}\right)^{\overline{k}_{x,\mu}^{\omega\omega'}}$$

- **introduce quark number densities**

$$n_x = \sum_{\omega,\omega'} k_{x,4}^{\omega\omega'} \qquad \overline{n}_x = \sum_{\omega,\omega'} \overline{k}_{x,4}^{\omega\omega'}$$

Full QCD

Gattringer & Marchis, NPB 916 (2017) 627
Marchis & Gattringer, PRD 97 (2018) 034508

- **dualization of Wilson quark determinant**

  introduce site and link variables

  $s_x, k_{x,\mu}, \overline{k}_{x,\mu} \in \{0,1\}$

- **integrate original *SU*(3) link variables *U***

$$Z(\beta,\mu) = \sum_{\{s,k,\overline{k}\}} \exp\Big(a\mu \sum_x \big(n_x(k) - \overline{n}_x(\overline{k})\big)\Big)\, W(k,\overline{k})$$

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi \prod_x \prod_\omega \left(-\overline{\psi}^\omega_x \psi^\omega_x\right)^{s^\omega_x} \prod_{\omega'} \prod_\mu \left(\overline{\psi}^\omega_x \psi^{\omega'}_{x+\hat\mu}\right)^{k^{\omega\omega'}_{x,\mu}} \left(\overline{\psi}^\omega_{x+\hat\mu} \psi^{\omega'}_x\right)^{\overline{k}^{\omega\omega'}_{x,\mu}}$$

- **constrain quark numbers in *V***

$$Z^e_{\partial V}(\beta,\mu) = \sum_{\{s,k,\overline{k}\}} \left[\prod_t \delta\Big(\sum_{\vec{x}\in V} (n_x - \overline{n}_x) = e \bmod 3\Big)\right] \exp\Big(a\mu \sum_x (n_x - \overline{n}_x)\Big) W(k,\overline{k})$$

$$\times \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \prod \cdots$$

(at all times *t*)

Quark in Finite Volume

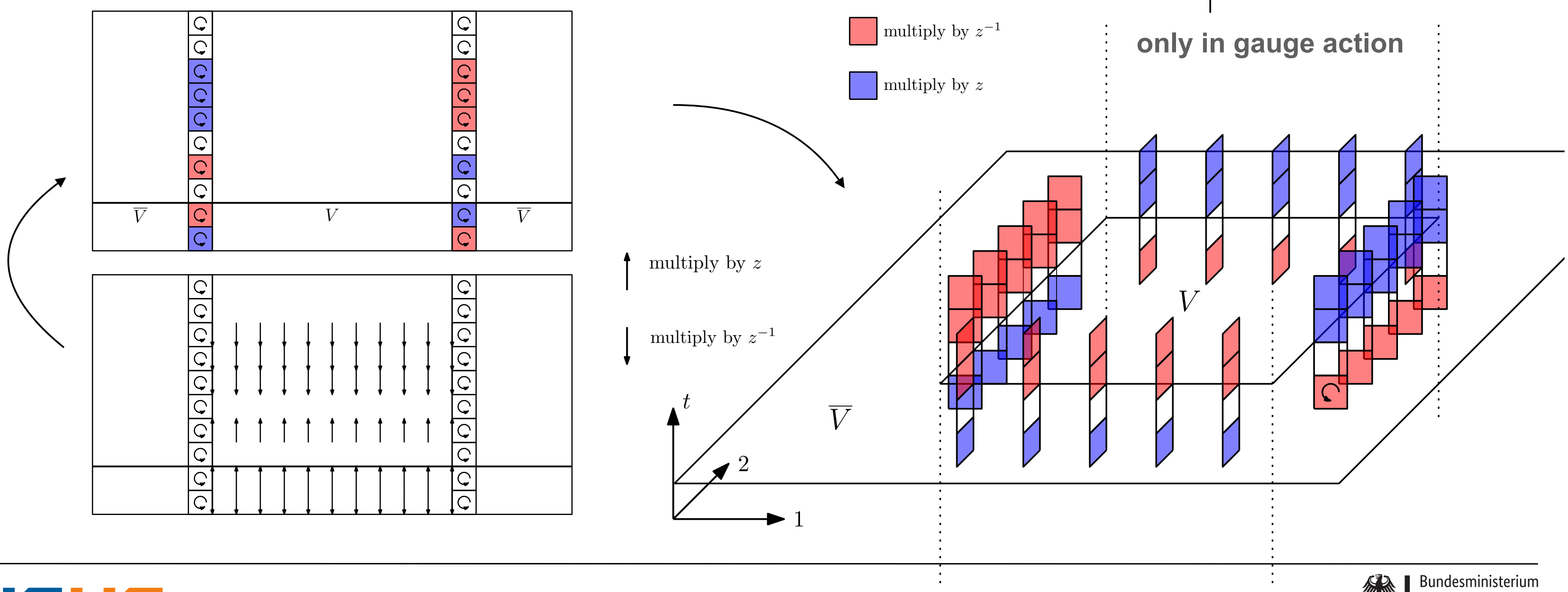
- as before arrive at

$$Z^e_{\partial V}(\beta,\mu) = \frac{1}{3^{N_t}} \sum_{\{z\in Z_3\}} \prod_t z_t^{-e}\, Z^z_{\partial V}(\beta,\mu,\{z\})$$

and show that

$$Z^z_{\partial V}(\beta,\mu,\{z\}) = \int \mathcal{D}[\ldots]\, e^{-S^z_G(U,\{z\})}\, e^{-S_F(\overline{\psi},\psi,U,\mu)}$$

only in gauge action



Summary

- **Electric fluxes in the pure gauge theory**

  from FT of ’t Hooft’s twisted b.c.’s

- **Free energy of quark in Volume *V***

  heavy-dense QCD ↔ flux-tube model

- **Deconfinement ↔ Percolation (with Gauss law)**

  oder parameter for finite density

- **Full QCD — dualization**

  closed center-vortex sheet in gauge action — fermion determinant unchanged

Outlook

- **Transfer-matrix formulation**

  fermion determinant needs modification as well!

- **Puzzle!**

  Equivalent? If not, which one is right?

- **Time for large-scale simulations?**

  expensive — find efficient ways

- **Entanglement entropy?**

  so far known only in pure gauge theory

Thank you for your attention!