A comparison of spectral reconstruction methods applied to non-zero temperature NRQCD meson correlation functions

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FASTSUM Collaboration

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FASTSUM Collaboration’s NRQCD Project

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Overview
Bottomonium from FASTSUM Collaboration

- FASTSUM lattice setup
  - anisotropic lattices
  - NRQCD to $O(v_b^4) \ | \ M_b$ set by spin average S-wave meson masses
- Towards chiral and continuum limits
  - $M_\perp = 392, 236, 140$ MeV
  - $a_T = 0.033, 0.017$ fm
- Spectral Reconstruction from 7 Methods
  - Maximum likelihood ($x^2$)
  - Moments
  - Bayesian ($x^2$)
  - Backus Gilbert
  - Kernel Ridge Regression
FASTSUM setup

Anisotropic Lattice:

\[ a_\tau < a_s \]

allowing for better resolution, particularly at finite temperatures, since

\[ T = \frac{1}{L_\tau} = \frac{1}{N_\tau a_\tau} \]
Lattice Parameters

(2+1) flavour
$a_s \sim 0.12$ fm

"fixed scale"

Aarts et al, JHEP 07 (2014) 097
Spectral Functions

$$G_{\pm}(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} K(\tau, \omega) \rho_{\pm}(\omega)$$

↑

Euclidean (Lattice) Spectral
Correlator Kernel Function

$$K(\tau, \omega) = e^{-\omega \tau}$$

ρ(ω)

Stable
Bound (decaying) States
Melted/Plasma
Melted/Plasma with Transport

No transport peak
Extracting Spectral Functions

Input Data:

\[ G(\tau); \tau = 1, 2, 3 \ldots O(10-100) \]

Output Data:

\[ \rho(\omega); \omega = \omega_0, \ldots O(1000) \]

\textit{ill-posed!} \quad \text{i.e.} \quad \infty \text{ solutions with } \chi^2 = 0

“Entropy” Factor \( P(F) \) breaks this degeneracy
Study of Numerical Methods

1. Exponential (Conventional δ f’ns)
2. Gaussian Ground State (+ δ f’n excited)
3. Moments of Correlation F’ns
4. BR Method
5. Maximum Entropy Method
6. Kernel Ridge Regression
7. Backus Gilbert

Maximum Likelihood (Minimise $\chi^2$)
Direct Method - “no” fit
Bayesian Approaches
Machine Learning from Geophysics
Moments

\[ G(\tau) = \int e^{-\omega \tau} \rho(\omega) d\omega \rightarrow \frac{dG(\tau)}{d\tau} = \int \omega e^{-\omega \tau} \rho(\omega) d\omega \]

\[ -\frac{1}{G(\tau)} \frac{dG(\tau)}{d\tau} = M_{eff}(\tau) = \frac{1}{G(\tau)} \int \omega e^{-\omega \tau} \rho(\omega) d\omega = \langle \omega \rangle e^{-\omega \tau} \rho(\omega) \]

Similarly, we can take a 2nd derivative to calculate

Variance (i.e. width):

\[ \Gamma^2 = \frac{1}{G(\tau)} \frac{d^2G(\tau)}{d\tau^2} - M_{eff}^2 = \langle (\omega - \langle \omega \rangle)^2 \rangle \]
Bayesian Approaches

Need to maximise $P(F|D)$

Bayes Theorem:

$$P(F \cap D) = P(F|D)P(D) = P(D|F)P(F)$$

i.e. $P(F|D) = \frac{P(D|F)P(F)}{P(D)}$

Note $P(D|F) \sim \chi^2$

So we should always include $P(F)$

$P(F)$ is encoded as an Entropy

BR and MEM use different Entropy definitions
Choice of Entropy Term

\[ P(F) \sim e^S \quad S = \text{Entropy} \]

**Maximum Entropy Method:**

**Shannon-Jaynes Entropy:**

\[
S = \int_0^\infty d\omega \left[ \rho(\omega) - m(\omega) - \rho(\omega) \ln \frac{\rho(\omega)}{m(\omega)} \right]
\]

Asakawa, Hatsuda, Nakahara, Prog.Part.Nucl.Phys. 46 (2001) 459

**BR Method:**

\[
S = \int_0^\infty d\omega \left[ 1 - \frac{\rho(\omega)}{m(\omega)} + \ln \frac{\rho(\omega)}{m(\omega)} \right]
\]

Direct comparison of Bayesian Approaches

BR Upsilon Preliminary

MEM Upsilon Preliminary

\[ \rho(\omega) \]

\[ \omega \ [\text{GeV}] \]

\[ T \ (\text{MeV}) \]

\[ \rho(\omega) \]

\[ \omega \ [\text{GeV}] \]

\[ T \ (\text{MeV}) \]
Kernel Ridge Regression

Machine Learning

- uses training data to determine an *alpha matrix* of parameters determined analytically using a cost function
- cost function includes a term to prevent overfitting
- training data set is $\mathcal{O}(10^4)$ mock data with 5 Gaussians
- difficult to produce systematic error estimate

$$C_{ij}(x_i, x_j) = \exp \left\{ -\gamma \sum_{\tau} \left[ \log(G_i(\tau)) - \log(G_j(\tau)) \right]^2 \right\}$$
Take \[ G(\tau) = \int \rho(\omega)e^{-\omega \tau} d\omega = \int \rho(\omega)K(\omega, \tau) \]

Generate \textit{averaging functions}: \[ A(\omega, \omega_0) = \sum_{\tau} c_{\tau}K(\omega, \tau) \]

(an approximation to the \(\delta\) f'n), such that

\[
\hat{\rho}(\omega_0) = \int A(\omega, \omega_0)\rho(\omega)d\omega \\
= \sum_{\tau} c_{\tau} G(\tau) \\
\approx \rho(\omega_0)
\]

Averaging coeiffs \( c_{\tau} \) determined by \textit{minimising the width of} \( A(\omega, \omega_0) \)
Results from all Methods
Generation 2L: Upsilon PRELIMINARY!
Summary
Bottomonium spectrum from FASTSUM Collaboration

• Towards chiral & continuum limits
  • $M_\pi = 392,236,140 MeV$
  • $a_\tau = 33,17 am$

• Spectral Reconstruction from 7 Methods
  • Max. Likelihood (x2)
  • Moments
  • Bayesian (x2)
  • Machine Learning
  • Backus-Gilbert

Towards Systematic Understanding of Bottomonium Spectrum from the Lattice