Fermion & gluon spectral functions far from equilibrium

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A Virtual Tribute to Quark Confinement and the Hadron Spectrum 2021, August 02, 2021

Talk mainly based on:

KB, Lappi, Mace, Schlichting, 2106.11319

Further literature (gluon spectral functions):
KB, Kurkela, Lappi, Peuron, 1804.01966; 2101.02715
1 Motivation

2 Method & setup

3 Nonperturbatively computed fermion spectral functions

4 Comparing to gluonic spectral functions

5 Conclusion
Highly occupied systems

Nonperturbatively strong bosonic fields can be encountered in
- Heavy-ion collisions at early times
- From instabilities during cosmological reheating
- IR sector of gluons and scalars

Understanding interactions with fermions important

Some applications in heavy-ion collisions:
- jets (energy loss, jet quenching)
- electromagnetic observables
Spectral functions

For microscopic description of observables and dynamics

Spectral function $\rho(\omega, p)$ of fermions encodes interactions

- Our approach: far from equilibrium
  - Highly occupied gluon plasma ($A \sim 1/g$), weak coupling ($g^2 \ll 1$)
  - Then nonperturbative and perturbative methods available!

- Classical-statistical lattice simulations vs. HTL, kinetic theory

- More generally, gluon & fermion spectral functions needed for
  - Heavy-ion collisions at early times
  - Comparison to thermal properties (general features?)
  - Nonperturbative extensions of kinetic theory, anisotropic HTL
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Theory

- **SU($N_c$) gauge theory** (simulations: $N_c = 2$, $U_j \approx \exp(ig a_s A_j)$)

\[
H_{YM} = \frac{1}{g^2 a_s} \sum_{\vec{x},i} \text{Tr}[E_i(t', \vec{x})^2] + \frac{1}{2} \sum_j \text{ReTr}[1 - U_{ij}(t', \vec{x})]
\]

\[
H_W = \frac{1}{2} \sum_{\vec{x}} \left[ \hat{\psi}^\dagger(t', \vec{x}), \gamma^0 (-i \not{\partial}_s[U] + m) \hat{\psi}(t', \vec{x}) \right]
\]

- $-i \not{\partial}_s[U] \hat{\psi}$ tree-level improved Wilson Dirac operator

- Neglect fermionic backreaction (suppressed), temporal gauge $A_0 = 0$

- Mode exp. $\hat{\psi}(t', \vec{x}) = \frac{1}{\sqrt{V}} \sum_{\lambda, \vec{p}} \hat{b}_{\lambda, \vec{p}}(t) \phi^u_{\lambda, \vec{p}}(t', \vec{x}) + \hat{d}_{\lambda, \vec{p}}^\dagger(t) \phi^v_{\lambda, \vec{p}}(t', \vec{x})$

- At reference time $t$:
  - Creation/annih. operators $\hat{b}$, $\hat{d}$ satisfy anti-commut. relations
  - Plane waves $\phi^u_{\lambda, \vec{p}}(t', \vec{x}) \bigg|_{t' = t} = u_{\lambda}(\vec{p}) e^{+i\vec{p}\cdot\vec{x}}$, $\phi^v_{\lambda, \vec{p}}(t', \vec{x}) \bigg|_{t' = t} = v_{\lambda}(\vec{p}) e^{-i\vec{p}\cdot\vec{x}}$

See also: Aarts, Smit (1999); Kasper, Hebenstreit, Berges (2014); Mace, Mueller, Schlichting, Sharma (2016); ...
Classical-statistical lattice simulations, algorithm

1. Set initial conditions for gluons at $t' = 0$, generating a configuration with $\langle E_T^*(t=0, \vec{p})E_T(t=0, \vec{q}) \rangle \propto p f(t=0, p) \delta_{jk}(2\pi)^3 \delta(\vec{p} - \vec{q})$

2. Solve classical EOMs of gluonic part for $t' \leq t$, set Coulomb-type gauge $\partial^j A_j \big|_t = 0$ at $t' = t$

3. For each momentum mode $\vec{p}$ initialize $\phi^{u/v}_{\lambda,\vec{p}}$, evolve for $t' > t$

4. Use leap-frog scheme to solve classical EOMs

   \[
   U_j(t', \vec{x}) = e^{ia t / a_s} E^j(t' - at/2, \vec{x}) U_j(t' - at, \vec{x})
   \]

   \[
   E^j(t' + at/2, \vec{x}) - E^j(t' - at/2, \vec{x}) = -\frac{a t}{a_s} \sum_{j \neq i} \left[ U_{ij}(t', \vec{x}) + U_{i(-j)}(t', \vec{x}) \right]_{ah}
   \]

   \[
   \phi^{u/v}_{\lambda,\vec{p}}(t' + at, \vec{x}) - \phi^{u/v}_{\lambda,\vec{p}}(t' - at, \vec{x}) = -2ia t \gamma^0 \left( -i\gamma^j D^s_{W,j} [U] + m \right) \phi^{u/v}_{\lambda,\vec{p}}(t', \vec{x})
   \]

5. Calculate fermionic spectral function $\rho(t', t, \vec{p})$ for each $\vec{p}$ mode
Calculating fermion spectral function nonperturbatively

- **Definition** $\rho^{\alpha\beta}(x, y) = \left\langle \left\{ \hat{\psi}^\alpha(t', \vec{x}), \hat{\bar{\psi}}^\beta(t, \vec{y}) \right\} \right\rangle$

- Plug in mode exp., Fourier transform, use plane waves at $t' = t$

$$
\rho^{\alpha\beta}(t', t, \vec{p}) = \frac{1}{V} \sum_{\lambda, \vec{q}} \left\langle \tilde{\phi}_{\lambda, \vec{q}}^{u, \alpha}(t', \vec{p}) \left( \tilde{\phi}_{\lambda, \vec{q}}^{u, \gamma}(t, \vec{p}) \right)^* + \tilde{\phi}_{\lambda, \vec{q}}^{v, \alpha}(t', \vec{p}) \left( \tilde{\phi}_{\lambda, \vec{q}}^{v, \gamma}(t, \vec{p}) \right)^* \right\rangle \gamma_0^{\gamma\beta}
$$

$$
= \frac{1}{V} \sum_{\lambda} \left\langle \tilde{\phi}_{\lambda, \vec{p}}^{u, \alpha}(t', \vec{p}) u_{\lambda, \gamma}^{\dagger}(\vec{p}) + \tilde{\phi}_{\lambda, -\vec{p}}^{v, \alpha}(t', \vec{p}) v_{\lambda, \gamma}^{\dagger}(-\vec{p}) \right\rangle \gamma_0^{\gamma\beta}
$$

- **Huge simplification** due to $\tilde{\phi}_{\lambda, \vec{q}}(t, \vec{p}) \propto \delta^3(\vec{p} - \vec{q})$

- **Classical-statistical average** over gluonic configurations
Benchmark for free fermions

- **Analytically:** \( \rho_{\text{free}}(\Delta t, \vec{p}) = \gamma^0 \cos(E_{\vec{p}} \Delta t) + i \left( \gamma^j \hat{p}^j / E_{\vec{p}} - m_{\vec{p}} / E_{\vec{p}} \right) \sin(E_{\vec{p}} \Delta t) \)

- **Our method:** set gauge fields to \( U^i(t', \vec{x}) = 1, \ E^i(t', \vec{y}) = 0 \)

- Extract \( \rho_V^0 = \frac{1}{4} \text{Tr}(\rho \gamma^0) \), \( \rho_V = -\frac{E_{\vec{p}} \hat{p}^j}{4 p^2} \text{Tr}(\rho \gamma^j) \), \( \rho_S = \frac{1}{4} \text{Tr}(\rho) \)

- **Nice agreement!**

- \( 64^3, a_s \vec{p} = (0.098, 0.195, 0.29), \text{mass } m_{a_s} = 0.003125; \)
  
  all other components of \( \rho \) at machine prec. \( \sim 10^{-16} \)
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Non-equilibrium state: self-similar turbulent attractor

**Figures:** attractor in 2+1D; KB, Kurkela, Lappi, Peuron, PRD 100, 094022 (2019)

- Gluonic $f_g(t=0, p \lesssim Q) \sim \frac{1}{g^2} \gg 1$ approach self-similar attractor

$$f(t, p) = (Qt)^\alpha f_s \left( (Qt)^\beta p \right)$$

- Universal exponents insensitive to details of initial conditions
  - ✓ 2+1D: $\beta = -1/5, \alpha = 3\beta$, KB, Kurkela, Lappi, Peuron (2019)
  - ✓ 3+1D: $\beta = -1/7, \alpha = 4\beta$, Berges, Scheffler, Sexty (2009); Kurkela, Moore (2011, 2012); Berges, Schlichting, Sexty (2012); Schlichting (2012); Berges, KB, Schlichting, Venugopalan (2014); ...

- We extract $\rho^{\alpha\beta}(t, \omega, p)$ at such a typical state in 3+1D
Fermion $\rho$ in 3+1D: non-trivial evolution

Spectral function $\rho^{\alpha\beta}(t+\Delta t, t, p)$

- Lattice $256^3$, $Qa_s = 0.75$, $m = 0.003125\, Q$, $Qt = 1500$

- Non-trivial $\rho^0_V = \frac{1}{4} \text{Tr}(\rho \gamma^0)$, $\rho_V = -\frac{E_{\beta} \rho^j}{4 p^2} \text{Tr}(\rho \gamma^j)$

- $\text{Re} \rho^0_V \approx e^{-\gamma(t,p)\Delta t} \cos(\omega(t,p)\Delta t)$

- $\text{Im} \rho_V \approx -e^{-\gamma(t,p)\Delta t} \sin(\omega(t,p)\Delta t)$

- All other components expected to vanish from symmetries; numerically suppressed by at least $\sim 10^{-2}$
Fermion $\rho$ in 3+1D: Comparison with HTL

$\rho_+(t, \omega, p) \equiv \rho_0^V + \rho_V$

**HTL:** Landau cut + q.p. peaks:
Braaten, Pisarski (1992); Rebhan (1992); Mrowczynski, Thomas (2000); Blaizot, Iancu (2002); ...

$$\rho_{+\text{HTL}}^V(\omega, p) = 2\pi \beta_+^V(\omega/p, p)$$
$$+ 2\pi [Z_+(p)\delta(\omega - \omega_+(p))$$
$$+ Z_-(p)\delta(\omega + \omega_-(p))]$$

**HTL+$\gamma$ extension**

$$\rho_{+\text{HTL}}^V(\omega, p) = 2\pi \beta_+^V(\omega/p, p)$$
$$+ \frac{2Z_+(p)\gamma_+(p)}{(\omega - \omega_+(p))^2 + \gamma_+^2(p)}$$
$$+ \frac{2Z_-(p)\gamma_-(p)}{(\omega + \omega_-(p))^2 + \gamma_-^2(p)}$$

**Very good description;**
$$\omega_{\pm}(t, p), Z_{\pm}(t, p), \gamma_{\pm}(t, p) \text{ fits}$$
Fermion $\rho$ in 3+1D: $\omega_{\pm}$, $Z_{\pm}$, $\gamma_{\pm}$

(top) $\omega_{\pm}$, (bottom) $Z_{\pm}$

- HTL dispersions and residues agree well with data
- First-principles insight into $p$-dependence of damping rates

$$m_f = \left[ C_F \int \frac{d^3p}{(2\pi)^3} \frac{g^2 f_g(p)}{p} \right]^{1/2}$$
Summary: fermion $\rho_+$ in isotropic 3+1D plasmas

We find:

- Lorentzian quasiparticle peaks + Landau cut ($\omega < p$)
- very good description by HTL
- full $p$-dependence of $\gamma_+(t, p)$
- (see Backup:) $\gamma_+(t, p=0) \sim m_F(t) \sim Q(Q t)^{-1/7}$
  vs. HTL expectation: $\gamma^{\text{HTL}}(t, p=0) \propto g^2 T^*(t) \sim Q(Q t)^{-3/7}$
Nonperturbative computation of gluonic $\rho$

Classical-statistical $SU(N_c)$ simulations + linear response theory

KB, Kurkela, Lappi, Peuron, *PRD 98, 014006 (2018)*, Editors’ suggestion

- Similar algorithm as for fermions
- Split $A(t, \vec{x}) \mapsto A(t, \vec{x}) + \delta A(t, \vec{x})$ at $t$, perturb with plane wave $j_0(\vec{p}) \delta(t' - t)$
- Response $\langle \delta A(t', \vec{p}) \rangle = G_R(t', t, \vec{p}) j_0(\vec{p})$
- Linearized EOM for $\delta A(t, \vec{x})$ such that Gauss law conserved (also in gauge-cov. formulation)

Kurkela, Lappi, Peuron, *EUJC 76 (2016) 688*

- $\theta(t' - t) \rho(t', t, \vec{p}) = G_R(t', t, \vec{p})$

Similar methods for scalars:

Aarts (2001); Piñeiro Orioli, Berges (2019); Schlichting, Smith, von Smekal (2020); KB, Piñeiro Orioli (2020); . . .

Kirill Boguslavski (TU Wien)  Fermion spectral functions  ConfXIV, 02.08.2021  17 / 22
Gluon spectral function in isotropic 3+1D plasmas

KB, Kurkela, Lappi, Peuron, PRD 98, 014006 (2018)

- Narrow Lorentzian q.p. peaks (position $\omega(p)$, width $\gamma(p)$)
- HTL at LO (black dashed) describes main features well
- Landau cut ($\omega < p$) and q.p. peak distinguishable

\[
\rho_T(\omega, p)
\]

\[
\text{Width } \propto (Qt)^{3/7} \gamma_T(t, p)
\]

- Peak width $\gamma(t, p) \ll \omega(t, p)$
- Full $p$ dep., $\gamma_T(t, p) \approx \gamma_L(t, p)$
- $\frac{\gamma_T(t,p)}{\omega(t,p=0)} \sim (Qt)^{-2/7}$ decreases
3+1D $\rightarrow$ 2+1D plasmas

- Isotropic 3+1D: $f(t, p)$
- 2+1D: $f(t, p_T, p_z=0)$
- can be regarded as extreme momentum anisotropy

\[ p_T = \sqrt{p_x^2 + p_y^2} \]
Vs. Gluon spectral function in 2+1D plasmas

KB, Kurkela, Lappi, Peuron, JHEP 05, 225 (2021)

\[ \dot{\rho}_T(\omega, p) \approx \omega \rho_T(\omega, p) \]

- Broad non-Lorentzian peaks
- HTL curves (green) agree poorly
- Landau cut and q.p. peak not distinguishable

\[ \text{Peak width } \gamma(t, p) \sim \omega_{pl}(t) \]

(\(\omega_{pl} \equiv \omega_T(p=0)\))

⇒ no quasiparticles for \(p \lesssim \omega_{pl}\)!
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Conclusion

- We have developed tools to extract spectral functions in highly occupied plasmas nonperturbatively.
- Applied to classical self-similar attractors: fermionic 3+1D, gluonic 3+1D, gluonic 2+1D.
- \( \rho \)'s in 3+1D generally well described by HTL, first principles determination of damping rates.

Outlook: applications to heavy-ion collisions

- \( \rho \) in Bjorken expanding systems or of heavy-flavor quarks.
- Anisotropic / expanding kinetic theory, HTL.
- Effects on transport coefficients.

Some talks with classical-statistical simulations:

- B. S. Kasmaei, We, 17:20 (track C)
- J. Peuron, Th, 10:10 (track C)
- D. Schuh, Th, 10:55 (track D)
Thank you for your attention!

Gluonic 3+1D

Fermionic 3+1D

Gluonic 2+1D
Backup slides
Universal (classical) attractors

- Rich nonequilibrium dynamics in gauge and scalar systems
- Share similar universal features:

**Nonthermal fixed point** (NTFP)

- Large initial occupancy
  \[ \Rightarrow \text{may approach attractor} \]
- System ‘forgets’ initial conditions
- Self-similar dynamics
  \[ f(t, p) = t^\alpha f_s(t^\beta p) \]

**Universal** \( \alpha, \beta, f_s(p) \)

**NTFP:** Micha, Tkachev (2004); Berges, Rothkopf, Schmid (2008)

**Universality:** Berges, KB, Schlichting, Venugopalan (2015); Piñeiro Orioli, KB, Berges (2015)

**Experimental observations:** Prüfer et al., Nature 563, 217 (2018); Erne et al., Nature 563, 225 (2018)
Self-similarity of 2+1D theory

*Figures:* attractor in 2+1D; KB, Kurkela, Lappi, Peuron, PRD 100, 094022 (2019)

- **Self-similar evolution**

\[ f(t, p) = (Qt)^\alpha f_s \left((Qt)^\beta p\right) \]

- **Via energy conservation:** \( \alpha = (d + 1)\beta \)
Fermion $\rho$ in 3+1D: Time evolution

KB, Lappi, Mace, Schlichting, arXiv:2106.11319

- $\rho_+(t, \omega, p=0)$ scales with fermion mass $m_f(t) \equiv \omega_\pm(t, p=0)$
- Expected from HTL: $\gamma^{\text{HTL}}(t, p=0) \propto g^2 T^*(t) \sim Q(Qt)^{-3/7}$
  $\Rightarrow$ observed $\gamma(t, p=0) \sim m_f(t)$ is surprising
Gluon HTL results: perturbative computation

- Hard loop (HTL) framework requires \( m_D/\Lambda \ll 1 \);
in thermal equ. for \( g \sim m_D/T \ll 1 \), Braaten, Pisarski (1990); Blaizot, Iancu (2002); . . .

- In 3+1D \( m_D^2 = 4N_c \int \frac{d^3p}{(2\pi)^3} \frac{g^2 f(t,p)}{p} \sim g^2 f \Lambda^2 \Rightarrow \text{HTL applicable} \)

- In 2+1D soft-soft interactions important

\[
m_D^2 \approx d_{\text{pol}} N_c \int \frac{d^2p}{(2\pi)^2} \frac{g^2 f(t,p)}{\sqrt{m^2 + p^2}} \sim g^2 f \Lambda \ln \left( \frac{\Lambda}{m_D} \right)
\]

\( \Rightarrow \text{HTL breaks down already at soft scale } p \sim m_D \)

- Comparison to HTL still useful to extract nonperturbative features
- Quasiparticles in \( \rho^{\text{HTL}}(\omega, p) \) as \( \sim \delta(\omega - \omega_{\alpha}^{\text{HTL}}(p)) \)
- All expressions depend only on \( m_D \), computed consistently in HTL
Gluon $\rho$ in 2+1D: Shape of the excitation peak

- **Left**: Different ways of computing the Fourier transform are consistent
- **Right**: Peak has non-Lorentzian shape (not Breit-Wigner)
Gluon $\rho$ in 2+1D: Dispersion relations, damping rates

- **Left:** Dispersions $\omega_\alpha(t, p)/\omega_{pl}(t)$
- **Right:** Peak width $\gamma_\alpha(t, p)/\omega_{pl}(t)$
- As functions of $p/\omega_{pl}(t)$ time independent $\Rightarrow \gamma(t, p) \sim \omega_{pl}(t)$
- Scalar excitation narrow for $p \lesssim m_D$, but same $t$ dependence
Gluon $\rho$ in 2+1D: Time dependence

- $\omega_{pl} \dot{\rho}(t, \omega/\omega_{pl}, p/\omega_{pl})$ is time independent
- This implies $\gamma_\alpha(t, p) \sim \omega_{pl}(t) \sim m_D \sim Q(Qt)^{-1/5}$
- No quasiparticles at low $p$ seems to be quite general in 2+1D
Classical thermal equilibrium \( f(p) \approx \frac{T}{\omega(p)} \)

\( \rho(\omega, p) \) qualitatively similar as far from equilibrium

- Broad gluonic excitations with \( \gamma(p) \sim \omega_{pl} \)
- HTL provides poor description
- For \( \omega \rightarrow 0 \), \( \dot{\rho}_T = \omega \rho_T \) finite at low \( p \)

Interpretation: Features seem generic in 2+1D gauge theories