Virtual Tribute to Confinement and Hadron structure 2021

On Boundary corrections of Lüscher-Weisz strings

Presenter

Mostafa N. Khalil

Department of Mathematics, Bergische Universität Wuppertal, Wuppertal, Germany.
Department of Physics, Al Azhar University, Cairo 11651, Egypt.
Department of Physics, University of Ferrara, Ferrara 44121, Italy.
Computation-based Science Research Center, Cyprus Institute, Nicosia 2121, Cyprus.

Co-authors:

BAKRY, A (Institute of Modern Physics, Chinese Academy of Sciences, Gansu 730000, China.) ;
CHEN, X (Institute of Modern Physics, Chinese Academy of Sciences, Gansu 730000, China.);
DELIYERGIYEV, Maksym (Institute of Physics, Jan Kochanowski University, Kielce, Poland.);
GALAL, Ahmed (Department of Physics, Al Azhar University, Cairo 11651, Egypt.);
KHALAF, Ali (Department of Physics, Al Azhar University, Cairo 11651,Egypt.);
WILLIAMS, A (ARC Center for Particle Physics, University of Adelaide, SA 5005, Australia.);
ZHANG, P (Department of physics, Sun Zhongshan University, Gaungdong 510275, China.)

Main points

Lüscher-Weisz (LW) effective string with boundary action.

The Casimir energy of the QCD flux-tube considering two boundary terms.

The modulation of the flux-tube profile by virtue of the boundary terms is derived at both b_2 and b_4 orders

The numerical simulations on the 4-dim pure SU(3) Yang-Mills lattice gauge theory at finite temperature are described.

The signatures of the two boundary terms of the string action on both the static $Q\bar{Q}$ are scrutinized.

Energy density in the presence of $Q\bar{Q}$ is fitted to the width of the Lüscher-Weisz string with two or one boundary term over distance scales $R \geq 0.5$ fm .

LW Effective action

The formation of a string-like condensate in the Yang-Mills vacuum spontaneously breaks the transnational and rotational symmetries.

The symmetry breaking generates $(d-2)$ massless transverse self-interacting modes by virtue of the Goldstone theorem.

An effective string action can be constructed from the derivative expansion of collective string co-ordinates satisfying Poincare and parity invariance.

One particular form is the Lüscher and Weisz (JHEP2002-07049) action which up to four-derivative term reads as

$$S^{\mathrm{LW}}[X] = \sigma_0 A + \frac{\sigma_0}{2}\int d^2\zeta \left[\left(\frac{\partial X}{\partial \zeta_\alpha}\cdot\frac{\partial X}{\partial \zeta_\alpha}\right) + \kappa_2 \left(\frac{\partial X}{\partial \zeta_\alpha}\cdot\frac{\partial X}{\partial \zeta_\alpha}\right)^2 + \kappa_3 \left(\frac{\partial X}{\partial \zeta_\alpha}\cdot\frac{\partial X}{\partial \zeta_\beta}\right)^2 \right] +$$
$$... + \gamma \int d^2\zeta \sqrt{g}\; \mathcal{R} + \alpha \int d^2\zeta \sqrt{g}\; \mathcal{K}^2 + S^b,$$

the vector $X^\mu(\zeta_0,\zeta_1)$ maps the region $\mathcal{C}\subset\mathbb{R}^2$ into $\mathbb{R}^4$.

The effective LW action

$$S^{\mathrm{LW}}[X] = \sigma_0 A + \frac{\sigma_0}{2}\int d^2\zeta \left[\left(\frac{\partial X}{\partial \zeta_\alpha} \cdot \frac{\partial X}{\partial \zeta_\alpha} \right) + \kappa_2 \left(\frac{\partial X}{\partial \zeta_\alpha} \cdot \frac{\partial X}{\partial \zeta_\alpha} \right)^2 + \kappa_3 \left(\frac{\partial X}{\partial \zeta_\alpha} \cdot \frac{\partial X}{\partial \zeta_\beta} \right)^2 \right] +$$

$$... + \gamma \int d^2\zeta \sqrt{g}\; \mathcal{R} + \alpha \int d^2\zeta \sqrt{g}\; \mathcal{K}^2 + S^b,$$

In the physical gauge $X^1 = \zeta_0, X^4 = \zeta_1$ which restrict the string fluctuations to transverse directions of the world-sheet $\mathcal{C}$.

g is the two-dimensional 'induced' metric on the world sheet embedded in the Eculidean background $\mathbb{R}^4$.

The world-sheet area A and the parameters σ_0, α and γ are the string tension, the rigidity and the self-interaction number, respectively.

Invariance under party transform would keep only terms of even number of derivatives.

The last two terms in the action

$$......+\gamma\int d^2\zeta\sqrt{g}\ \mathcal{R}+\alpha\int d^2\zeta\sqrt{g}\ \mathcal{K}^2+.....,$$

The extrinsic curvature is defined as

$$\mathcal{K}=\frac{1}{\sqrt{g}}\partial_\alpha[\sqrt{g}g^{\alpha\beta}\partial_\beta]X.$$

and inner curvatures

$$\mathcal{R}=\left[g^{\alpha\beta}g^{\gamma\eta}-g^{\alpha\eta}g^{\beta\gamma}\right]\nabla_\alpha\nabla_\beta X_\mu(\nabla_\gamma)^2X^\mu,$$

The two geometrical terms satisfy the Poincare and parity invariance and lies within the general class of (LW) string actions.

The effects on the two terms on the energy-density profile and static potential versus abel;ian and Yang-Mills data have been reported in detail in

M. Caselle, M. Panero, R. Pellegrini, D. Vadacchino (2015) JHEP 01:105.
BB. Brandt (2017) JHEP 07:008.
A. Bakry, M. Deliyergiyev, A. Galal, M. Khalil, A. Williams, [arXiv:hep- lat/2001.04203], [arXiv:hep-lat/2001.02392], [arXiv:hep-th/1709.09446].

The boundary action

$$S_{b_i} = \int_{\partial\Sigma} d\zeta^0 \mathcal{L}_i,$$

with the Lagrangian density $\mathcal{L}_i$ for each effective low-energy coupling b_i.

The leading-order corrections due to second boundary terms with the coupling b_2 appears at the four derivative term in the bulk.

$$\mathcal{L}_2 = b_2 \partial_0 \partial_1 \mathbf{X} \cdot \partial_0 \partial_1 \mathbf{X} \ ,$$

The variation of the boundary actions with Lorentz transform of the Lagrangian densities entails a vanishing value for $b_1 = 0$ and $b_3 = 0$.

The next order, $\mathcal{L}_4$, the general effective Lagrangian on the boundary is

$$\mathcal{L}_4 = b_4 \partial_0^2 \partial_1 \mathbf{X} \cdot \partial_0^2 \partial_1 \mathbf{X}.$$

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(Billo M, Caselle M, Gliozzi F, Meineri M, Pellegrini R (2012) The Lorentz-invariant boundary action of the confining string and its universal contribution to the inter-quark potential. JHEP 05:130)

The Static $Q\bar{Q}$ potential

The Casimir energy is extracted from the string partition function as

$$V(R,T) = -\frac{1}{L_T}\log(Z(R,L_T)).$$

The partition function of the NG model in the physical gauge is a functional integrals over all the world sheet, of area $A = L_T R$ configurations swept by the string

$$Z(R,T) = \int_{\mathcal{C}} [D\,X]\,\exp(\,-S_{\ell o}(X)).$$

The static potential (JHEP2002 07049) at leading order is given by

$$V_{\ell o}^{\rm NG}(R,L_T) = \sigma_0 R + (d-2)L_T\,\log\eta\,(\tau) + \mu,$$

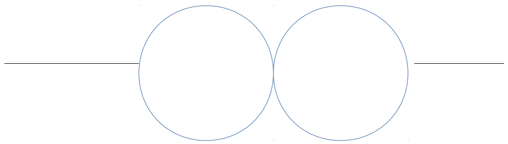
where μ is a UV-cutoff and η is the Dedekind η function defined on the real axis as

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n).$$

The second term on the right hand side encompasses the Lüscher term of the interquark potential.

The partition function of the next to leading order term is extracted the explicit calculation of the two-loop approximation using the ζ regularization scheme.

$$Z = \int DX e^{-S^{NG}_{\ell o} - S^{NG}_{n\ell o}}$$



The static potential of NG string at second loop order is given by

$$V^{\mathrm{NG}}_{\mathrm{n\ell o}}(R, L_T) = \sigma_0 R + (d-2) L_T \, \log \eta\,(\tau) \\ - L_T \log \left(1 - \frac{L_T}{R^3} \frac{(d-2)\pi^2}{1152\sigma_0} \left[2E_4(\tau) + (d-4)E_2^2(\tau) \right] \right) + \mu,$$

with E_2 and E_4 are the second and forth-order Eisenstein series. *

K. Dietz and T. Filk Phys. Rev. D 27(12)(1983):29442955.

$$Z = \int DX e^{-S^{NG}_{\ell o} - S_{b_2} - S_{b_4}}$$

Expanding around the free action yields

$$Z = Z^0 \left((1 - \langle S_{b_2} \rangle - \langle S_{b_4} \rangle) + \frac{1}{2} \left\langle (S_{b_2} + S_{b_4} + ...)^2 \right\rangle \right) + ...$$

The Lorentzian-Invariance imply $b_1 = 0, b_3 = 0$, the next two non-vanishing Lorentzian-Invariant terms come at order four and six derivative terms at coupling b_2

$$\langle S_{b_2} \rangle = b_2 \int_{\partial\Sigma} d\zeta_0 \, \langle \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X \ \rangle \, ,$$

and coupling b_4

$$\langle S_{b_4} \rangle = b_4 \int_{\partial\Sigma} d\zeta_0 \, \langle \partial_0^2 \partial_1 X \cdot \partial_0^2 \partial_1 X \rangle \, .$$

The modification to the potential received when considering Dirichlet boundary condition are given by

$$V^{b_2} = b_2 (d-2) \frac{\pi^3 L_T}{60 R^4} E_4 \left(q \right) ,$$

$$V^{b_4} = \frac{-b_4 (d-2) \pi^5 L_T}{126 R^6} \mathbf{E}_6 \left(\tau \right) .$$

*

(Billo M, Caselle M, Gliozzi F, Meineri M, Pellegrini R (2012) The Lorentz-invariant boundary action of the confining string and its universal contribution to the inter-quark potential. JHEP 05:130)

Perturbative width of the Energy-density

The mean-square width of the string is defined as the second moment of the field with respect to the center of mass of the string X_0

$$W^2(\zeta) = \frac{\int DX(\mathbf{X}(\zeta^1,\zeta^0) - \mathbf{X_0})^2 e^{-S[\mathbf{X}]}}{\int D\mathbf{X} e^{-S[\mathbf{X}]}},$$

Expanding around the free-string action

$$W^2(\zeta^1) = W^2_{\ell o}(\zeta^1) - \langle \mathbf{X}^2(\zeta^0,\zeta^1) S^{\mathrm{Pert}} \rangle_0 + 2\gamma \langle \partial_\alpha \mathbf{X}^2(\zeta^0,\zeta^1)\rangle_0 + \gamma^2 \langle \partial^2_\alpha \mathbf{X}^2(\zeta^0,\zeta^1)\rangle_0 - \frac{\beta^2}{L_T R} \int d\zeta^0 \; d\zeta^1 \; d\zeta^{0'} \; d\zeta^{1'} \langle \partial^2_\alpha \mathbf{X}(\zeta^0,\zeta^1) \cdot \partial^2_{\alpha'} \mathbf{X}(\zeta^{0'},\zeta^{1'})\rangle_0.$$

where the vacuum expectation value $\langle ... \rangle_0$ **is with respect to the free-string partition function,** γ **is a low energy parameter of effective action.**

The next-to-leading NG term combined with the expansion of the surface term on the boundary

$$S^{\mathrm{Pert}}[\mathbf{X}] = S_{n\ell o} + S_b, \\ = (S_{n\ell o} + S_{b_2} + S_{b_4} + ..) + ...,$$

The Green-function defines the two-point free propagator

$$G\left(\zeta^0,\zeta^1;\zeta^{0'},\zeta^{1'}\right)=\langle\mathbf{X}(\zeta^0,\zeta^1)\cdot\mathbf{X}(\zeta^{0'},\zeta^{1'})\rangle,$$

which is the solution of Laplace equation on a cylindrical sheet of surface area RL_T.
The expectation value of the mean-square width corresponds to the Green-function correlator of the free bosonic string theory in two dimensions

$$W^2_{\ell o}(\zeta_1,\tau)=\frac{d-2}{2\pi\sigma_0}\log\left(\frac{R}{R_0(\zeta_1)}\right)+\frac{d-2}{2\pi\sigma_0}\log\left|\frac{\vartheta_2(\pi\,\zeta_1/R;\tau)}{\vartheta_1'(0;\tau)}\right|,$$

where θ are Jacobi elliptic functions

$$\theta_1(\zeta;\tau)=2\sum_{n=0}^{\infty}(-1)^n q_1^{n(n+1)+\frac{1}{4}}\sin((2n+1)\,\zeta),$$

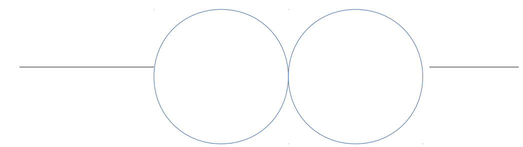
$$\theta_2(\zeta;\tau)=2\sum_{n=0}^{\infty}q_1^{n(n+1)+\frac{1}{4}}\cos((2n+1)\zeta),$$

with $q_1=e^{\frac{i\pi}{2}\tau}$, and R_0^2 is the UV cutoff which has been generalized to be dependent on distances from the sources.
Allais A, Caselle M (2009) On the linear increase of the flux tube thickness near the deconfinement transition. JHEP 01:073.

The low-energy parameter expansion of the mean-square width

$$
\begin{aligned}
W^2(R,L_T) =& \left\langle (\mathbf{X}^2(\zeta^1,\zeta^0) S_{\ell o}) \right\rangle \\
& - \left\langle (\mathbf{X}^2(\zeta^1,\zeta^0) S_{n\ell o} + \mathbf{X}^2(\zeta^1,\zeta^0) S_{b_2} + \mathbf{X}^2(\zeta^1,\zeta^0) S_{b_4} + \ldots) \right\rangle , \\
=& W^2_{\ell o}(R,L_T) + W^2_{n\ell o} + W^2_b(R,L_T) + \ldots
\end{aligned}
$$

$$
W^2(\zeta) = \frac{\int DX(\mathbf{X}(\zeta^1,\zeta^0) - \mathbf{X_0})^2 e^{-S_{n\ell o}[\mathbf{X}]}}{\int D\mathbf{X} e^{-S[\mathbf{X}]}},
$$



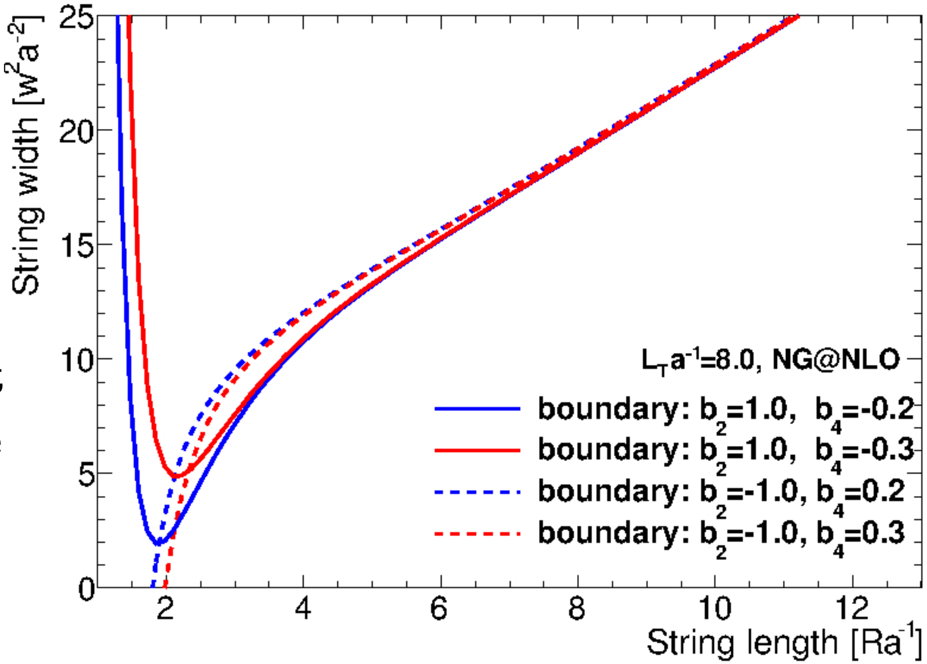
The width due to the two loop self-interaction is modified by

$$
\begin{aligned}
W^2_{n\ell o} =& \frac{(d-2)\pi}{12\sigma^2 R^2} \left\{ \tau \left(q\frac{d}{dq} - \frac{d-2}{12} E_2(\tau) \right) [E_2(2\tau) - E_2(\tau)] - \frac{d-2}{8\pi} E_2(\tau) \right\} \\
& + \frac{\pi}{12\sigma R^2} [E_2(\tau) - 4E_2(2\tau)] \left(W^2_{lo} - \frac{d-2}{4\pi\sigma} \right).
\end{aligned}
$$

Gliozzi F, Pepe M, Wiese UJ (2010) The Width of the Color Flux Tube at 2-Loop Order. JHEP 11:053. Gliozzi F, Pepe M, Wiese UJ (2010) The Width of the Confining String in Yang-Mills Theory. Phys. Rev. Lett. 104:232001.

The mean-square width $W_{b_2}^2$

$$\mathbf{W}_{b_2}^2 = \frac{-\pi b_2(d-2)}{4R^3\sigma^2}\left(\frac{1}{8} - \frac{1}{24}E_2(2\tau)\right),$$



On adjancent figure we compare modificat profile received from the depicted positive of the coupling b_2 and b_4.

The boundary term S_{b_2} in the effective string action may increase or decrease the mean-square width of NG string depending on whether positive or negative values of the coupling parameter b_2 are considered.

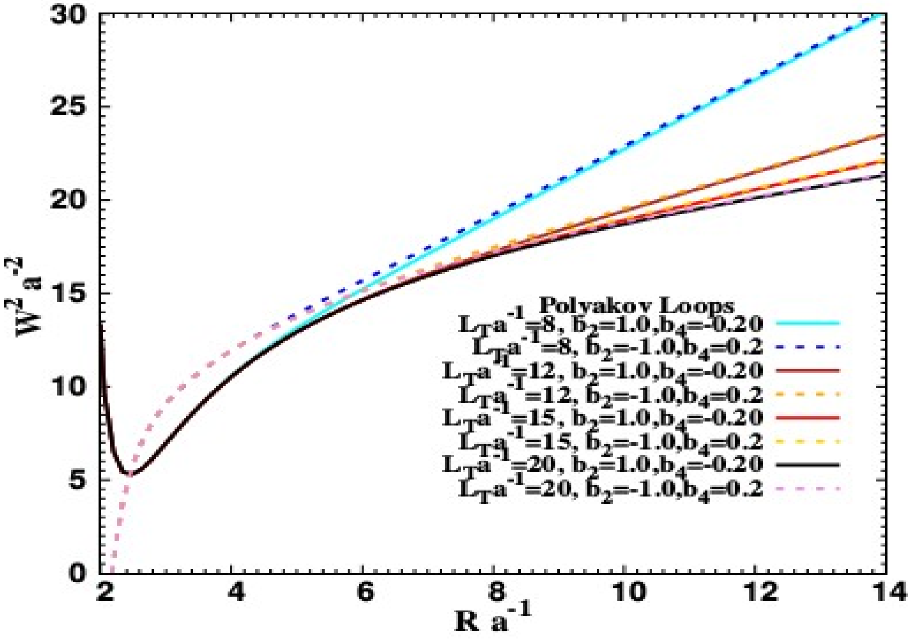
However, the resultant effect on the width depends on a compromisation between the value of the two couplings b_2 and b_4 with signs.

A. S. Bakry, M. A. Deliyergiyev, A. Galal, M. N. Khalil, and A. G. Williams (2019), Boundary action and profile of effective bosonic strings, arXiv:1912.13381 [hep-th]

The modification of the width at fourth order is

$$W^2_{b_4} = \frac{-\pi^3(d-2)b_4}{32R^5\sigma^2}\left(\mathbf{E}_2(\tau)-\frac{5}{4}\right)\left(\frac{11\mathbf{E}_2(\tau/2)}{36}-\frac{5\mathbf{E}_2(\tau)}{9}-\frac{55}{12}\right),$$

As expected from the dimensional considerations the remarkable effects of the corrections occur over the intermediate and short string length scale.



The diffusion of the interaction from the sources at the boundaries along the string is more stringent for relatively short string length.

For the values of b_4 considered the S_{b_4} boundary term in the action appears to dominate and can fine tune the width at shorter distances.

Numerical Results

Simulation Setup

We perform simulations on large enough lattice sizes to gain high statistics. The two lattices employed in this investigation are of a typical spatial size of 3.6^3 fm^3 with a lattice spacing $a = 0.1$ fm.

We choose to perform our analysis with lattices with temporal extents of $N_t = 8$, and $N_t = 10$ slices at a coupling of value $\beta = 6.00$. The two lattices correspond to temperatures $T/T_c = 0.9$ just before the deconfinement point, and $T/T_c = 0.8$ near the end of QCD plateau.

The gauge configurations were generated using the standard Wilson gauge-action employing a pseudo-heatbath (FHKP) updating to the corresponding three $SU(2)$ subgroup elements(Marinari). Each update step/sweep consists of one heatbath and 5 micro-canonical reflections.

The gauge configurations are thermalized following 2000 sweeps. The measurements are taken on 500 bins. Each bin consists of 4 measurements separated by 70 sweeps of updates.

(a) Static $Q\bar{Q}$ potential

The Monte-Carlo evaluation of the $Q\bar{Q}$ potential at R and $T = \frac{1}{L_T}$ is

$$\mathcal{P}_{2\mathrm{Q}} = \int d[U]\, P(0)\, P^{\dagger}(R)\, \exp(-S_w), \\ = \exp(-V(R,T)/T),$$

with the Polyakov loop defined as

$$P(\vec{r}_i) = \frac{1}{3}\mathbf{Tr}\left[\prod_{n_{t=1}}^{N_t} U_{\mu=4}(\vec{r}_i, n_t)\right].$$

The correlator equation is evaluated after integrating the time links

$$\bar{U}_t = \frac{\int dU U e^{-Tr(Q\,U^{\dagger}+U\,Q^{\dagger})}}{\int dU e^{-Tr(Q\,U^{\dagger}+U\,Q^{\dagger})}}.$$

P. deForcrand, C. Roiesnel, Phys. Lett. B151, 7780 (1985); G. Parisi, et al, Phys. Lett. B128, 418 (1983),

The $Q\bar{Q}$ potential reads

$$V_{Q\bar{Q}}(R) = -\frac{1}{T}\log\langle P(x)P(x+R)\rangle.$$

(b) Width of energy density

The action density is related to the chromo-electromagnetic fields via $\frac{1}{2}(E^2 - B^2)$ and is evaluated via a three-loop improved lattice field-strength tensor.

A scalar field characterizing the action density distribution in the Polyakov vacuum or in the presence of color sources can be defined as

$$\mathcal{C}(\vec{\rho};\vec{r}_1,\vec{r}_2) = \frac{\langle \mathcal{P}_{2Q}(\vec{r}_1,\vec{r}_2)\, S(\vec{\rho})\,\rangle}{\langle\, \mathcal{P}_{2Q}(\vec{r}_1,\vec{r}_2)\,\rangle\ \langle S(\vec{\rho})\,\rangle},$$

with the vector $\vec{\rho}$ referring to the spatial position of the energy probe with respect to some origin, and the bracket $\langle ... \rangle$ stands for averaging over gauge configurations and lattice symmetries.

We im lement a number of $n_{sw} = 20$ of improved cooling sweeps to elimi ate statistical fl ctuations. That ought to keep the physical observable intact over source separation distances $R \geq 0.5$ fm.

A. Bakry, X. Cjhen, PM Zhang, Noise reduction by combining smearing with multi-level integration methods, Int.J.Mod.Phys.E 23 (2014) 1460008 Contribution to:ISPP 13

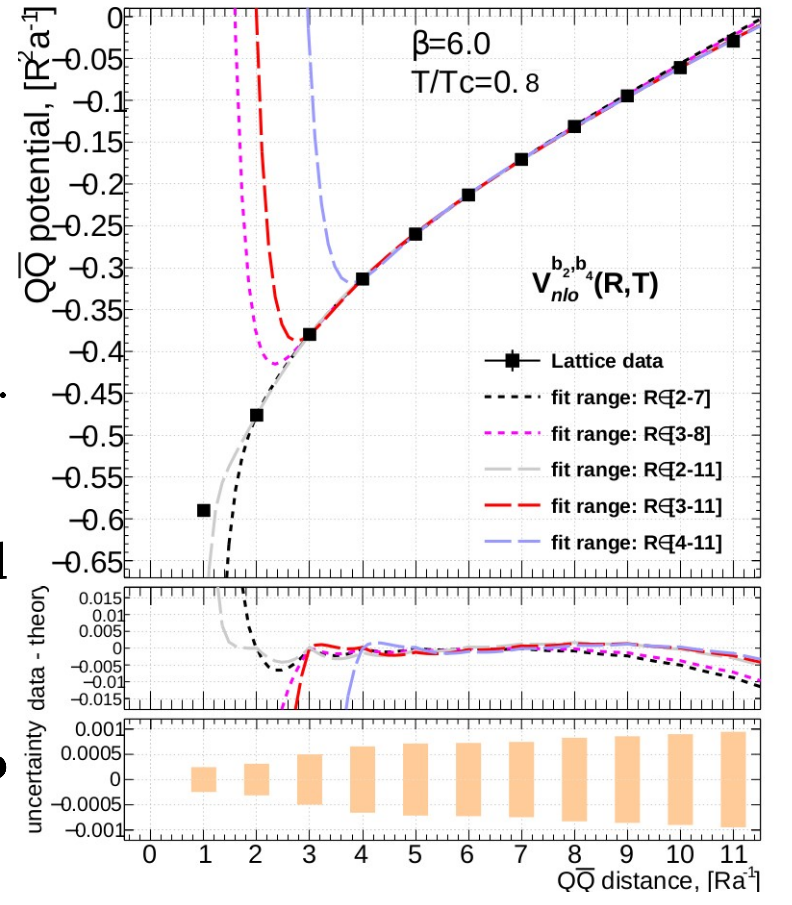
$T = 0.8\,T_c$

The $Q\bar{Q}$ potential data are fitted using

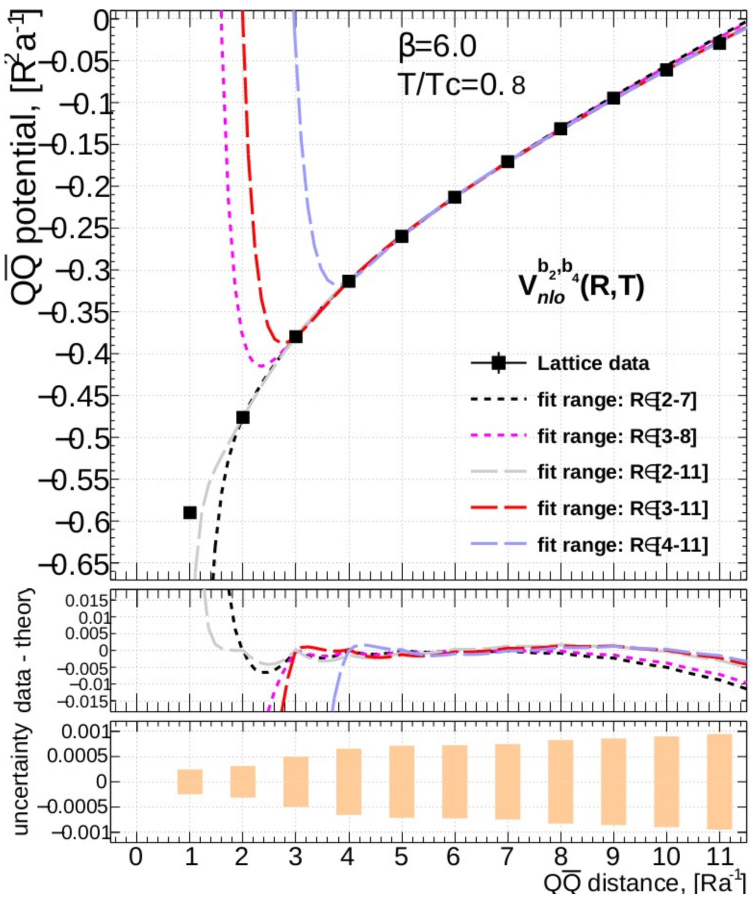
$$V^{b_2}_{\ell o, n\ell o} = V_{\ell o, n\ell o} + V^{b_2},$$
$$V^{b_2,b_4}_{\ell o, n\ell o} = V_{\ell o, n\ell o} + V^{b_2} + V^{b_4}.$$

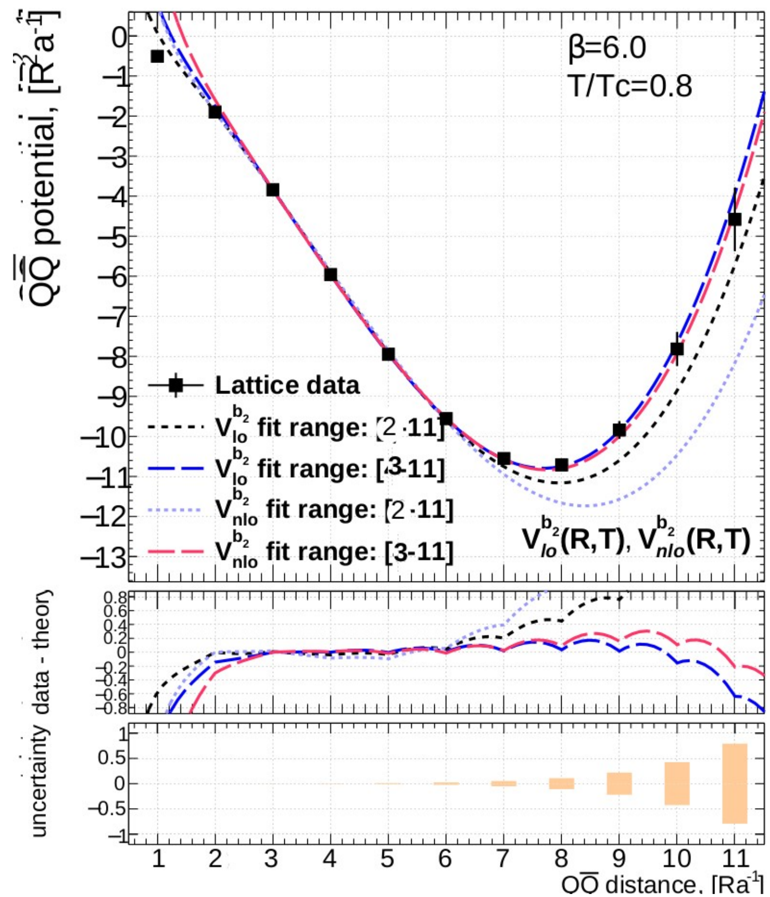
The minimal residuals of the fits over the interval $R \in [0.2, 1.1]$ fm, $\chi^2_{dof} = 1.37$, is produced using the ansatz V^{b_2,b_4} with two boundary terms.

The string self-interaction $V_{n\ell o}$ with have no significant reduction of the residuals over most source separation.



		Fit Interval $R \in I$	Fit Parameters, $T/T_c = 0.8$ $\sigma_0 a^2$	μ(LU)	b_2(LU)	b_4(LU)	χ^2_{dof}
$V^{b_2}_{\ell o}$	(a)						
		[2,11]	0.0422(1)	-0.4767(6)	-0.148(1)	0.0	197.566
		[3,11]	0.0452(2)	-0.489(1)	-0.607(9)	0.0	5.91231
$V^{b2,b_4}_{\ell o}$	(b)						
		[2,11]	0.0452(3)	-0.496(2)	-0.44(2)	0.10(1)	1.37428





The figure compares considering the fitted potential using one V^{b_2} and two V^{b_2,b_4} boundary terms.

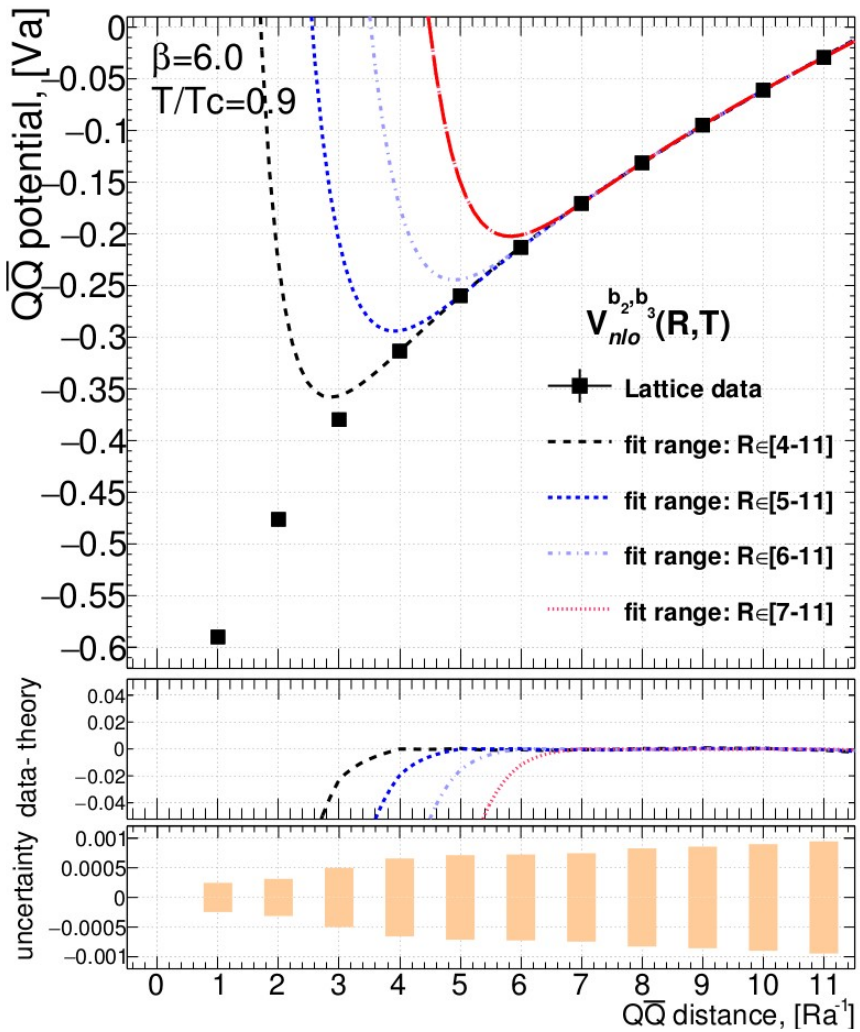
$$V_{\ell o}^{b_2} = V_{\ell o} + V^{b_2},$$
$$V_{\ell o}^{b_2,b_4} = V_{\ell o} + V^{b_2} + V^{b_4}.$$

The residuals noticeably decrease over the fit intervals including points at short distances such as $R \in [0.3, 1.1]$ fm.

A. Bakry, M. Deliyergiyev, A. Galal, M. Khalil, A. Williams, On QCD Strings beyond non-interacting model, [arXiv:hep- lat/2001.04203]

$T = 0.9T_c$

Reduction in the values of χ^2 observed with V^{b_2,b_4} compared to one boundary term V^{b_2}

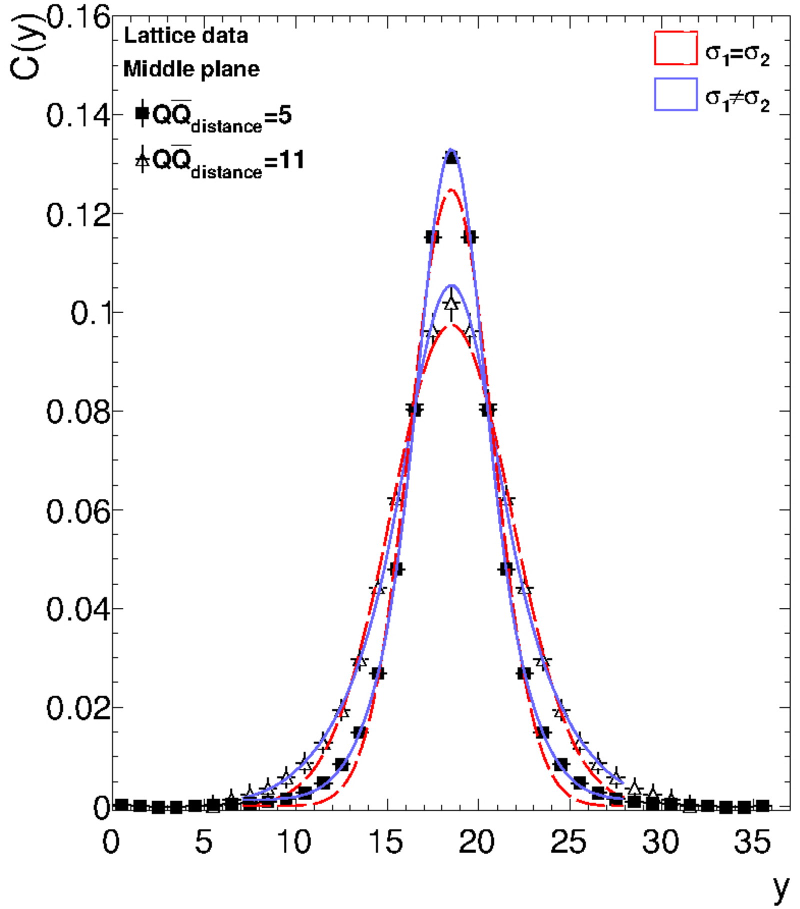


Fits are not returning $\sigma_0 a^2 = 0.045$, thus, other effects such as rigidity ought to be considered.

The χ^2 values and the corresponding fit parameters b_2, b_4 and μ returned from fits to the next-to-leading order (NLO) static potential with boundary terms $V_{n\ell o}^{b_2,b_4}$.

$V_{n\ell o}^{b_2,b_4}$	Fit Interval $R \in I$	Fit Parameters, $T/T_c = 0.9$ $\sigma_0 a^2$	μ(LU)	b_2(LU)	b_4(LU)	χ^2
$V_{n\ell o} + V^{b_2} + V^{b_4}$	$[R_m, R_M]$					
	[2,11]	0.04123(1)	-0.3991(2)	-0.236(5)	0.023(2)	2452.4
	[3,11]	0.04091(2)	-0.332(2)	2.61(9)	-1.84(6)	1511.9
	[4,11]	0.04053(2)	0.97(5)	58.1 (2.1)	-44.7(1.6)	805.8
	[5,11]	0.0400(3)	27.7 (12.4)	1185.4 (520.8)	-948.9 (417.7)	336.6
	[6,11]	0.0394(5)	553.3 (334.4)	23246.0 (14037.0)	-18790.0 (11352.0)	89.2
	[4,9]	0.04142(4)	0.12(6)	22.3 (2.5)	-17.2 (1.9)	41.3

Energy-density profile



We implement a number of $n_{sw} = 20$ of improved cooling sweeps to eliminate statistical fluctuations. That ought to keep the physical observable intact* over source separation distances $R \geq 0.5$ fm.

We choose a double Gaussian function of the same amplitude A_0 and mean $\mu_0 = 0$

$$G(r, \theta; z) = A(e^{-r^2/\sigma_1^2} + e^{-r^2/\sigma_2^2})$$

to estimate the mean-square width defined as

$$W^2(z) = \frac{\int dr\, r^3\, G(r, \theta; z)}{\int dr\, r\, G(r, \theta; z)}, \qquad (2)$$

where the loci of the color sources correspond to $z = 0$ or $z = R$, respectively.

A. Bakry, X. Chen, PM Zhang, Noise reduction by combining smearing with multi-level integration methods, Int.J.Mod.Phys.E 23 (2014) 1460008 Contribution to:ISPP 13

We investigate the pattern of growth of the width of the energy-density as the color sources are pulled apart.

We draw a comparison between the analytic estimate of the mean-square width of Lüscher-Weisz (LW) string action with the boundary corrections.

We consider some possibly interesting combinations of boundary corrections with both the leading and next-to-leading Nambu-Goto string's width

$$
\begin{aligned}
W^2 &= W^2_{NG_{(\ell o)}} + W^2_{b_2}, \\
W^2 &= W^2_{NG_{(\ell o)}} + W^2_{NG_{(n\ell o)}} + W^2_{b_2}, \\
W^2 &= W^2_{NG_{(\ell o)}} + W^2_{b_2} + W^2_{b_4}, \\
W^2 &= W^2_{NG_{(\ell o)}} + W^2_{NG_{(n\ell o)}} + W^2_{b_2} + W^2_{b_4}.
\end{aligned}
$$

with

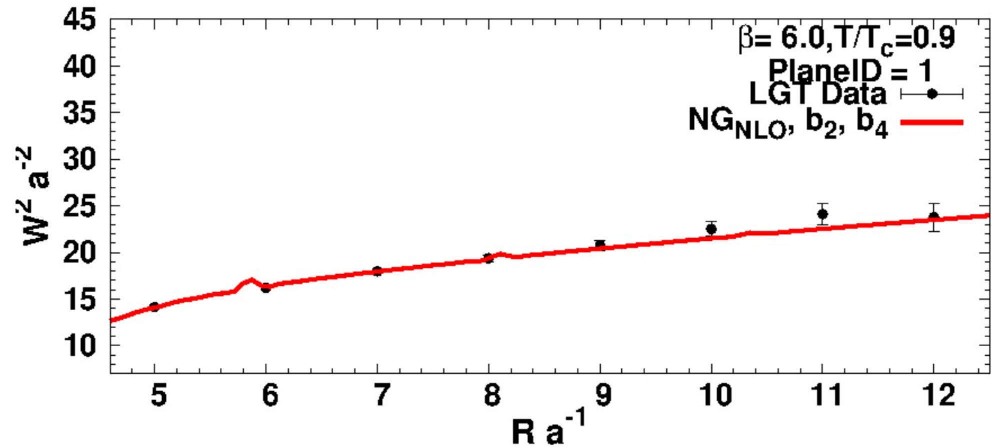
$$
W^2_{b_2} = \frac{-\pi b_2 (d-2)}{4R^3\sigma^2}\left(\frac{1}{8} - \frac{1}{24}E_2(2\tau)\right),
$$

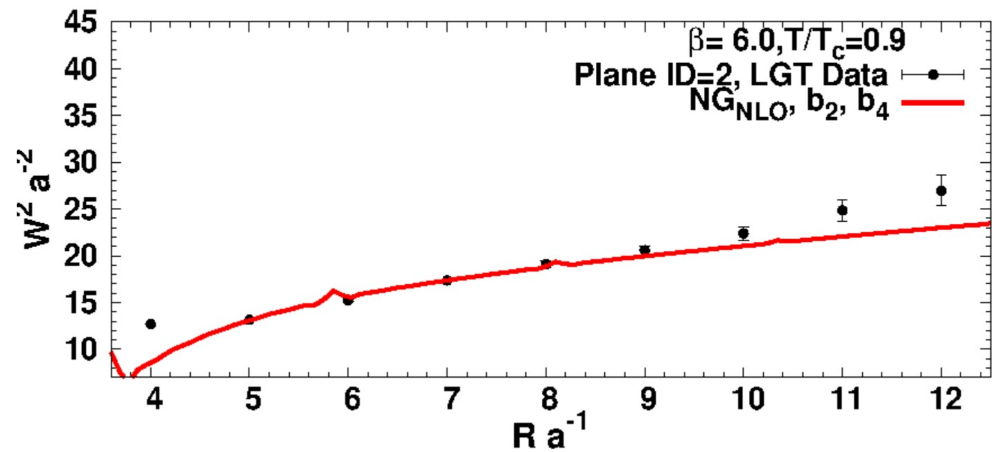
and

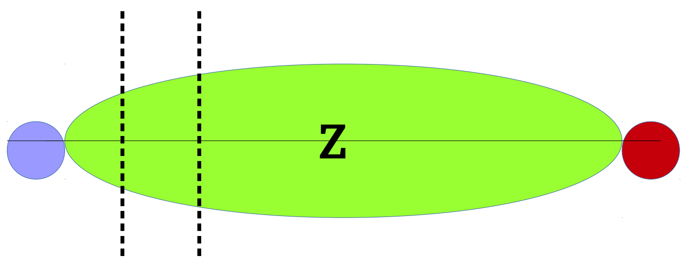
$$
W^2_{b_4} = \frac{-\pi^3 (d-2) b_4}{32 R^5 \sigma^2}\left(\mathbf{E}_2(\tau) - \frac{5}{4}\right)\left(\frac{11\mathbf{E}_2(\tau/2)}{36} - \frac{5\mathbf{E}_2(\tau)}{9} - \frac{55}{12}\right),
$$

The consideration of $W_{b_4}^2$ correction had improved the fits over intervals from $R = 0.4$ fm, even without considering the NLO term from NG string.

For longer distances the inclusion of the boundary corrections sensibly removes deviations from the lattice data at planes $z = 2, z = 3$.

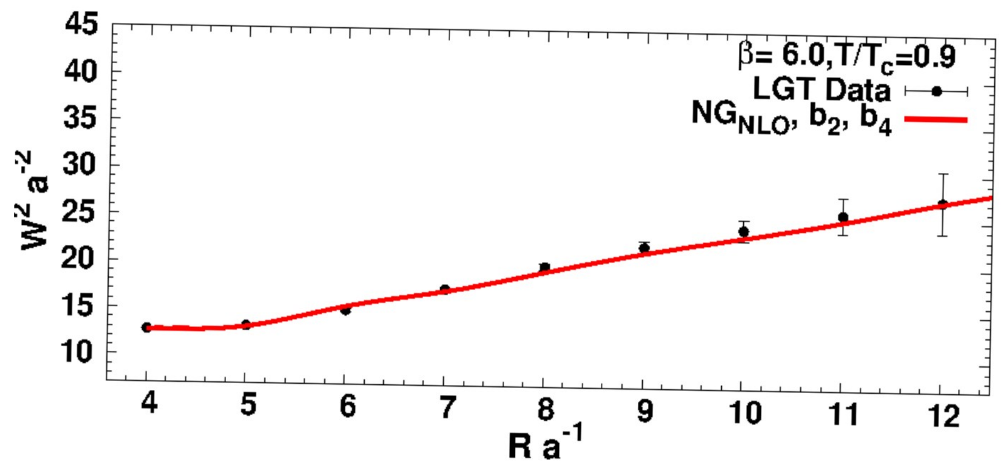




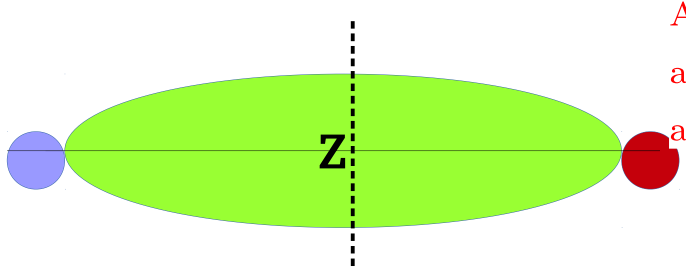


	$R \in [0.5, 1.2]$ fm				$R \in [0.6, 12]$ fm			
Plane,NG-order	χ^2	R_0	b_2	b_4	χ^2	R_0	b_2	b_4
$z = 1, NG_{\ell o}$	52.2	12(1.7)	33(31)	-18(27.8)	0.70	25(2.5)	-121(38.0)	131(34.8)
$NG_{n\ell o}$	5.66	14(2.2)	-11(34.5)	7(28.7)	1.43	17(2.5)	-36(36.7)	31(30.8)
$z = 2, NG_{\ell o}$	92.5	4.8(5)	62(27.2)	-45(24.3)	1.5	22(2.4)	-153(35)	161(32.5)
$NG_{n\ell o}$	27.5	7(1.9)	4(30.8)	-0.5(25.7)	3.08	14(2.4)	-67(34.1)	63(28.7)
$z = \frac{R}{2}, NG_{\ell o}$	24.20	-4(1.7)	137(29.9)	-119(26.6)	0.29	12(3.6)	-59(49.9)	67(46.3)
$\tilde{\ } NG_{n\ell o}$	6.96	-4(2.3)	108(35.7)	-89(29.7)	0.07	3(3.6)	25(47.5)	-17(40.3)

The resultant χ^2 and fit parameters are enlisted below indicates better match when including the first boundary correction $W^2_{b_2}$ compared to the fits along the pure leading (NG) model.



The values of χ^2 is further reduced upon switching on the string's self-interaction together with the boundary corrections.



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	$R \in [0.5, 1.2]$ fm				$R \in [0.6, 12]$ fm			
Plane,NG-order	χ^2	R_0	b_2	b_4	χ^2	R_0	b_2	b_4
$z=1$, $NG_{\ell o}$	52.2	12(1.7)	33(31)	-18(27.8)	0.70	25(2.5)	-121(38.0)	131(34.8)
$NG_{n\ell o}$	5.66	14(2.2)	-11(34.5)	7(28.7)	1.43	17(2.5)	-36(36.7)	31(30.8)
$z=2$, $NG_{\ell o}$	92.5	4.8(5)	62(27.2)	-45(24.3)	1.5	22(2.4)	-153(35)	161(32.5)
$NG_{n\ell o}$	27.5	7(1.9)	4(30.8)	-0.5(25.7)	3.08	14(2.4)	-67(34.1)	63(28.7)
$z=\frac{R}{2}$, $NG_{\ell o}$	24.20	-4(1.7)	137(29.9)	-119(26.6)	0.29	12(3.6)	-59(49.9)	67(46.3)
~ $NG_{n\ell o}$	6.96	-4(2.3)	108(35.7)	-89(29.7)	0.07	3(3.6)	25(47.5)	-17(40.3)

Concluding remarks

(Static $Q\bar{Q}$ potential)

Near the end of QCD Plateau region the boundary corrections show good agreement with the static $Q\bar{Q}$ potential for color source separation as short as $R = 0.3$ fm.

As we approach the deconfinement point $T/T_c = 0.9$, the fits show reduction of the residuals by virtue of of the Lüscher-Weisz corrections. However, deviations from the returned string tension $\sigma_0 a^2$ persist.

(Energy-density profile)

For the energy width profile at $T/T_c = 0.9$, We find that the consideration of either one boundary term with Lüscher-Weisz action at two-loop order; or otherwise two boundary terms with the bulk action at one-loop order displays a good match with the LGT data $R \geq 0.5$ fm.