Non-invertible symmetry, and string tensions beyond N-ality

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based on 2101.02227, 2104.01824

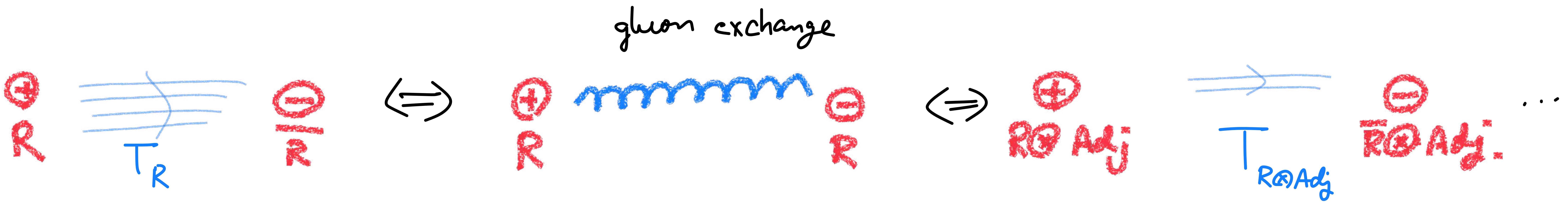
Spectrum of confining strings : N-ality

T_R (string tension) are important quantities characterizing confinement.

$\Rightarrow$ How do T_R's depend on the gauge representation R ?

<u>N-ality</u> σ_R depends only on #(boxes of Young tab.) mod N.

(Believed to be true for pure $SU(N)$ YM for 3 & 4 dim.)



1-form symmetry (center symmetry)

In modern understandings, for relativistic QFTs, (cf. Gaiotto, Kapustin, Seiberg, Willett '14)

$$\boxed{\text{Symmetry} \overset{\text{def}}{:=} \text{existence of "topological" operators } U(M_{d-p-1})}$$

<u>e.g.</u> For conti. sym., $U(M_{d-1}) = \exp\left(i\alpha \int_{M_{d-1}} \underset{\sim}{j}\right)$ — Noether current.

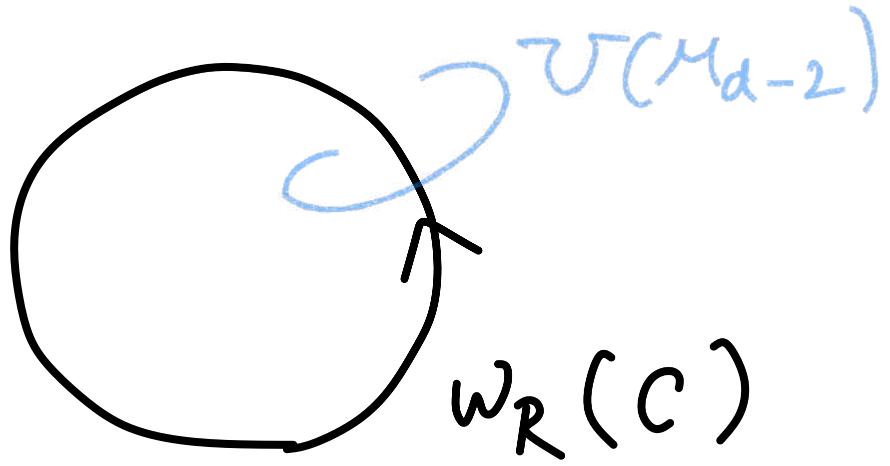
<u>$\mathbb{Z}_N$ 1-form sym.</u>

For SU(N) YM, there are codim-2 topological operator $U(M_{d-2})$.

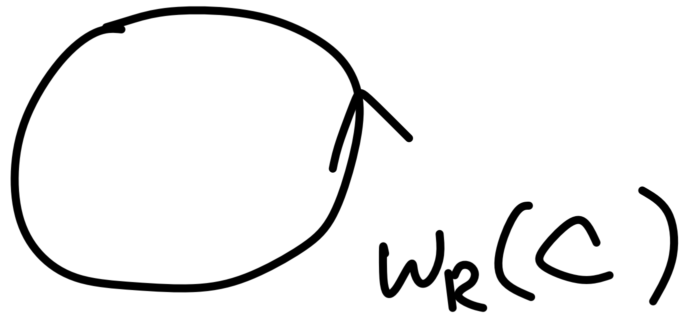
[Gukov-Witten operator]

$$W_R(C) \longmapsto e^{\frac{2\pi i}{N}\cdot|R|}\, W_R(C).$$

⇒ Nice explanation of N-ality rule.



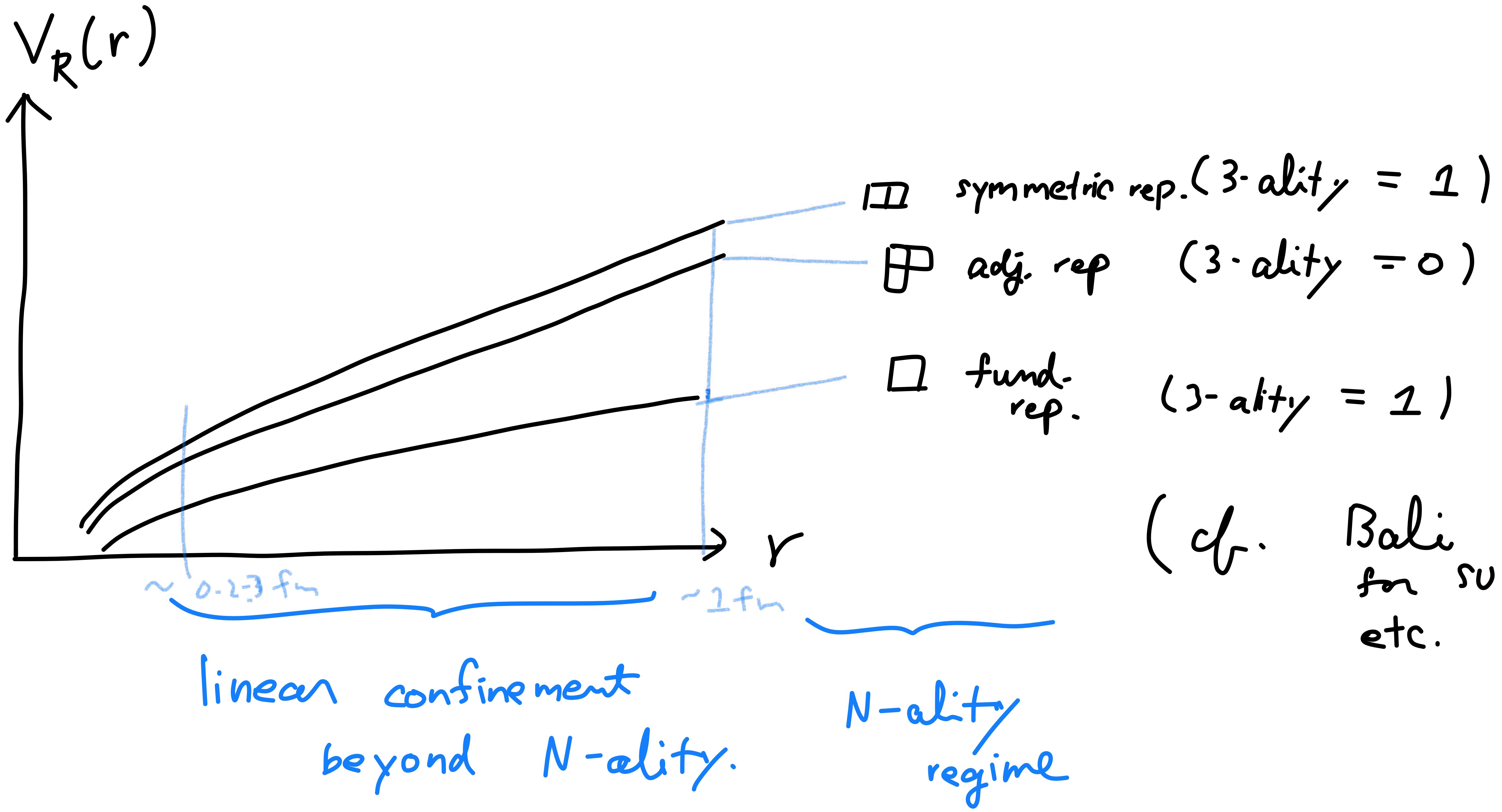
$= e^{\frac{2\pi i}{N}|R|}$



Beyond N-ality

N-ality is an important feature in the deep IR regime.

However, it does not kick in immediately after the linear confinement occurs.



(cf. Bali 2000 for SU(3) YM etc.)

Is there some "nice" way to understand the beyond N-ality regime?

⇒ For some toy models, it does!!

(2+1)d $U(1)^{N-1}$ gauge model with S_N global sym

Gauge field $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$ with $\mathrm{tr}(\vec{a}) = \sum_{i=1}^{N} a_i = 0$.

(Using simple roots α_i of $SO(N)$, we can rewrite it as

$$\vec{a}(x) = \sum_{i=1}^{N-1} \tilde{a}_i(x)\,\alpha_i .$$
)

The Lagrangian

$$\mathcal{L} = \frac{1}{g^2} \vec{f}_{\mu\nu} \cdot \vec{f}_{\mu\nu}$$

is symmetric under the global S_N permutation

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \longmapsto \begin{pmatrix} a_{\sigma(1)} \\ a_{\sigma(2)} \\ \vdots \\ a_{\sigma(N)} \end{pmatrix} \qquad (\sigma \in S_N).$$

(* Here, I write down the continuum Lagrangian,

but, in the actual computations, we used the lattice model.)

Our model & Polyakov model

Idea is very similar to the Polyakov model:

3d $SU(N)$ YM + Adj. scalar Φ

$\xRightarrow{\text{Higgsing by } \langle \Phi \rangle}$ 3d $U(1)^{N-1}$ gauge theory + monopoles.

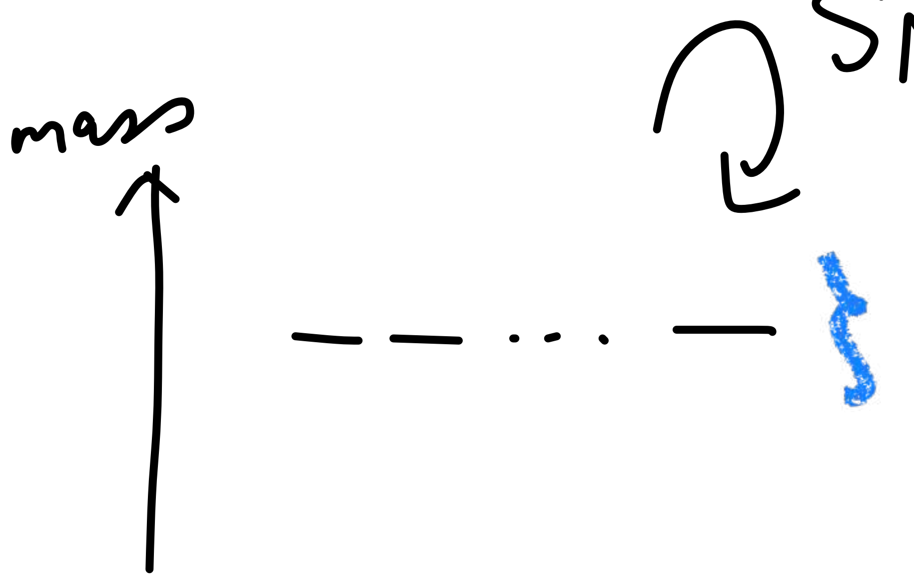
$$\mathcal{L}_{eff} \sim |d\vec{\sigma}|^2 + e^{-\frac{\#}{g^2}} \sum_{\alpha:\,\text{positive simple roots}} \left(1 - \cos(\alpha \cdot \vec{\sigma})\right)$$

The Polyakov model, however, does not realize S_N symmetry at low-energy, while our model does.

Our model $\quad V_{eff} \sim \sum_{\alpha:\text{root}} (1-\cos(\alpha\cdot\sigma))$

S_N ✓

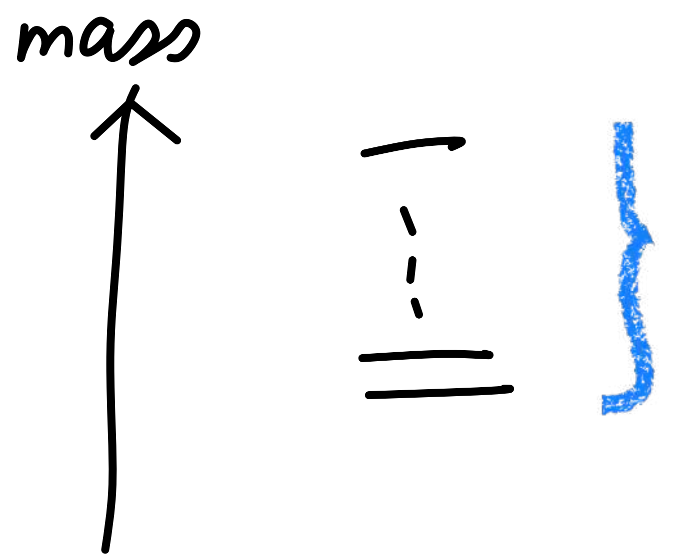
S_N permutation



(N-1) dual photons have degenerate mass

Polyakov model $\quad V_{eff} \sim \sum_{\alpha:\text{simple root}} (1-\cos(\alpha\cdot\sigma))$

S_N ✗



(N-1) dual photons have different masses.

Gauging S_N : $U(1)^{N-1} \rtimes S_N$ gauge theory

As our $U(1)^{N-1}$ model has a manifest S_N symmetry, let us gauge it !

$\Rightarrow$ Physical operators must be local S_N invariant.

Wilson formulation

On the lattice, we can realize it by saying that

link variable

$$U_\ell = \underbrace{P_\ell}_{\substack{N\times N \text{ matrix} \\ \text{for } S_N \text{ perm.}}} \cdot \underbrace{C_\ell}_{= \begin{pmatrix} e^{i a_1} & & \\ & \ddots & \\ & & e^{i a_N} \end{pmatrix} \in U(1)^{N-1}}$$

$$S = \underbrace{\beta_1 \, \mathrm{tr}\Big(\mathbb{1}_N - \prod_{\ell \in \partial p} (P_\ell \cdot C_\ell)\Big)}_{U(1)^{N-1} \rtimes S_N \text{ plaquette action}} + \underbrace{\beta_2 \, \mathrm{tr}\Big(\mathbb{1}_N - \prod_{\ell \in \partial p} P_\ell\Big)}_{S_N \text{ plaquette action}} .$$

1-form symmetries of semi-Abelian theory

Gauge invariance

$$U(1)^{N-1} \xrightarrow{S_N\text{-gauging}} U(1)^{N-1} \rtimes S_N \quad \text{"semi-Abelian"}$$

Center of the gauge group

$$Z(U(1)^{N-1}) = U(1)^{N-1}$$

$$\Downarrow$$

$U(1)^{N-1}$ 1-form sym.

⇝ Infinitely many string tensions can be expected.

Very similar to $SU(N)$!

$$Z(U(1)^{N-1} \rtimes S_N) = \mathbb{Z}_N.$$

$$\Downarrow$$

$\boxed{\mathbb{Z}_N \text{ 1-form sym.}}$

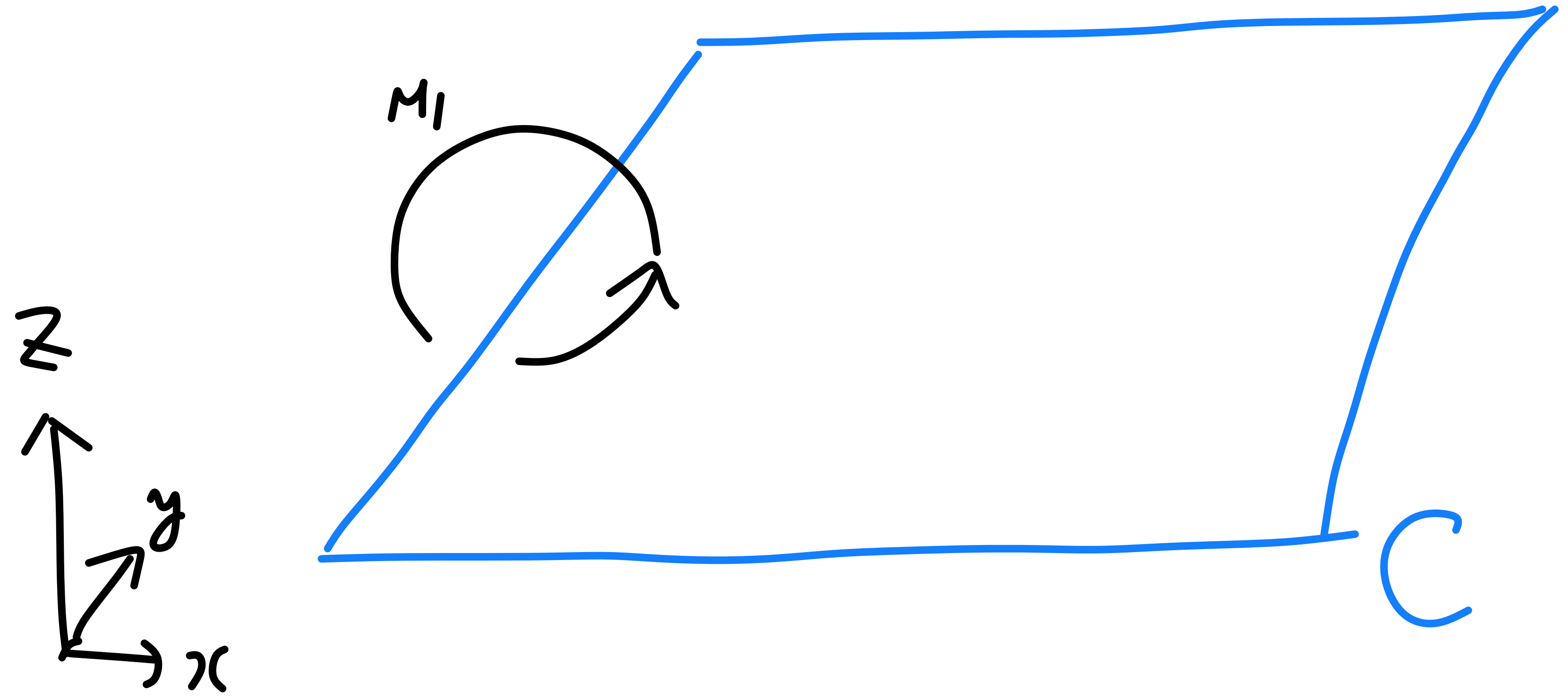
⇝ Only N-distinct strings can be naturally explained.

String tensions

In order to compute string tensions, we must know how the Wilson loops can be realized in the monopole theory

$$\mathcal{L}_{eff} = |d\vec{\sigma}|^2 + e^{-\frac{\#}{g^2}} \sum_{\alpha} \left(1 - \cos(\alpha \cdot \vec{\sigma})\right).$$

Defect operator



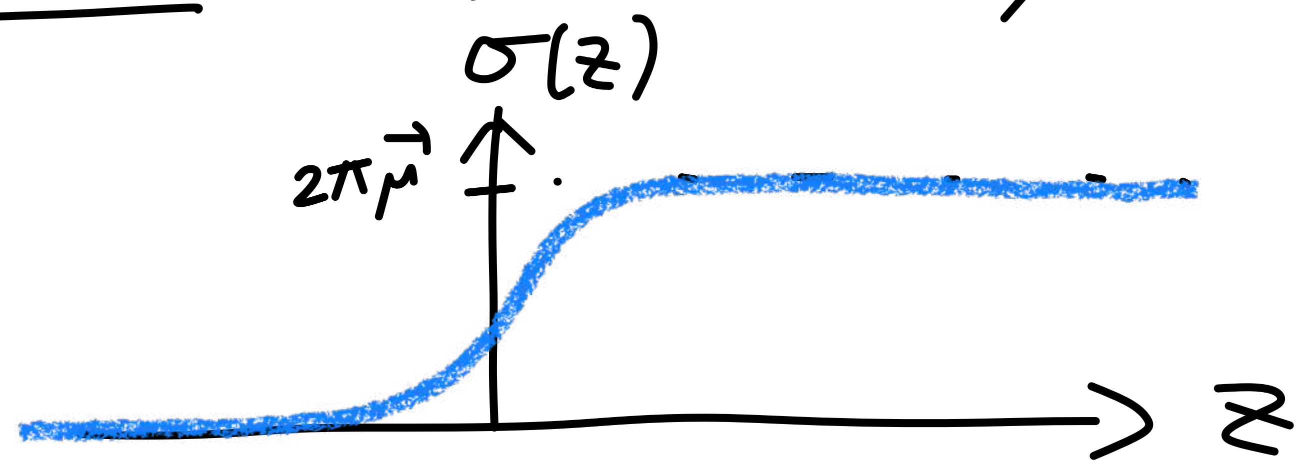
$$\oint_{M_1} d\vec{\sigma} = 2\pi \vec{\mu} \quad (\vec{\mu} \text{ is a weight vector})$$

$$\Updownarrow$$

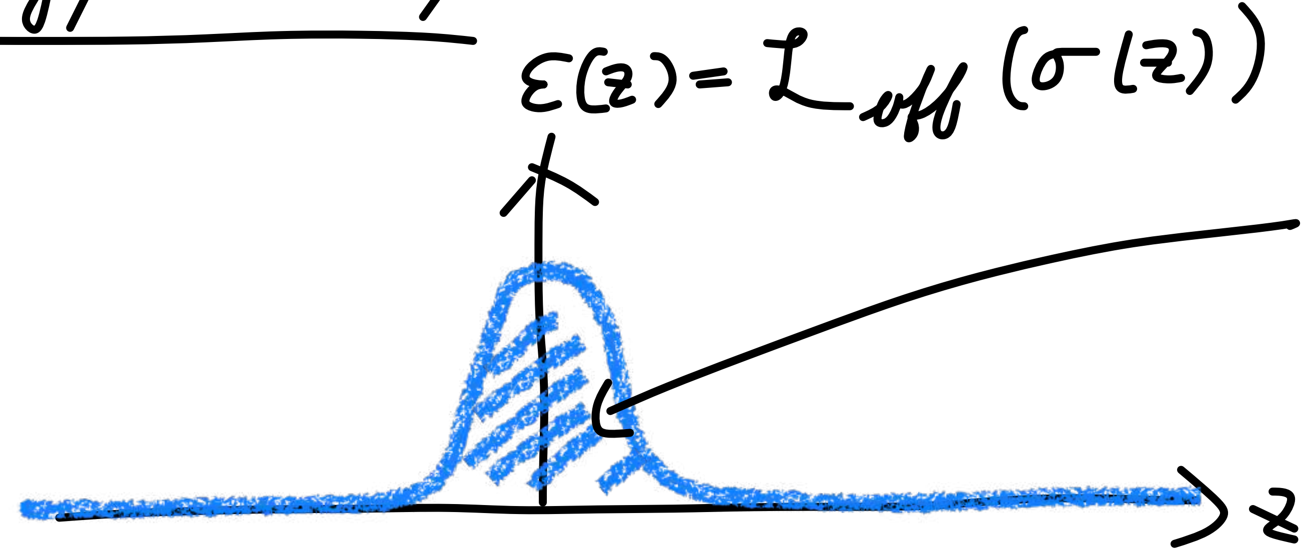
$$W_{\vec{\mu}}(C) = e^{i\vec{\mu}\cdot\oint_C \vec{a}}.$$

σ- profile

Inside C, it looks like



Energy density



string tension $T_{\vec{\mu}}$.

String tensions beyond N-ality

Under a reasonable anatz, we obtain the string tensions for various charges:

$\underline{\mu_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \frac{1}{N}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}$ (SU(N) fundamental. N-ality $= 1$)

$$T_{\mu_1} \left(= \# e^{-\frac{\#}{g^2}} \times \frac{N-1}{\sqrt{N}} \right) \neq 0.$$

$\underline{\mu_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \frac{2}{N}\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}$ (SU(N) 2-index anti-sym. N-ality $=2$)

$$T_{\mu_2} = \frac{2(N-2)}{N-1} T_{\mu_1} \quad (< 2 T_{\mu_1})$$

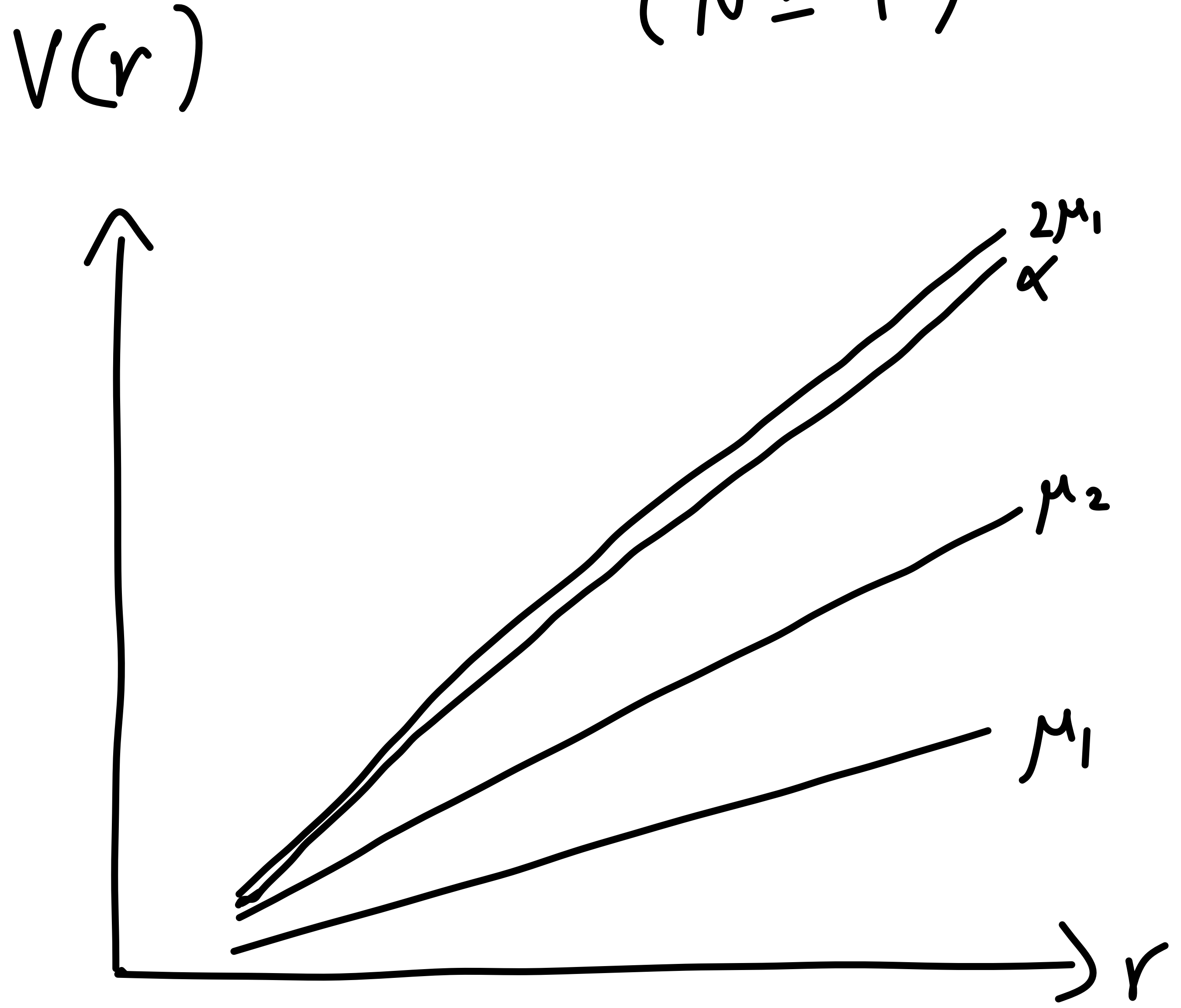
$\underline{2\mu_1}$ (SU(N) 2-index sym. N-ality $= 2$)

$$T_{2\mu_1} = 2 T_{\mu_1}$$

$\underline{\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}$ (SU(N) adjoint. N-ality $= 0$)

$$T_\alpha \simeq 2 T_{\mu_1}.$$

$(N \geq 4)$



Natural or unnatural ?

For $U(1)^{N-1}$ model, this is natural. We have $U(1)^{N-1}$ 1-form sym. generated by

$$U_\theta^{(k)}(M_1) = \exp\left(i \frac{\theta}{2\pi} \oint_{M_1} \alpha_k \cdot d\vec{\sigma} \right) \qquad (k=1,\cdots, N-1).$$

$\Rightarrow$ Infinitely many strings can be explained by center symmetry.

For $U(1)^{N-1} \rtimes S_N$ model, these operators are not gauge invariant.

We must consider a special combination

$$U_n(M_1) = \prod_{k=1}^{N-1} U^{(k)}_{\frac{2\pi}{N}nk}(M_1)$$

$$= \exp\left(i \frac{n}{N} \oint_{M_1} (\alpha_1 + 2\alpha_2 + \cdots + (N-1)\alpha_{N-1}) \cdot d\sigma \right).$$

This obeys the $\mathbb{Z}_N$ multiplication law :

$$U_n \, U_m = U_{n+m \bmod N}.$$

$\Rightarrow$ Only N distinct strings can be expected. Unnatural situation ??

Non-invertible symmetry

(= Non-group like)

We can construct another class of S_N-inv. operators from $U_\theta^{(k)}(M_1)$:

$$J_\theta(M_1) \equiv \frac{1}{N!} \sum_{P \in S_N} P\, U_\theta^{(1)}(M_1)\, P^{-1}$$

$$= \frac{1}{N(N-1)} \sum_{\alpha:\ \text{roots}} \exp\left(i \frac{\theta}{2\pi} \oint_{M_1} \alpha \cdot d\sigma\right).$$

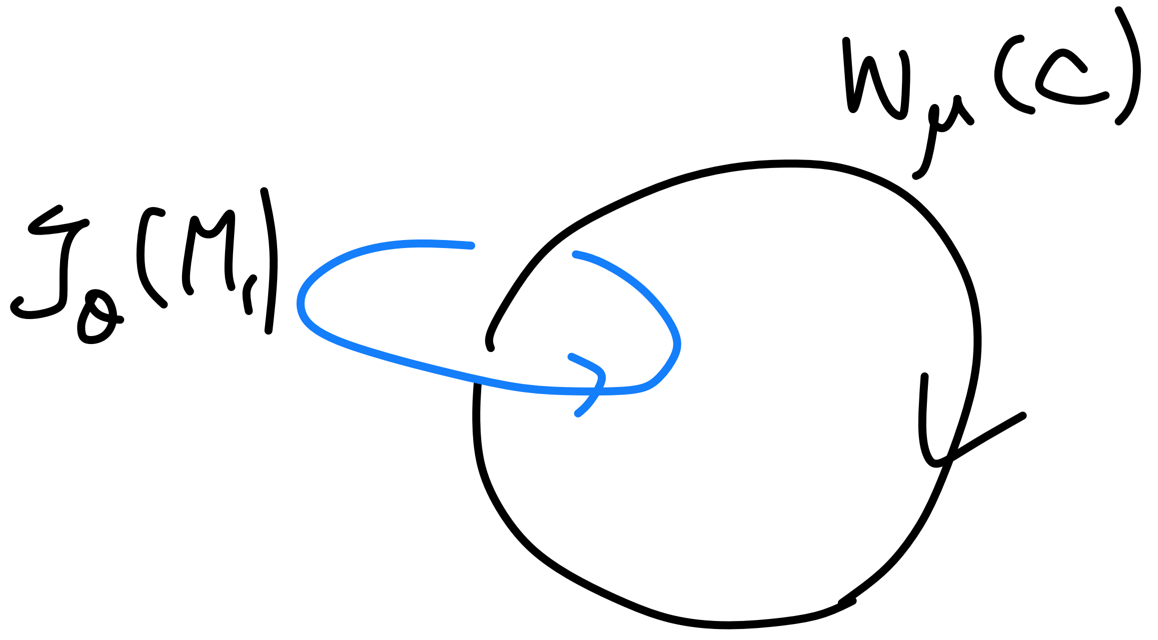
- $J_\theta(M_1)$ is topological $\Rightarrow$ "Symmetry"

- $J_\theta(M_1)$ does not form a group. $J_\theta \cdot J_{\theta'} \neq J_{\theta+\theta'}$
- Moreover, $J_\theta(M_1)$ does not correspond to unitary/anti-unitary operations.

Even though the last two properties are very unusual as symmetries, we regard J_θ as generators of "non-invertible symmetry".

(cf. Bhardwaj, Tachikawa '17, Buican, Gromov, '17, Thorngren, Wang '19, Komargodski, Ohmori, Roumpedakis, Seifnashri '20 etc.)
(for 2d QFTs)

Transformation by J_θ



$$= \left(\frac{1}{N(N-1)}\sum_\alpha e^{i\theta\,\vec{\alpha}\cdot\vec{\mu}}\right) \times W_\mu(C)$$

$\mu = \mu_1$ (N-ality 1)

$$W_{fd.} \longmapsto \frac{N-2(1-\cos(\theta))}{N} W_{fd.}$$

As J_θ is not unitary, its "eigenvalue" is smaller than 1.
Moreover, it can be 0 in some cases. ("non-invertible")

$\mu = \alpha_1$ (Adjoint rep. N-ality 0)

$$W_{adj.} \longmapsto \frac{(N-2)(N-3)+4(N-2)\cos(\theta)+2\cos(2\theta)}{N(N-1)} W_{adj.}$$

$\rightsquigarrow$ J_θ can distinguish the adj. rep. from the trivial rep.
Consistent with $T_\alpha \neq 0$.

Summary

① Construction of a toy model (3d semi-Abelian gauge theory)

- $G_{gauge} = U(1)^{N-1} \rtimes S_N$
- $\mathbb{Z}_N$ 1-form sym.

② Studying its string tensions (string tensions beyond N-ality)

- $T_{Adj} \simeq 2 T_{fd.} \neq 0$
- $U_1(M_1) = \exp\left(i \frac{1}{N} \oint_{M_1} (\alpha_1 + 2\alpha_2 + \cdots + (N-1)\alpha_{N-1}) \cdot d\sigma\right)$ detects N-ality.

③ "Symmetry" explanation on beyond N-ality (non-invertible sym. in 3d)

- $\mathcal{J}_\theta(M_1) = \frac{1}{N(N-1)} \sum_{\alpha:\text{roots}} \exp\left(i \frac{\theta}{2\pi} \int_{M_1} \alpha \cdot d\sigma\right)$

  is a topological, gauge-inv. operator.

- $\mathcal{J}_\theta \cdot W_{adj} = (\theta\text{-dep. factor}) \times W_{adj}$.

  $\Rightarrow$ Trivial N-ality can have nonzero string tensions.

Applicable to $SU(N)$ YM ? (This story is true for 2d pure YM. $\leftarrow$ 2104.01824)