Leading hadronic contribution to the muon magnetic moment from lattice QCD

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[2002.12347]

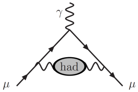
Sz. Borsanyi, Z. Fodor, J. N. Guenther, C. Hoelbling, S. D. Katz, L. Lellouch, T. Lippert, K. Miura, L. Parato, K. K. Szabo, F. Stokes, B. C. Toth, Cs. Torok, L. Varnhorst

Outline

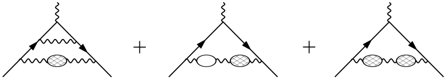
Introduction

Strong contributions

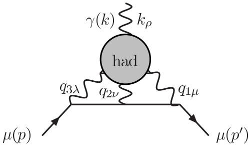
- LO hadron vacuum polarization (LO-HVP, $(\frac{\alpha}{\pi})^2$)



- NLO hadron vacuum polarization (NLO-HVP, $(\frac{\alpha}{\pi})^3$)



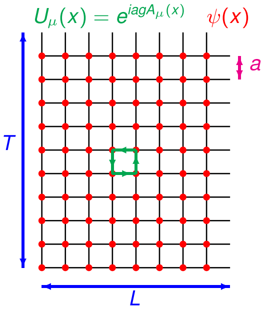
- Hadronic light-by-light (HLbL, $(\frac{\alpha}{\pi})^3$)



- pheno $a_\mu^{\text{HLbL}} = 9.2(1.9)$
  [Colangelo, Hoferichter, Kubis, Stoffer et al '15–'20]
- lattice a_μ^{HLbL}=7.9(3.1)(1.8) or 10.7(1.5)
  [RBC/UKQCD '19 and Mainz '21]

Lattice QCD

- Lattice gauge theory: systematically improvable, non-perturbative, 1st principles method
  - Discretize space-time with lattice spacing: a
  - Take a finite volume: $V = L^3 \times T$



- Quarks: on sites
  Gluons: on links
- Have to discretize action + operators

$$\int \mathrm{d}^4 x \longrightarrow a^4 \sum_x$$

$$\partial_\mu \longrightarrow \text{finite differences}$$

- Momentum $p \leq \frac{\pi}{a} \implies$ natural UV cutoff.

- To get physical results, need to perform:
  1. Infinite volume limit $(V \to \infty) \longrightarrow$ numerically or analytically
  2. Continuum limit $(a \to 0) \longrightarrow$ min. 3 different a

Lattice QCD

- Compute path integral in Euclidean spacetime

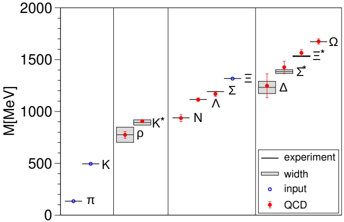
$$\int [\mathrm{d}U]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}\psi]\;\mathcal{O}\;e^{-S_{\mathrm{g}}(U)-\overline{\psi}\,M(U)\,\psi} = \int [\mathrm{d}U]\;\mathcal{O}_{\mathrm{Wick}}\;\det(M(U))\,e^{-S_{\mathrm{g}}(U)}$$

- $L^3 \times T = 96^3 \times 144 \quad\longrightarrow\quad \approx 4 \cdot 10^9$ dimensional integral
- $\det(M(U))\,e^{-S_{\mathrm{g}}(U)}$ positive measure $\longrightarrow$ stochastic integration

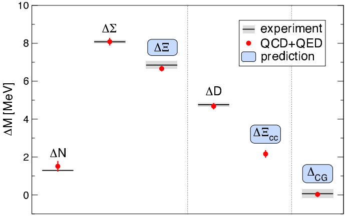
- 100000 years for a laptop $\longrightarrow$ 1 year for supercomputer

Lattice QCD: examples

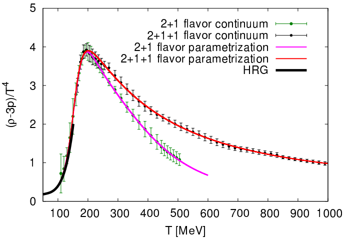
- Dürr et.al, *Ab-initio Determination of Light Hadron Masses*, Science 322 (2008) 1224-1227



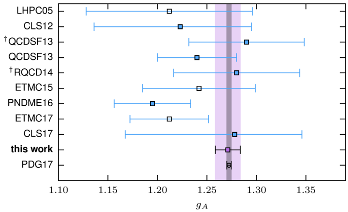
- Borsanyi et.al, *Ab initio calculation of the neutron-proton mass difference*, Science 347 (2015) 1452-1455



- Borsanyi et.al, *Lattice QCD for Cosmology*, Nature 539 (2016) 7627, 69-71

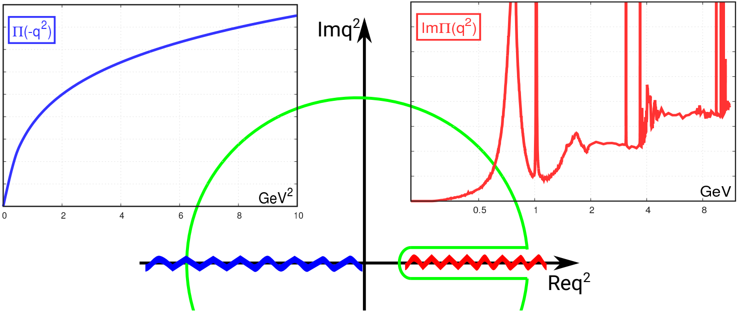


- Chang et.al, *A per-cent-level determination of the nucleon axial coupling from quantum chromodynamics*, Nature 558 (2018) 7708, 91-94



Hadronic vacuum polarization

- $\Pi_{\mu\nu}(q) = \left(q_\mu q_\nu - g_{\mu\nu} q^2\right) \Pi(q^2)$ analytic + branch-cut



- Minkowski from R-ratio experiments
- Euclidean from lattice QCD or exp. like MUonE
- Minkowski → Euclidean via dispersion relation ($Q^2 = -q^2$)

$$\Pi(Q^2) = \int_{s_{\text{th}}}^{\infty} ds \, \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \text{Im}\Pi(s)$$

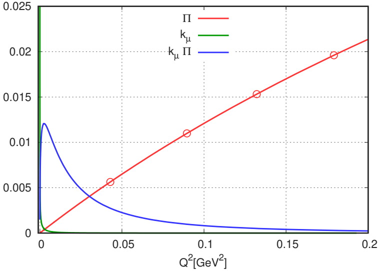
$a_\mu^{\text{LO-HVP}}$ from lattice QCD

- get Π from Euclidean current-current correlator [Blum '02]

$$\Pi_{\mu\nu} = \int dx\ e^{iQx} \langle J_\mu(x) J_\nu(0) \rangle = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2 \right) \Pi(Q^2)$$

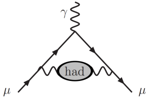
- Q is available at discrete momenta only
- smooth interpolation in Q and prescription for $\Pi(0)$

[Bernecker,Meyer '11], [HPQCD'14], ...



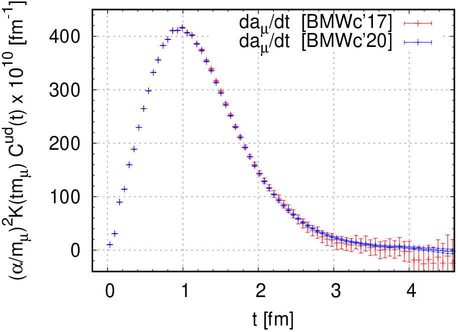
$$a_\mu^{\text{HVP}} = \frac{\alpha^2}{\pi^2} \int dQ^2\ k_\mu(Q^2)\ \Pi(Q^2)$$

$k_\mu(Q^2)$ describes the leptonic part of diagram



$a_\mu^{\text{LO-HVP}}$ from lattice QCD

- $a_\mu^{\mathrm{LO-HVP}} = \alpha^2 \int_0^\infty dt\ K(t)\ C(t)$



with the weight function

$$K(t) = \int_0^{Q^2_{\text{max}}} \frac{dQ^2}{m_\mu^2}\, \omega\left(\frac{Q^2}{m_\mu^2}\right) \left[t^2 - \frac{4}{Q^2}\sin^2\left(\frac{Qt}{2}\right)\right]$$

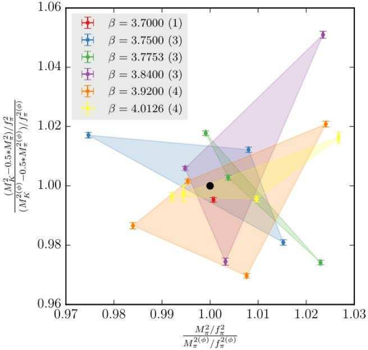
$$\omega(r) = [r + 2 - \sqrt{r(r+4)}]^2/\sqrt{r(r+4)}$$

- only integrate up to $Q^2_{\text{max}} = 3\,\text{GeV}^2$
- $Q^2 > Q^2_{\text{max}}$: perturbation theory

Simulation setup

Simulations

- Tree-level Symanzyk gauge action
- $N_f = 2+1+1$ staggered fermions
- stout smearing 4 steps, $\varrho = 0.125$
- $L \sim 6\,\text{fm}$, $T \sim 9\,\text{fm}$
- M_π and M_K are around physical point

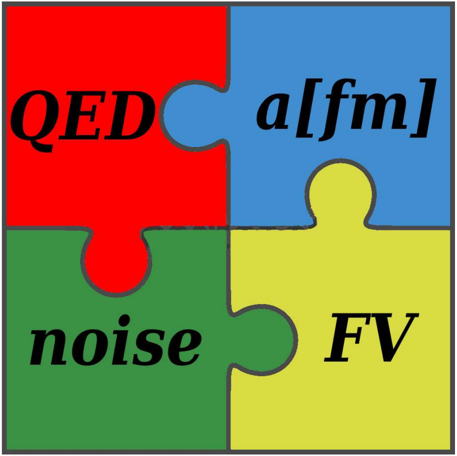


β	a[fm]	$L \times T$	#conf
3.7000	0.1315	48×64	904
3.7500	0.1191	56×96	2072
3.7753	0.1116	56×84	1907
3.8400	0.0952	64×96	3139
3.9200	0.0787	80×128	4296
4.0126	0.0640	96×144	6980

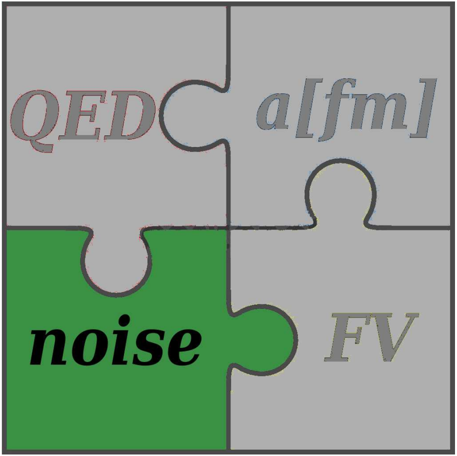
- Ensembles for dynamical QED

β	a[fm]	$L \times T$	#conf
3.7000	0.1315	24×48	716
		48×64	300
3.7753	0.1116	28×56	887
3.8400	0.0952	32×64	4253

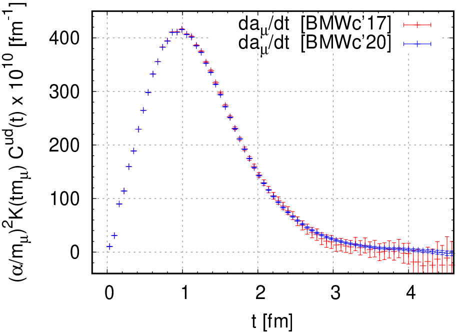
Challenges



Noise reduction



Upper and lower bounds on $\langle JJ \rangle$



- decrease noise by replacing $C(t)$ by upper/lower bounds above t_c

[Lehner 2016] [Borsanyi *et.al.* 2017]

$$0 \leq C^{\text{light}}(t) \leq C^{\text{light}}(t_c) e^{-E_{2\pi}(t-t_c)} \qquad t_c = 4.0\,\text{fm}$$

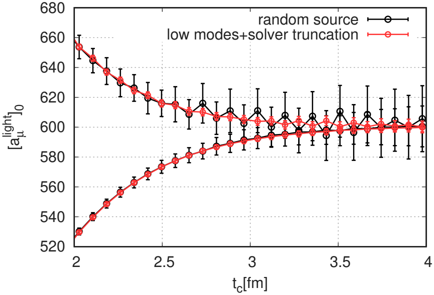
$$0 \leq -C^{\text{disc}}(t) \leq \frac{1}{10} C^{\text{light}}(t_c) e^{-E_{2\pi}(t-t_c)} + C^{\text{strange}}(t) + C^{\text{charm}}(t) \qquad t_c = 2.5\,\text{fm}$$

Low Mode Averaging

- Treat lowest eigenmodes of Dirac operator exactly

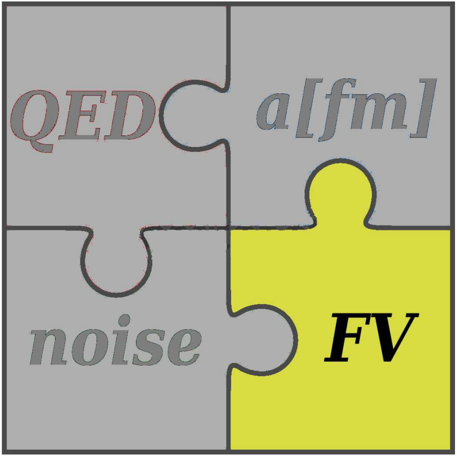
[Neff *et.al.* 2001] [Giusti et.al. 2004] [Li *et.al.* 2010] ...

- $L = 6\,\text{fm}$ ≈ 1000 eigenvectors up to $\approx m_s/2$
- $L = 11\,\text{fm}$ ≈ 6000 eigenvectors



- factor 5 gain in precision
- bounding t_c: $3\,\text{fm} \to 4\,\text{fm}$
- few permil accuracy on each ensemble

Finite volume corrections

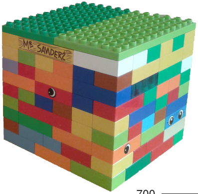


FV: lattice

- FV correction in two steps

$$a_\mu(\infty) - a_\mu(\text{ref}) = [a_\mu(\text{big}) - a_\mu(\text{ref})]_{\text{4HEX}} + [a_\mu(\infty) - a_\mu(\text{big})]_{\text{XPT}}$$

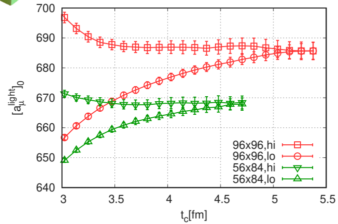
$L_{\text{ref}} = 6.272\,\text{fm}$
$T_{\text{ref}} = \frac{3}{2} L_{\text{ref}}$



$L_{\text{big}} = 10.752\,\text{fm}$
$T_{\text{big}} = 10.752\,\text{fm}$

1. $a_\mu(\text{big}) - a_\mu(\text{ref})$
   Choose action with small taste splitting
   - 4 steps of HEX smearing
   - DBW2 gauge action
   - $\beta = 0.73,\ a = 0.112\,\text{fm}$
   - $M_\pi = 104\,\text{MeV}$ and $M_\pi = 121\,\text{MeV}$
   - Interpolate to $M_\pi = 110\,\text{MeV}$

$$\longrightarrow \quad M_{\pi,\text{HMS}}^{-2} \equiv \frac{1}{16} \sum_\alpha M_{\pi,\alpha}^{-2} = M_{\pi^0,\text{phys}}^{-2}.$$



$$a_\mu(\text{big}) - a_\mu(\text{ref}) = 18.1(2.0)_{\text{stat}}(1.4)_{\text{cont}}$$

FV: non-lattice

Comparison to non-lattice approaches

- NLO and NNLO Chiral perturtabion theory (XPT)

  [Gasser & Leutwyler 1985] [Bijnens *et.al.* 1999]

- MLLGS-model [Gounaris & Sakurai 1968] [Lellouch & Lüscher 2001] [Meyer 2011] [Francis *et.al.* 2013]
- Hansen–Patella approach [Hansen & Patella 2019,2020]
- Rho-pion-gamma model (RHO) [Sakurai 1960], [Jegerlehner & Szafron 2011] [Chakraborty *et.al.* 2017]

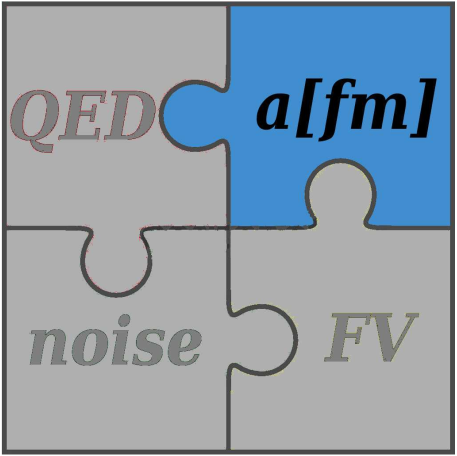
	NLO XPT	NNLO XPT	MLLGS	HP	RHO
$a_\mu(\text{big}) - a_\mu(\text{ref})$	11.6	15.7	17.8	16.7	15.2

$$a_\mu(\text{big}) - a_\mu(\text{ref}) = 18.1(2.0)_{\text{stat}}(1.4)_{\text{cont}}$$

2. $a_\mu(\infty) - a_\mu(\text{big})$
   - NLO XPT: 0.3
   - NNLO XPT: 0.6

$$a_\mu(\infty) - a_\mu(\text{ref}) = 18.7(2.0)_{\text{stat}}(1.4)_{\text{cont}}(0.3)_{\text{big}}(0.6)_{I=0}(0.1)_{\text{qed}}[2.5]$$

Scale setting

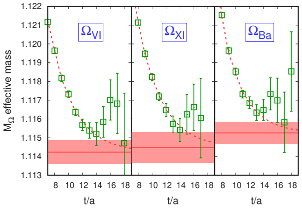


Scale determination

Lattice spacing a enters into a_μ determination:

- physical value of m_μ
- physical values of m_π, m_K

$\longrightarrow$ $\Delta_{\text{scale}} a_\mu \sim 2 \cdot \Delta(\text{scale})$

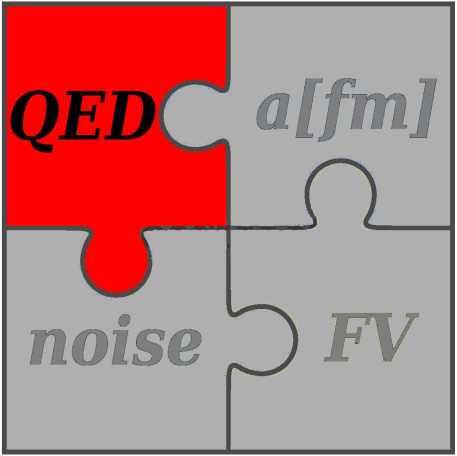


1. For final results: M_Ω scale setting $\longrightarrow$ $a = (aM_\Omega)^{\text{lat}} / M_\Omega^{\text{exp}}$
   - Experimentally well known: 1672.45(29) MeV [PDG 2018]
   - Moderate m_q dependence
   - Can be precisely determined on the lattice
2. For separation of isospin breaking effects: w_0 scale setting
   - Moderate m_q dependence
   - Can be precisely determined on the lattice
   - No experimental value

     $\longrightarrow$ Determine value of w_0 from $M_\Omega \cdot w_0$

$$w_0 = 0.17236(29)(63)[70] \text{ fm}$$

Isospin breaking



QCD+QED

- Reach sub-percent level: include isospin breaking effects for
    - $\langle jj \rangle$
    - masses
    - scale
- Rewrite dynamical QED as quenched QED expectation values

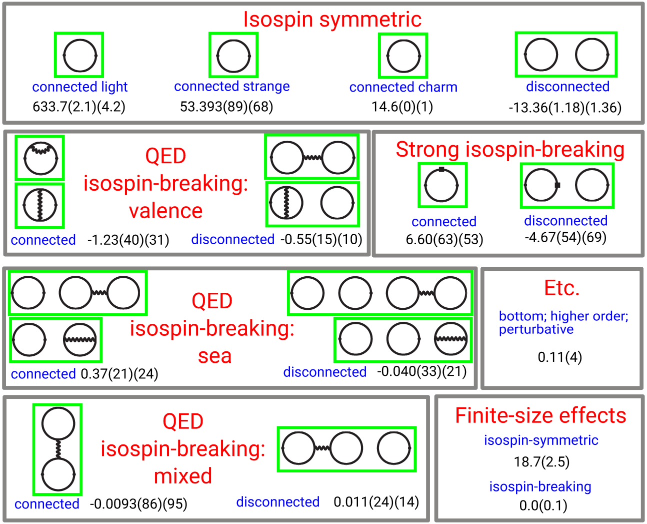
$$\langle \mathcal{O} \rangle_{\text{QCD+unquenched QED}} = \frac{\left\langle \left\langle \mathcal{O}(U,A) \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{\text{quenched QED}} \right\rangle_{\text{QCD}}}{\left\langle \left\langle \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{\text{quenched QED}} \right\rangle_{\text{QCD}}}$$

- Take isospin symmetric gluon configurations: U
- Compute derivatives

$$m_l \frac{\partial X}{\partial \delta m} \qquad \frac{\partial X}{\partial e} \qquad \frac{1}{2} \frac{\partial^2 X}{\partial e^2}$$

- Hybrid approach:
    - sea effects: derivatives [De Divitiis *et.al.* 2013]
    - valence effects: finite differences [Eichten *et.al.* 1997]

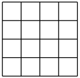
Overview of contributions

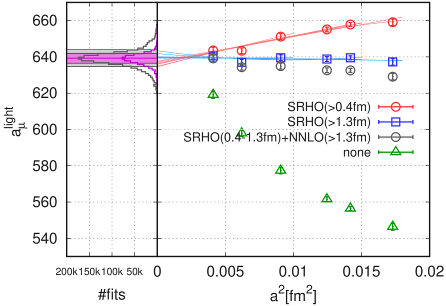


$$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

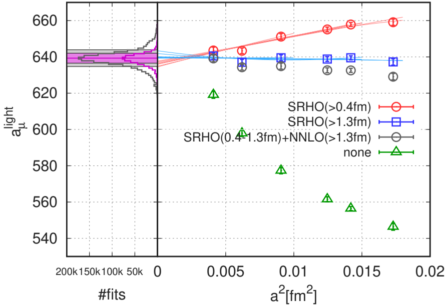
Results for $a_\mu^{\text{LO-HVP}}$

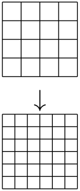
Continuum limit



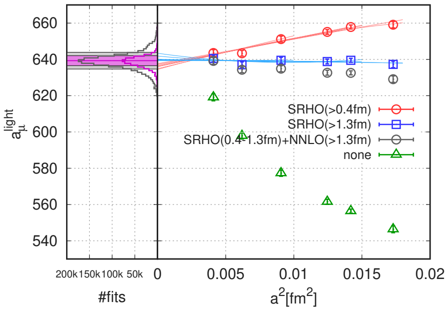


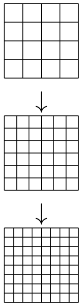
Continuum limit



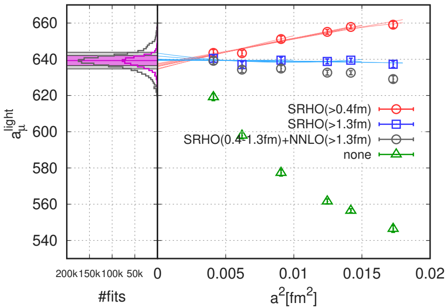


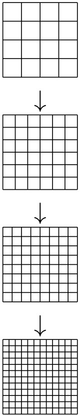
Continuum limit



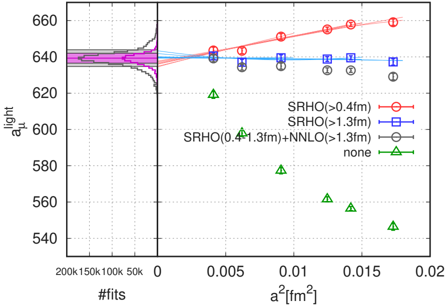


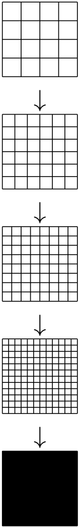
Continuum limit





Continuum limit



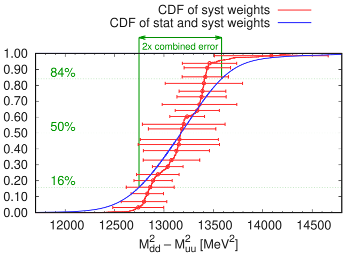


Global fit procedure

- For full result: physical point is set via
  $M_\Omega, \quad M^2_{K_\chi} = \frac{1}{2}\left(M^2_{K_0} + M^2_{K_+} - M^2_{\pi+}\right), \quad \Delta M^2_K, \quad M^2_{\pi_0}$ $\longleftarrow$ Type-I
- For IB-decomposition: physical point is set via
  $w_0, \quad M^2_{ss}, \quad \Delta M^2 = M^2_{dd} - M^2_{uu}, \quad M^2_{\pi_\chi} = \frac{1}{2}\left(M^2_{uu} + M^2_{dd}\right)$ $\longleftarrow$ Type-II
- Expand observable around physical point

$$Y = A + BX_l + CX_s + DX_{\delta m} + Ee_v^2 + Fe_v e_s + Ge_s^2$$

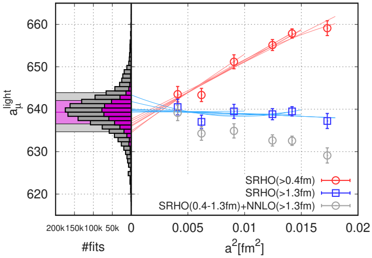
- Combined χ^2 fit for all components
- Several thousand analyses, combined using histogram method
  - functional form:
    linear vs quadratic in $a^2\alpha_s(1/a)^n$
    a^2 vs $a^2\alpha_s(1/a)^3$ [Husung *et.al* 2020]
    ...
  - cuts in lattice spacing
  - hadron mass fit ranges
  - starting t_c for taste improvement
  - ...



Results

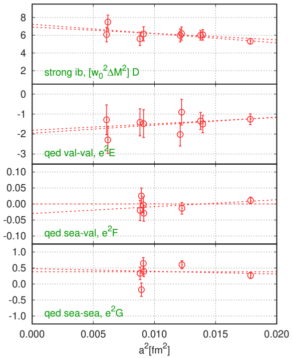
- Continuum limit of a_μ^{light} – 516096 fits

phys. point: $M_\Omega, M^2_{K_\chi}, \Delta M^2_K$



$$a_\mu^{\text{light}}(L_{\text{ref}}, T_{\text{ref}}) = 639.3(2.0)(4.2)[4.6]$$

phys. point: $w_0, M^2_{ss}, \Delta M^2$



- $a_\mu^{\text{strange}}(L_{\text{ref}}, T_{\text{ref}}) = 53.379(89)(67)[111]$ – 32256 fits
- $a_\mu^{\text{disc}}(L_{\text{ref}}, T_{\text{ref}}) = -18.61(1.03)(1.17)[1.56]$ – 55296 fits

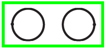
Results

Isospin symmetric

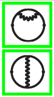

connected light
633.7(2.1)(4.2)

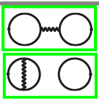

connected strange
53.393(89)(68)


connected charm
14.6(0)(1)


disconnected
-13.36(1.18)(1.36)

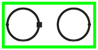
QED isospin-breaking: valence


connected -1.23(40)(31)

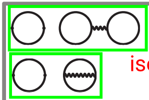

disconnected -0.55(15)(10)

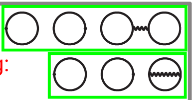
Strong isospin-breaking


connected
6.60(63)(53)


disconnected
-4.67(54)(69)

QED isospin-breaking: sea


connected 0.37(21)(24)

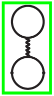

disconnected -0.040(33)(21)

Etc.

bottom; higher order; perturbative

0.11(4)

QED isospin-breaking: mixed


connected -0.0093(86)(95)


disconnected 0.011(24)(14)

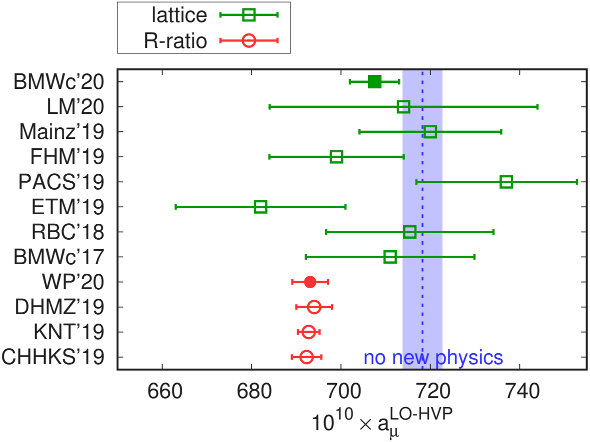
Finite-size effects

isospin-symmetric
18.7(2.5)

isospin-breaking
0.0(0.1)

$$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

Comparison with other determinations

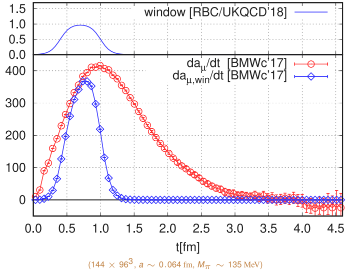


Window

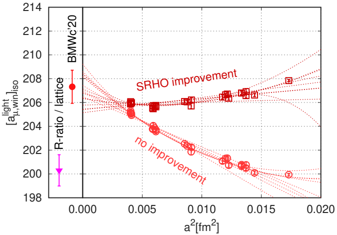
Window observable

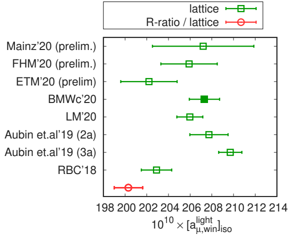
- Restrict correlator to window between $t_1 = 0.4\,\text{fm}$ and $t_2 = 1.0\,\text{fm}$

[RBC/UKQCD'18]



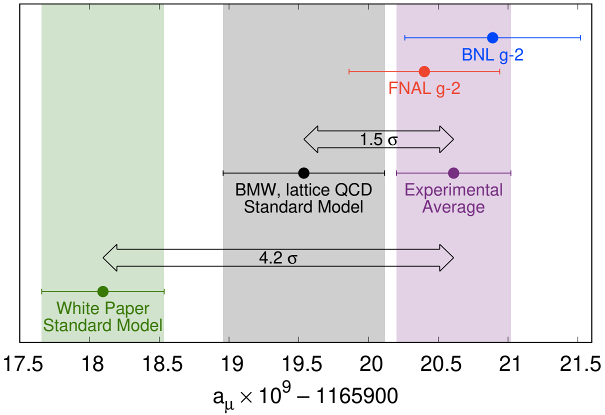
- Less challenging than full a_μ
  - signal/noise
  - finite size effects
  - lattice artefacts (short & long)





Conclusions & Outlook

Conclusions



Do our results imply NP @ EW scale?

- Passera et al '08: first exploration of connection $a_\mu^{\text{LO-HVP}} \leftrightarrow \Delta_{\text{had}}^{(5)}\alpha(M_Z^2)$
- Crivellin et al '20, most aggressive scenario (see also Keshavarzi et al '20, Malaescu et al '20): our results suggest a 4.2σ overshoot in $\Delta_{\text{had}}^{(5)}\alpha(M_Z^2)$ compared to result of fit to EWPO
- Assume same 2.8% relative deviation in R-ratio as we found in $a_\mu^{\text{LO-HVP}}$
- Hypothesis is not consistent w/ BMWc '17 nor new result