# A comparison of spectral reconstruction methods applied to non-zero temperature NRQCD meson correlation functions

Thomas Spriggs [Swansea University]

FASTSUM Collaboration

**A Virtual Tribute to Quark Confinement and the Hadron Spectrum 2021**, August 2021

# FASTSUM Collaboration's NRQCD Project

**Gert Aarts, Chris Allton, Tim Burns, Sam Offler, Benjamin Page, TS**
*Swansea University*
**Benjamin Jäger**
*Southern Denmark University*
**Seyong Kim**
*Sejong University, Korea*
**Maria Paola Lombardo**
*INFN, Florence*
**Sinead Ryan**
*Trinity College, Dublin*
**Jon-Ivar Skullerud**
*National University of Ireland, Maynooth, Ireland*

# Overview

## Bottomonium from FASTSUM Collaboration

- FASTSUM lattice setup
  - anisotropic lattices
  - NRQCD to $O(v_b^4)$ | $M_b$ set by spin average S-wave meson masses
- Towards chiral and continuum limits
  - $M_\pi$=392, 236, 140 MeV
  - $a_\tau$=0.033, 0.017 fm
- Spectral Reconstruction from 7 Methods
  - Maximum likelihood (x2)
  - Moments
  - Bayesian (x2)
  - Backus Gilbert
  - Kernel Ridge Regression

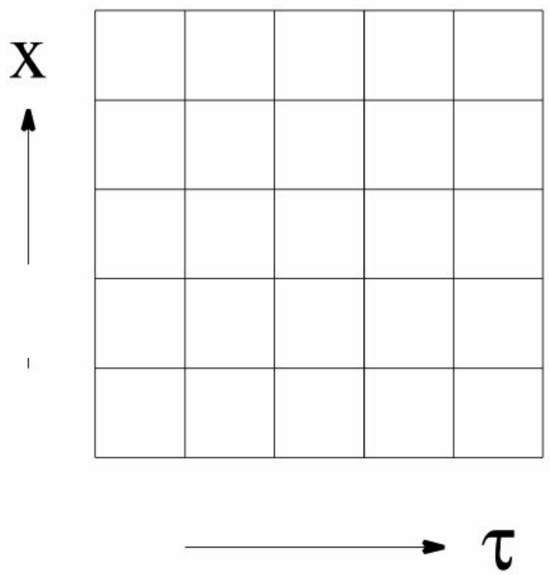
# FASTSUM setup

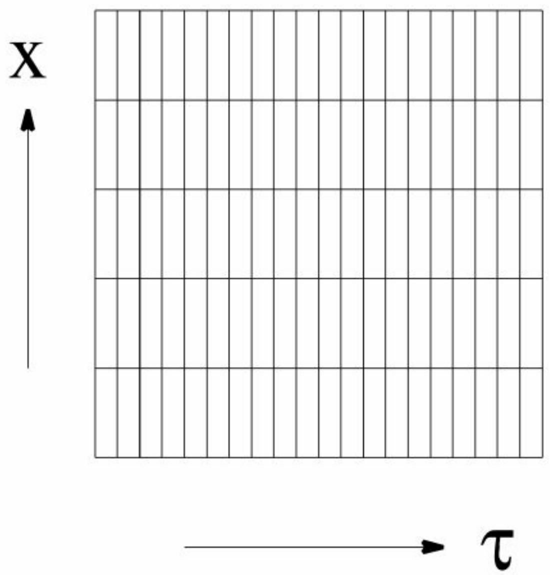
Anisotropic Lattice:

$a_\tau < a_s$

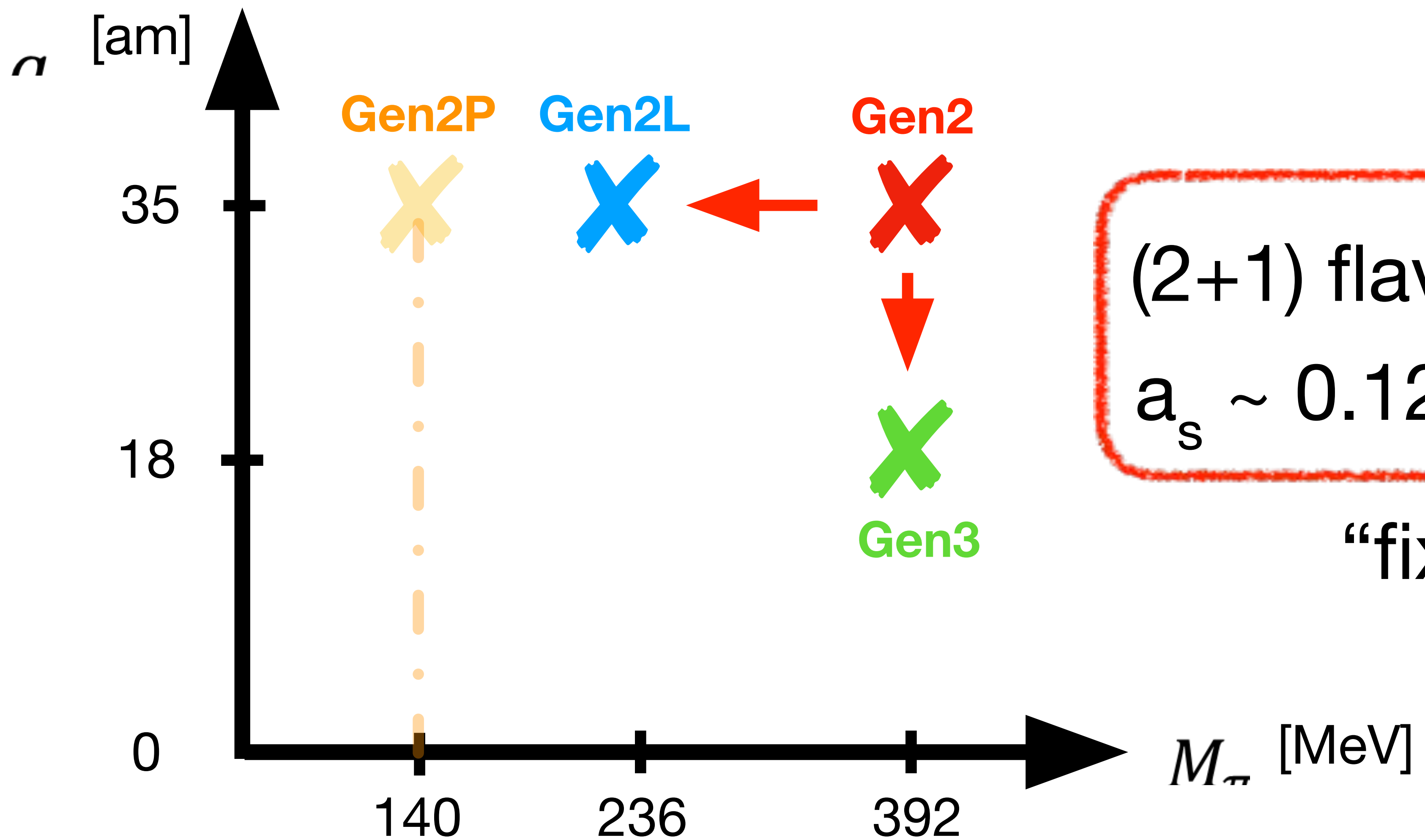
allowing for better resolution, particularly at finite temperatures, since

$$T = \frac{1}{L_\tau} = \frac{1}{N_\tau a_\tau}$$



# Lattice Parameters




(2+1) flavour

$a_s$ ~ 0.12 fm

“fixed scale”

Aarts *et al*, JHEP 07 (2014) 097

# Spectral Functions

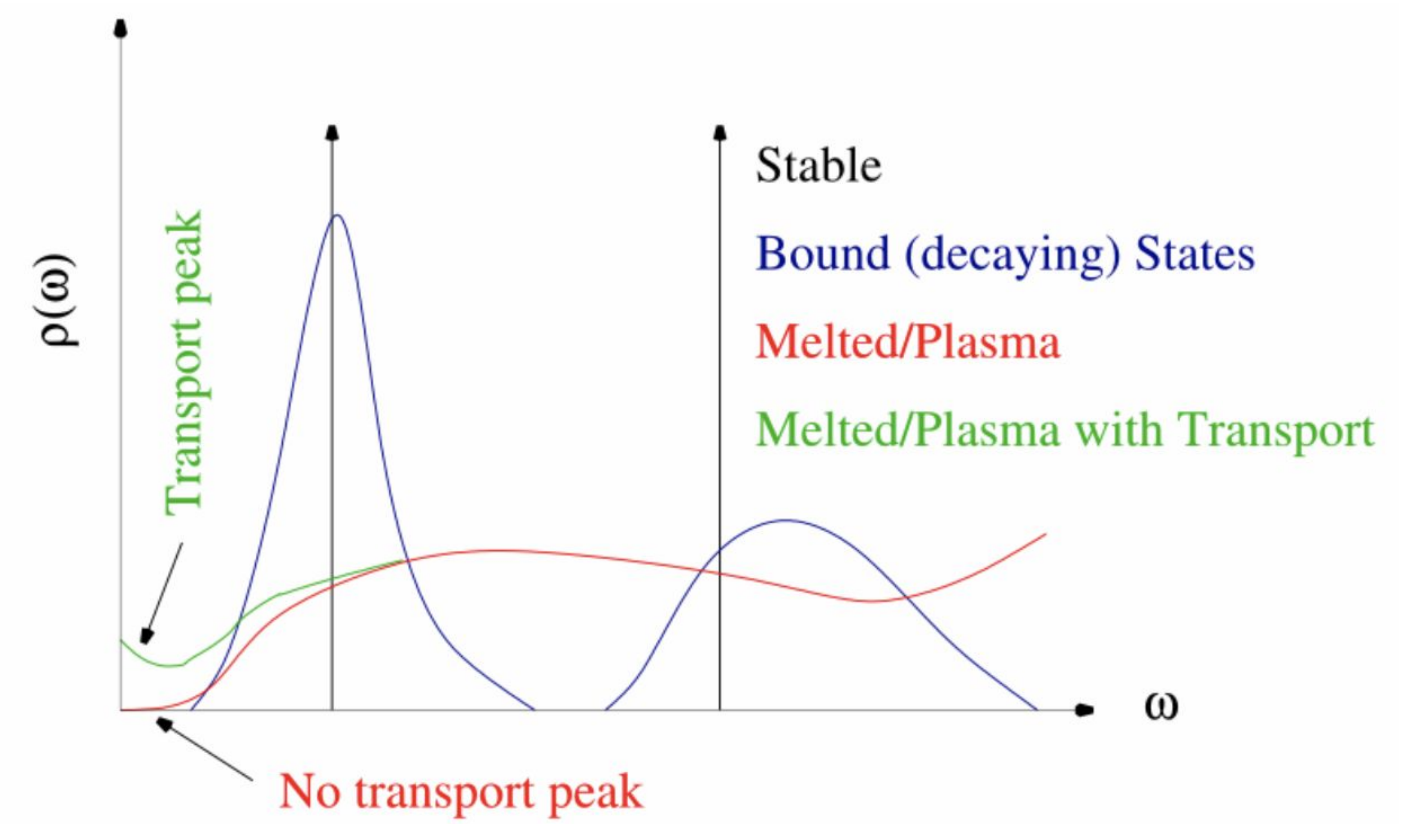
$$G_{\pm}(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} K(\tau,\omega)\rho_{\pm}(\omega)$$

$\uparrow$ Euclidean Correlator

$\nearrow$ (Lattice) Kernel

$\downarrow$ Spectral Function

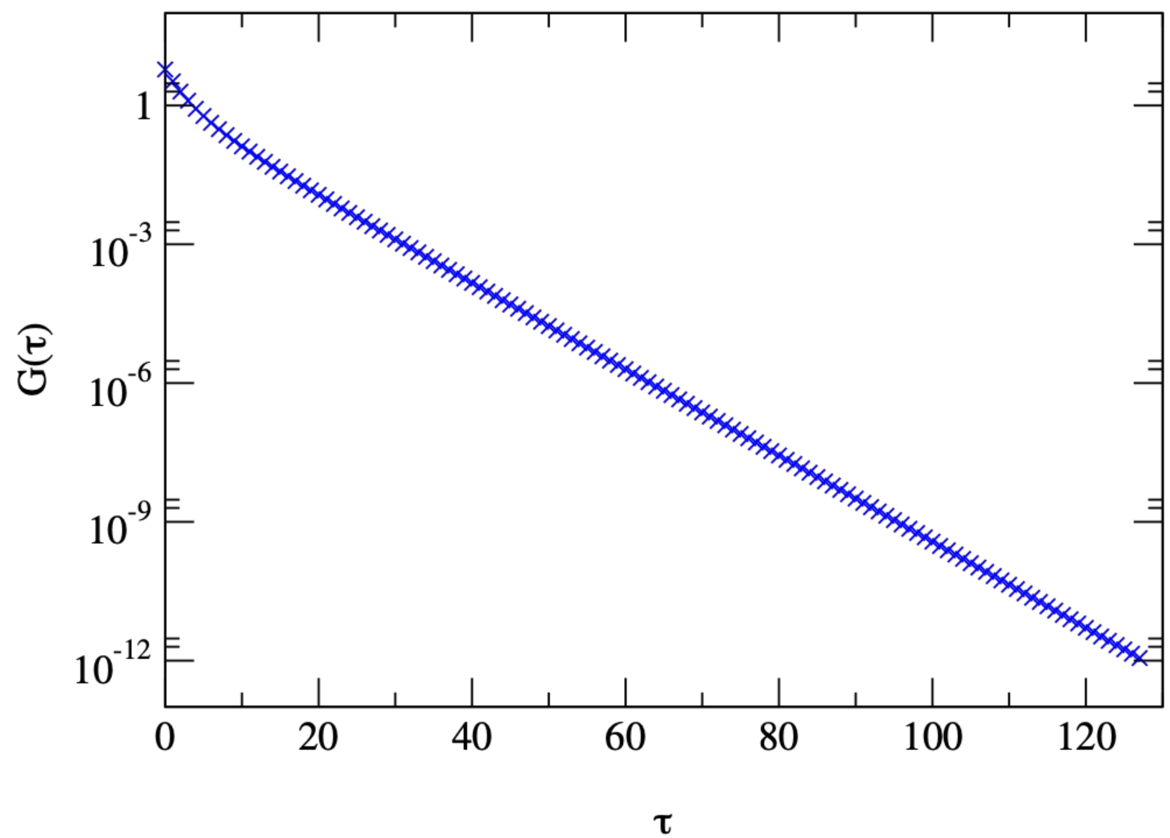
$$K(\tau,\omega) = e^{-\omega\tau}$$



# Extracting Spectral Functions

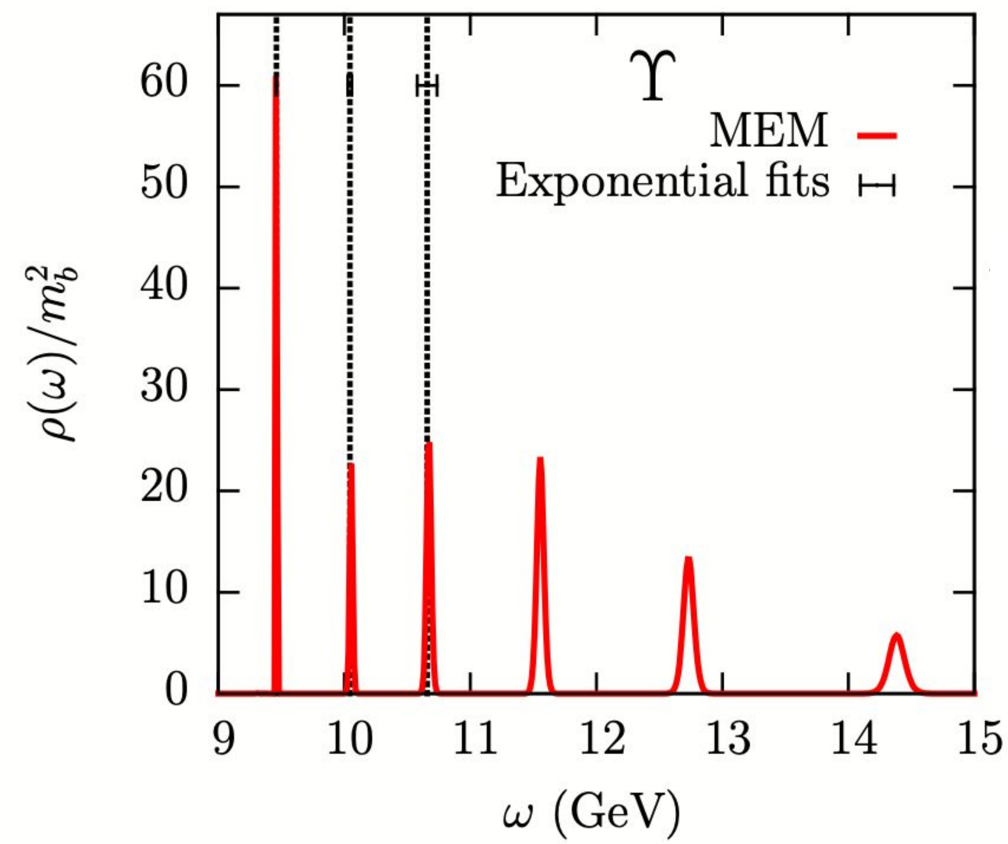
Input Data:

G(τ); τ = 1,2,3...O(10-100)




Output Data:

ρ(ω); ω = $\omega_0$,...O(1000)




***ill-posed !*** ***i.e. ∞ solutions with $\chi^2 = 0$***

"Entropy" Factor $P(F)$ breaks this degeneracy

# Study of Numerical Methods

1. Exponential (Conventional $\delta$ f'ns)
2. Gaussian Ground State (+ $\delta$ f'n excited)

} Maximum Likelihood (Minimise $\chi^2$)

3. Moments of Correlation F'ns

Direct Method - "no" fit

4. BR Method
5. Maximum Entropy Method

} Bayesian Approaches

6. Kernel Ridge Regression

Machine Learning

7. Backus Gilbert

from Geophysics

# Moments

$$G(\tau) = \int e^{-\omega\tau}\rho(\omega)d\omega \longrightarrow \frac{dG(\tau)}{d\tau} = \int \omega e^{-\omega\tau}\rho(\omega)d\omega$$

$$-\frac{1}{G(\tau)}\frac{dG(\tau)}{d\tau} = M_{eff}(\tau) = \frac{1}{G(\tau)}\int \omega e^{-\omega\tau}\rho(\omega)d\omega = \langle\omega\rangle_{e^{-\omega\tau}\rho(\omega)}$$

Similarly, we can take a 2nd derivative to calculate

Variance (i.e. width):

$$\Gamma^2 = \frac{1}{G(\tau)}\frac{d^2G(\tau)}{d\tau^2} - M_{eff}^2 = \langle(\omega - \langle\omega\rangle)^2\rangle$$

# Bayesian Approaches

Need to maximise $P(F|D)$

Bayes Theorem :

$$P(F \cap D) = P(F|D)P(D) = P(D|F)P(F)$$

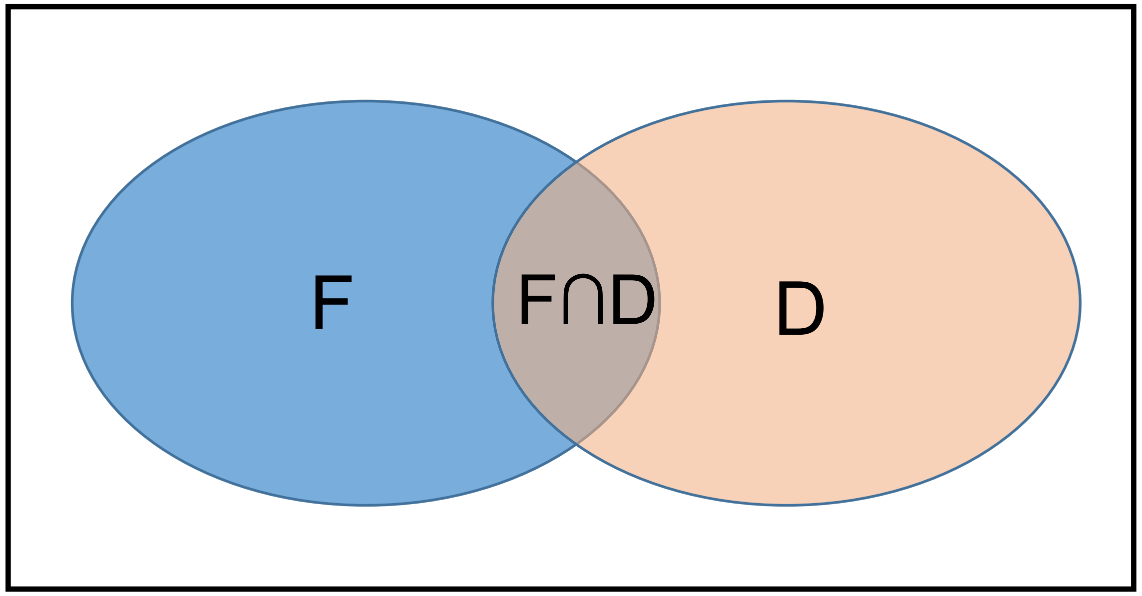
i.e. $P(F|D) = \frac{P(D|F)P(F)}{P(D)}$

Note $P(D|F) \sim \chi^2$

So we should always include $P(F)$

$P(F)$ is encoded as an *Entropy*

BR and MEM use different Entropy definitions



# Choice of Entropy Term

$P(F) \sim e^{S}$ *S = Entropy*

**Maximum Entropy Method:**

*Shannon-Jaynes Entropy:* $$S = \int_0^\infty d\omega \left[ \rho(\omega) - m(\omega) - \rho(\omega) \ln \frac{\rho(\omega)}{m(\omega)} \right]$$

Asakawa, Hatsuda, Nakahara, Prog.Part.Nucl.Phys. 46 (2001) 459

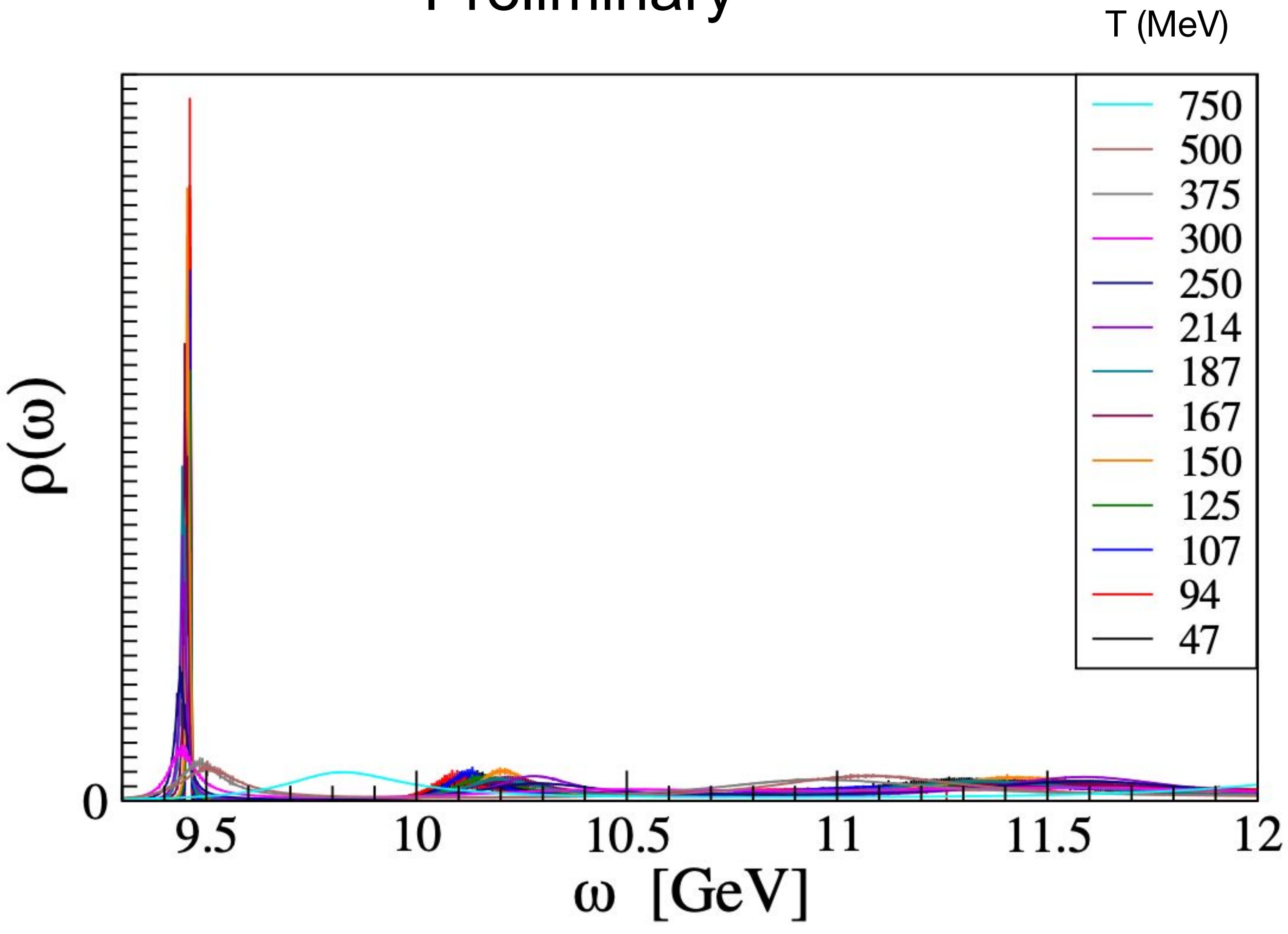
**BR Method:**

$$S = \int_0^\infty d\omega \left[ 1 - \frac{\rho(\omega)}{m(\omega)} + \ln \frac{\rho(\omega)}{m(\omega)} \right]$$

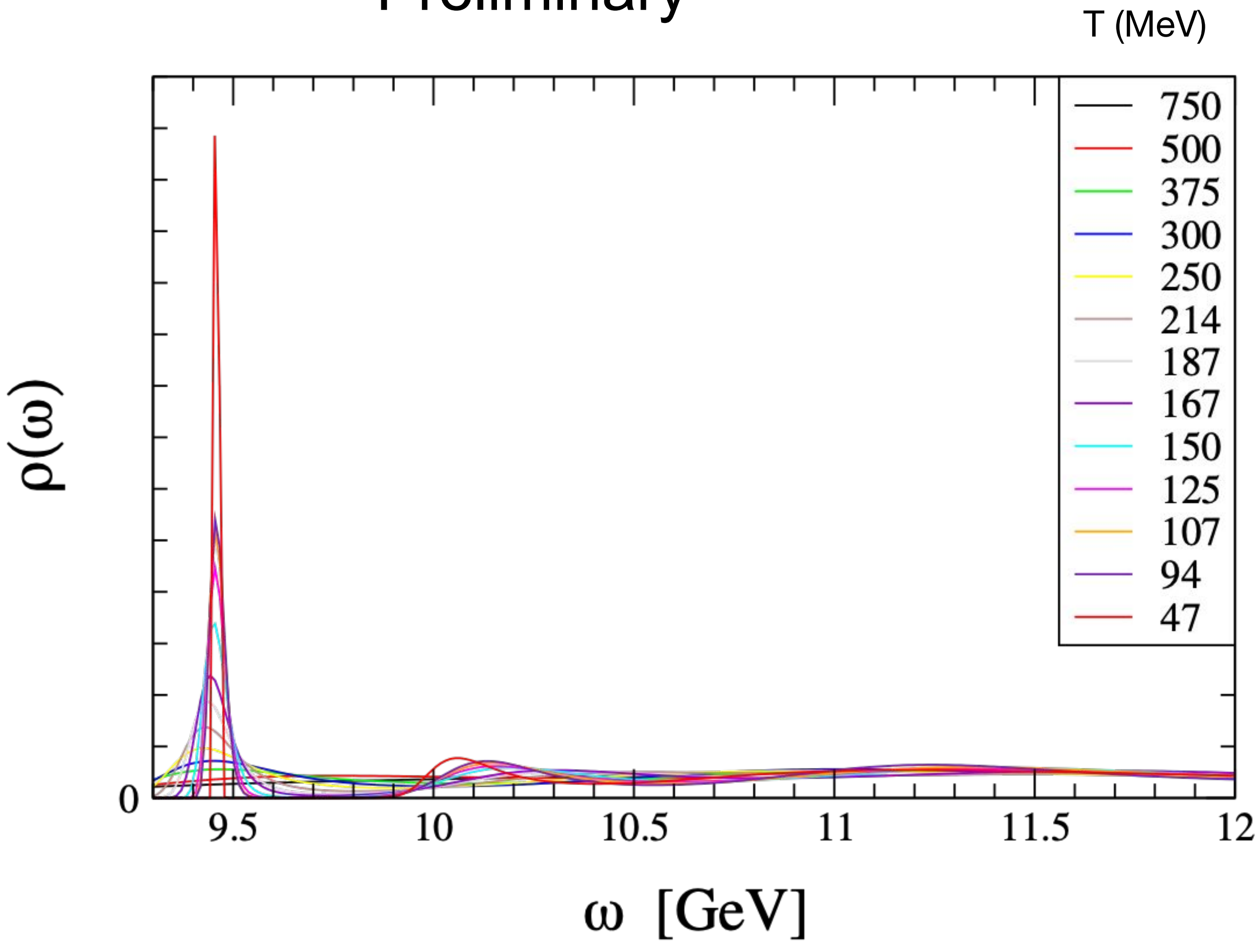
Burnier & Rothkopf Phys.Rev.Lett. 111 (2013) 182003

# Direct comparison of Bayesian Approaches

BR Upsilon
Preliminary




MEM Upsilon
Preliminary



# Kernel Ridge Regression

## Machine Learning

- uses training data to determine an *alpha matrix* of parameters determined analytically using a cost function
- cost function includes a term to prevent overfitting
- training data set is $\mathcal{O}(10^4)$ mock data with 5 Gaussians
- difficult to produce systematic error estimate

$$C_{ij}(x_i, x_j) = \exp\left\{-\gamma \sum_{\tau} \left[\log(G_i(\tau)) - \log(G_j(\tau))\right]^2\right\}$$

# Backus Gilbert

Take $G(\tau) = \int \rho(\omega) e^{-\omega\tau} d\omega = \int \rho(\omega) K(\omega, \tau)$

Generate *averaging functions:* $A(\omega, \omega_0) = \sum_\tau c_\tau K(\omega, \tau)$

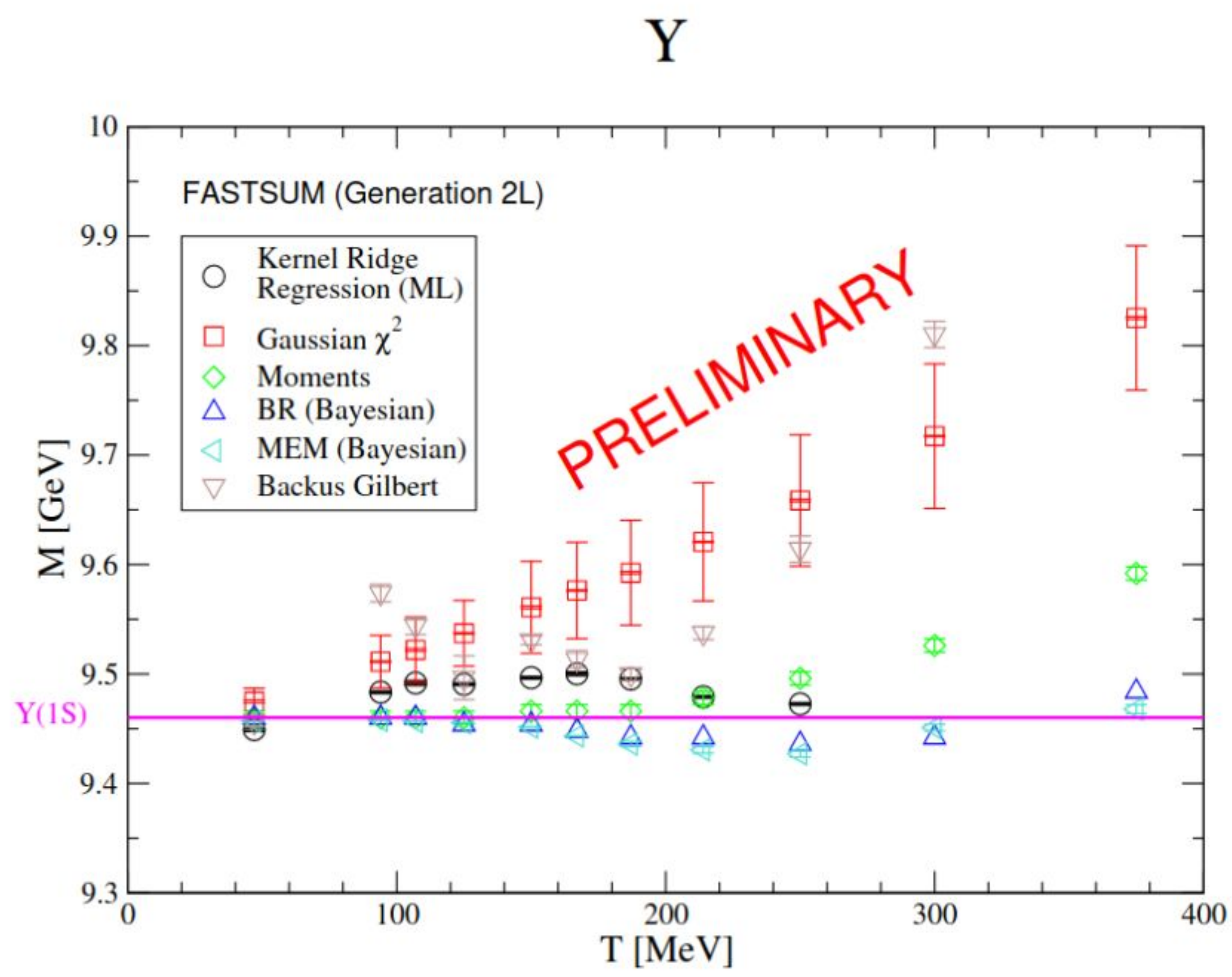
(an approximation to the $\delta$ f'n), such that

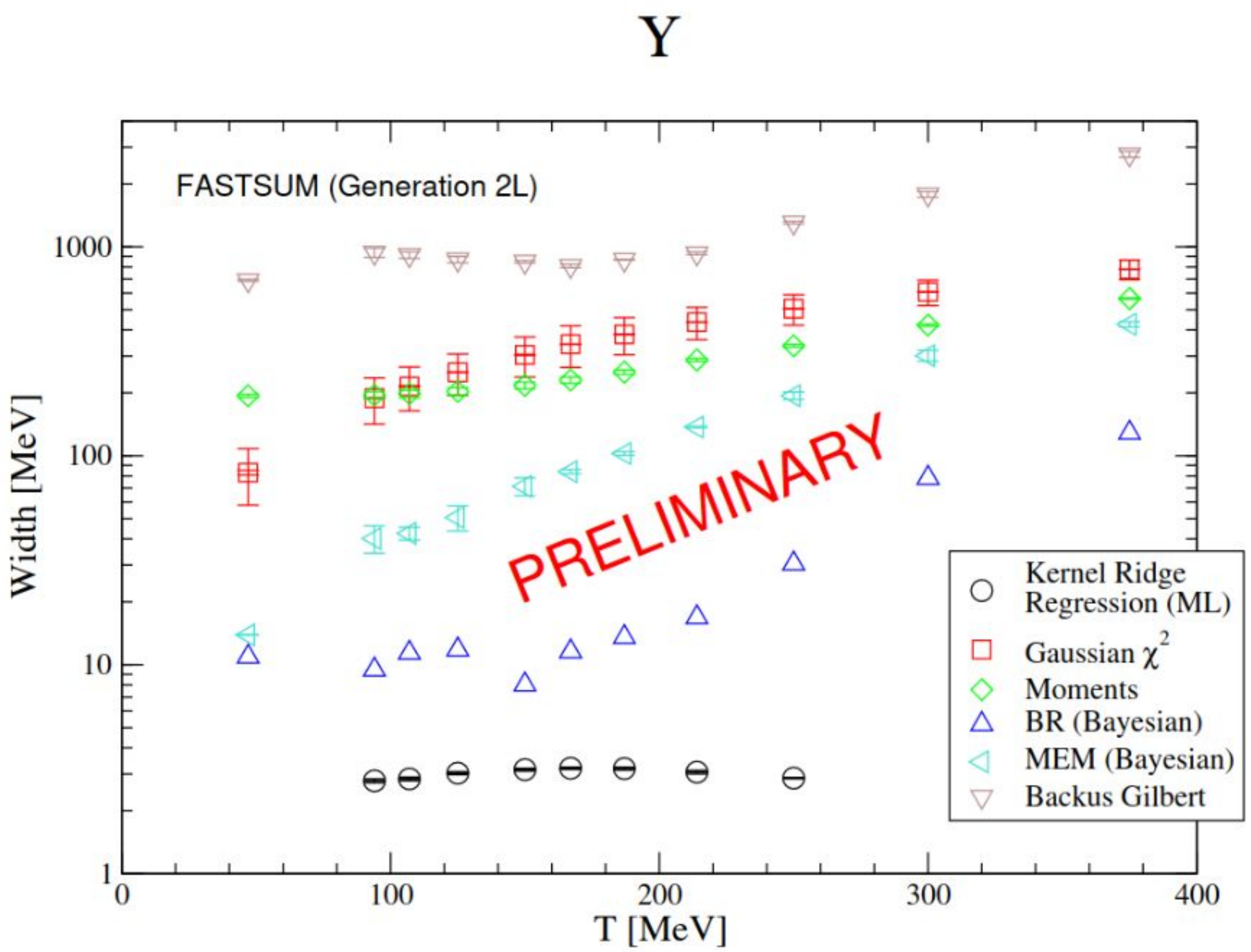
$$\hat{\rho}(\omega_0) = \int A(\omega, \omega_0) \rho(\omega) d\omega$$
$$= \sum_\tau c_\tau G(\tau)$$
$$\approx \rho(\omega_0)$$

Averaging coeffts $c_\tau$ determined by *minimising the width of* $A(\omega, \omega_0)$

# Results from all Methods

## Generation 2L: Upsilon PRELIMINARY!

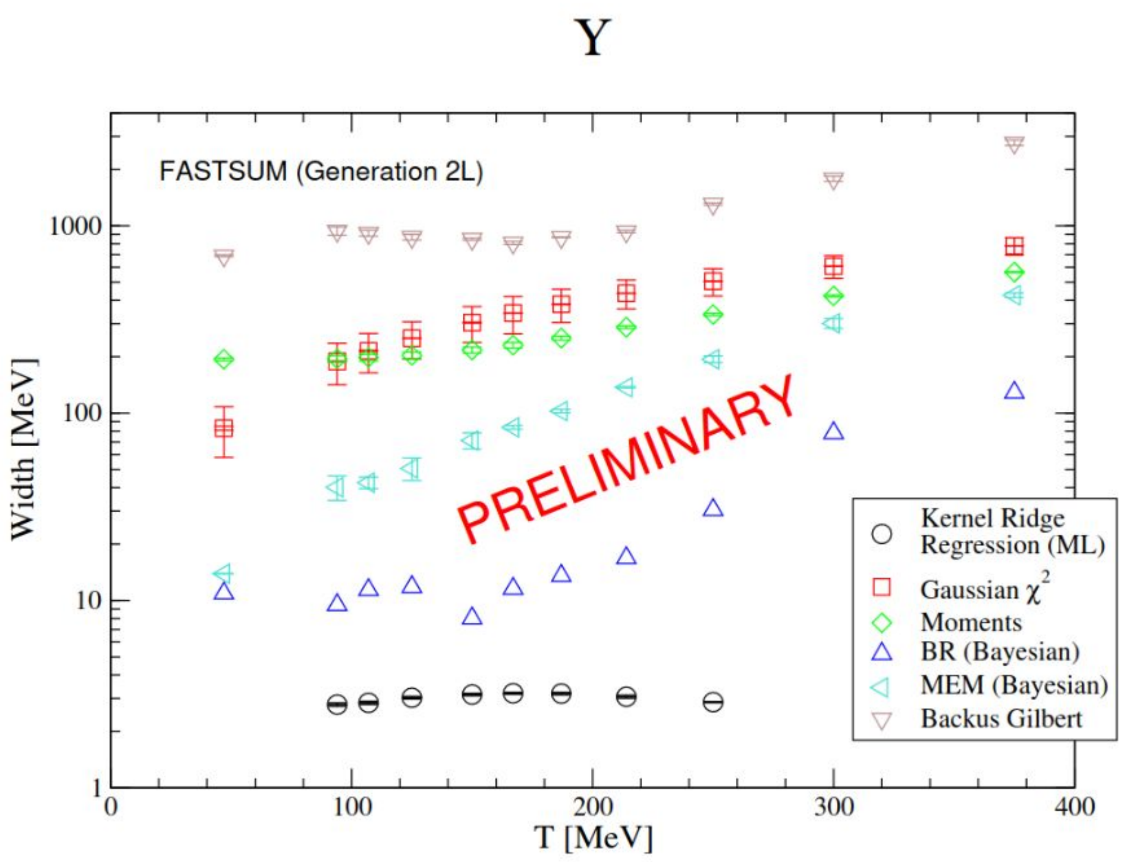


# Summary

## Bottomonium spectrum from FASTSUM Collaboration

- Towards chiral & continuum limits
  - $M_\pi = 392{,}236{,}140 MeV$
  - $a_\tau = 33{,}17am$
- Spectral Reconstruction from 7 Methods
  - *Max.Likelihood (x2)*
  - *Moments*
  - *Bayesian (x2)*
  - *Machine Learning*
  - *Backus-Gilbert*




*Towards Systematic Understanding of Bottomonium Spectrum from the Lattice*