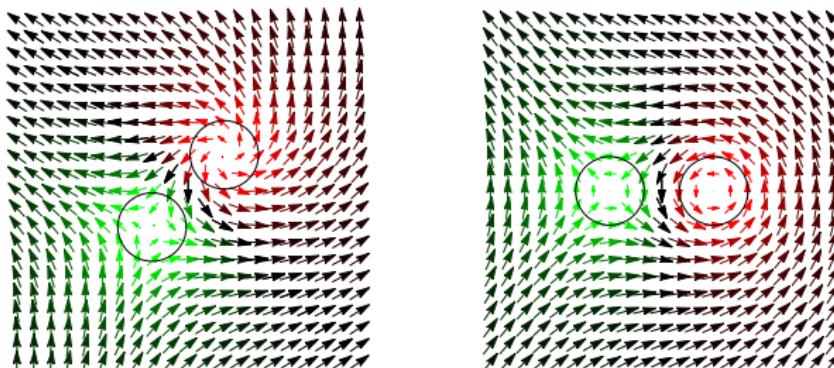
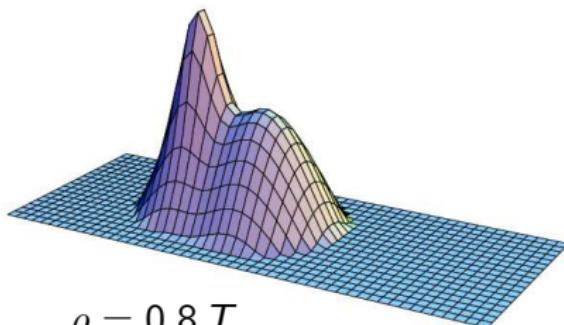


Calorons, monopoles and stable, charged solitons

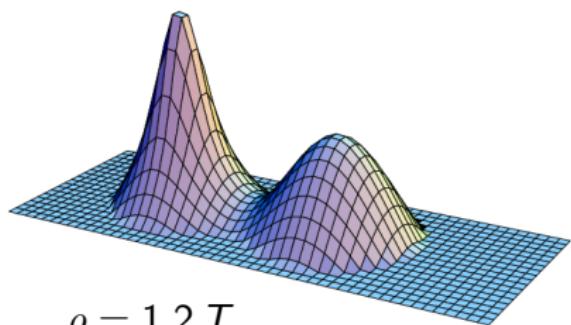
Manfried Faber
Atominsttitut, Technische Universität Wien



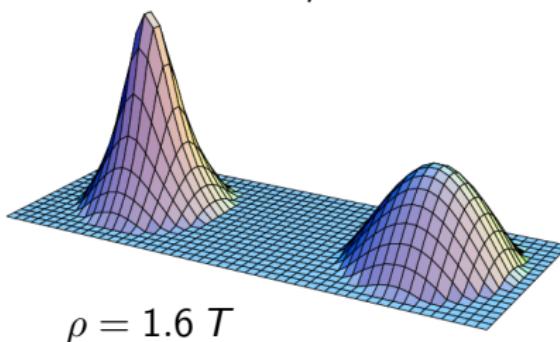
action density of SU(2) Calorons



$$\rho = 0.8 T$$



$$\rho = 1.2 T$$



$$\rho = 1.6 T$$

from: Thomas C Kraan and Pierre van Baal
Nuclear Physics B 533 (1998) 627–659

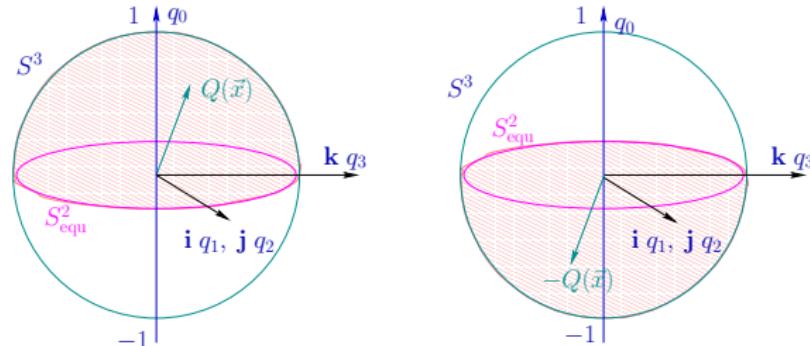
Caloron = instantons at finite temperature

in $SU(N)$ instantons separable in N dyons, quarks or monopoles,
calorons are characterised by Polyakov matrices

$$Q(\vec{x}) := \prod_{i=1}^{N_t} U_4(\vec{x}) = q_0(\vec{x}) - i\vec{\sigma}\vec{q}(\vec{x})$$

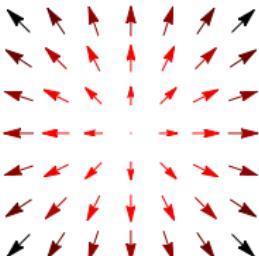
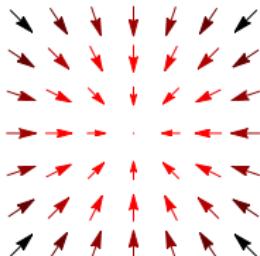
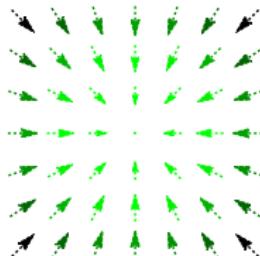
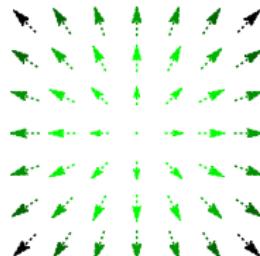
size of monopoles depends on $q_0(\infty)$, the asymptotic (an)holonomy,
specialise:

- ▶ $SU(2)$
- ▶ $q_0 = 0 \cdots$ equal sizes of monopoles



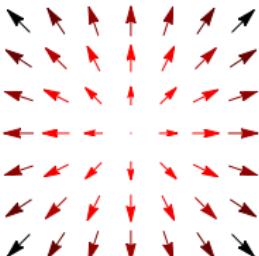
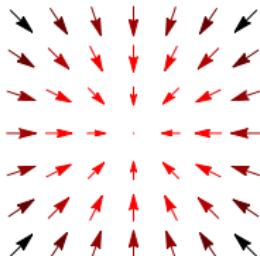
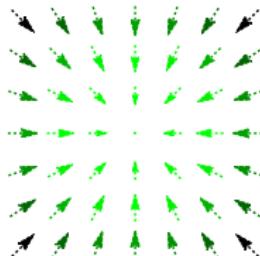
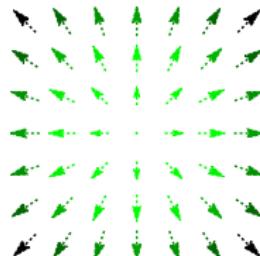
$$Q(\vec{x}) = q_0(\vec{x}) - i\vec{\sigma}\vec{q}(\vec{x}) = \cos \alpha(\vec{x}) - i\vec{\sigma}\vec{n}(\vec{x}) \sin \alpha(\vec{x}), \quad q_0^2 + \vec{q}^2 = 1$$

4 Types of monopoles

$\mathcal{T} = 1$	$\mathcal{T} = \Pi$	$\mathcal{T} = z$	$\mathcal{T} = z\Pi$
M	\bar{M}	L	\bar{L}
$e = 1$	$e = 1$	$e = -1$	$e = -1$
$m = 1$	$m = -1$	$m = 1$	$m = -1$
			
$q_0 \geq 0$	$q_0 \geq 0$	$q_0 \leq 0$	$q_0 \leq 0$

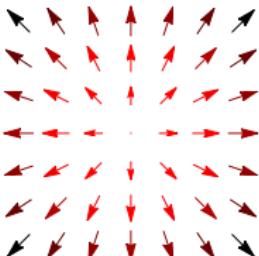
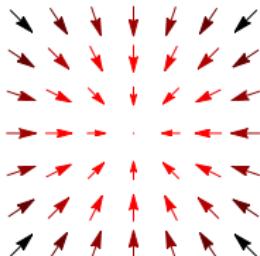
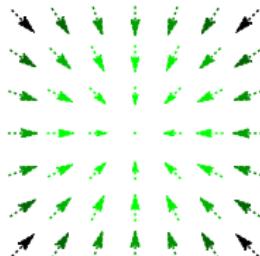
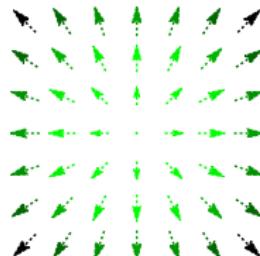
arrows indicate imaginary part $\vec{q}(x) = \sin \alpha(x) \vec{n}(x)$ of quaternions $Q(x)$.

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 try real monopoles:

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arrows indicate imaginary part $\vec{q}(x) = \sin \alpha(x) \vec{n}(x)$ of quaternions $Q(x)$.
 try real monopoles:

just with scalar field $Q(x)$: finite, electric, with charge and spin?

Lagrangian

$$Q(x) \rightarrow \vec{\Gamma}_\mu(x) \rightarrow \vec{R}_{\mu\nu}(x) \rightarrow \mathcal{L}(x)$$

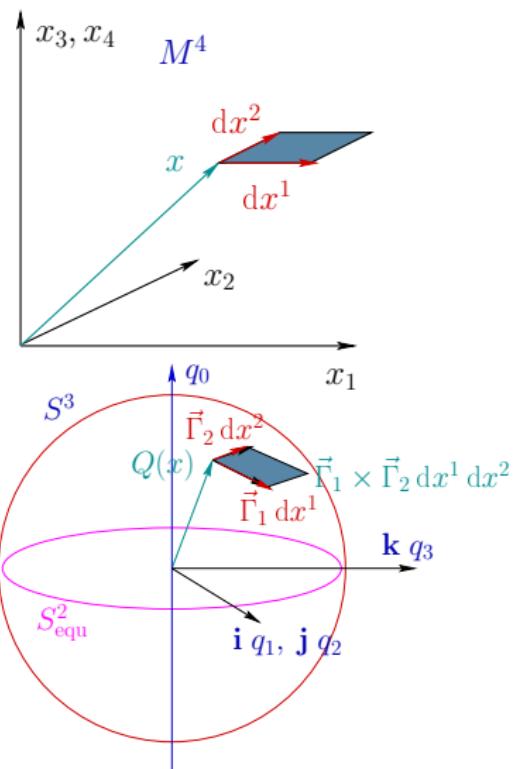
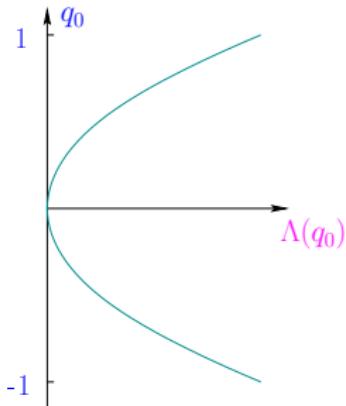
Connection one-form = dual photon field

$$\partial_\mu Q(x) Q^\dagger(x) =: -i \vec{\Gamma}_\mu(x) \vec{\sigma}$$

$$\text{Curvature: } \vec{R}_{\mu\nu}(x) := \vec{\Gamma}_\mu(x) \times \vec{\Gamma}_\nu(x)$$

$$\text{Lagrangian: } \mathcal{L} = -\frac{\alpha_f \hbar c_0}{4\pi} \left(\frac{1}{4} \vec{R}_{\mu\nu} \vec{R}^{\mu\nu} + \Lambda \right)$$

$$\text{potential term: } \Lambda(x) = q_0^6/r_0^4$$



2D degeneracy of vacuum \rightarrow two Goldstone bosons = photons

Calorons and monopoles

Manfried Faber

Relation to other models

- ▶ a 3D generalisation of the Sine-Gordon model
 $1+1\text{D} \rightarrow 3+1\text{D}$, 1 dof \rightarrow 3 dofs
- ▶ a model for soft dual Dirac monopoles
no Dirac string, no singularity in the origin
- ▶ a modification of the Skyrme model
short range \rightarrow long-range interaction

Relation to nature:

$${}^*\vec{F}_{\mu\nu} := -\frac{e_0}{4\pi\varepsilon_0 c} \vec{R}_{\mu\nu} = \begin{pmatrix} 0 & \vec{B}_1 & \vec{B}_2 & \vec{B}_3 \\ -\vec{B}_1 & 0 & \frac{\vec{E}_3}{c} & -\frac{\vec{E}_2}{c} \\ -\vec{B}_2 & -\frac{\vec{E}_3}{c} & 0 & \frac{\vec{E}_1}{c} \\ -\vec{B}_3 & \frac{\vec{E}_2}{c} & -\frac{\vec{E}_1}{c} & 0 \end{pmatrix}. \quad (1)$$

Stable minima of energy (topological Solitons)

► hedgehog ansatz: $\vec{n}(x) = \frac{\vec{r}}{r}$,

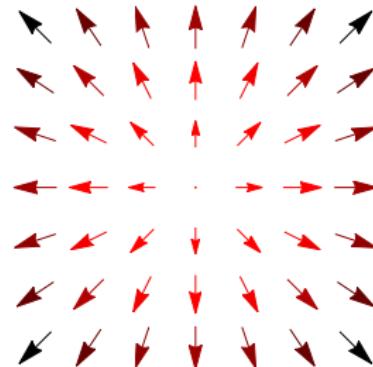
$$\vec{q}(x) = \vec{n}(x) \sin \alpha(x), \quad q_0 = \cos \alpha(x),$$

$$\alpha = \alpha(\rho), \quad \rho = r/r_0,$$

$$q_0^2 + \vec{q}^2 = 1,$$

$$Q(x) = q_0(x) + i\vec{\sigma}\vec{q}(x),$$

soliton covers half of S^3



► minimisation of energy $q_0 = \pm \frac{r_0}{\sqrt{x^2+r_0^2}} \quad q_i = \pm \frac{x_i}{\sqrt{x^2+r_0^2}}$

► energy of soliton $E = \frac{\alpha_f \hbar c}{r_0} \frac{\pi}{4}$

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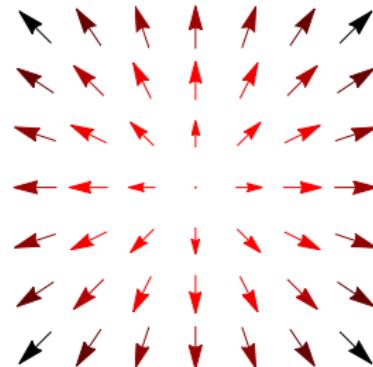
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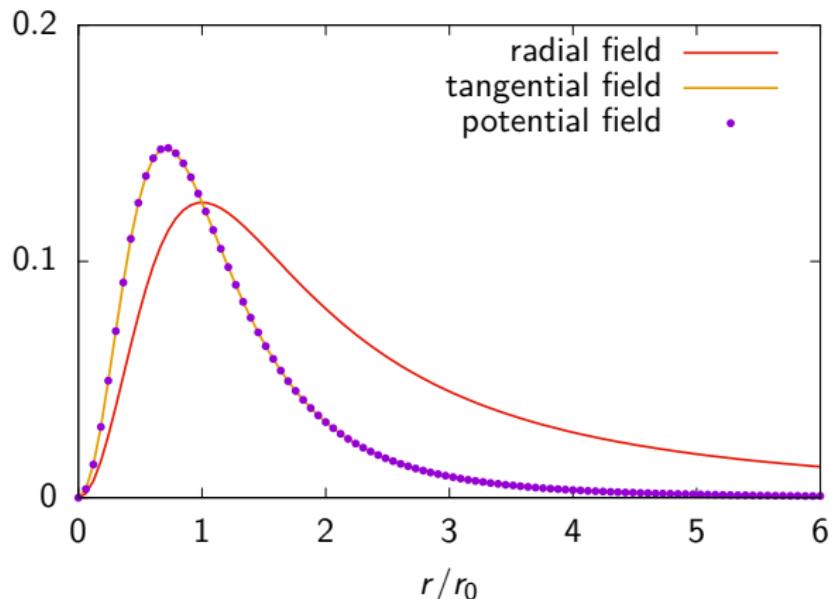
► compare with monopoles? with (non) existing?

$$\alpha_f \hbar c = 1.44 \text{ MeV fm}, \quad m_e c^2 = 0.511 \text{ MeV}, \quad r_0 = 2.21 \text{ fm}$$

Energy densities No Divergencies!

$$\mathcal{L} = -\frac{\alpha_f \hbar c}{4\pi} \left(\frac{1}{4} \vec{R}_{\mu\nu} \vec{R}^{\mu\nu} + \frac{q_0^6}{r_0^4} \right), \quad q_0(\rho) = \cos \alpha(\rho) = \frac{1}{\sqrt{1+\rho^2}}$$

radial energy densities



particle and field are indistinguishable

Field variables in 3+1D

describe field of rotations of spatial Dreibein in $\mathbb{M}^4 \ni x$

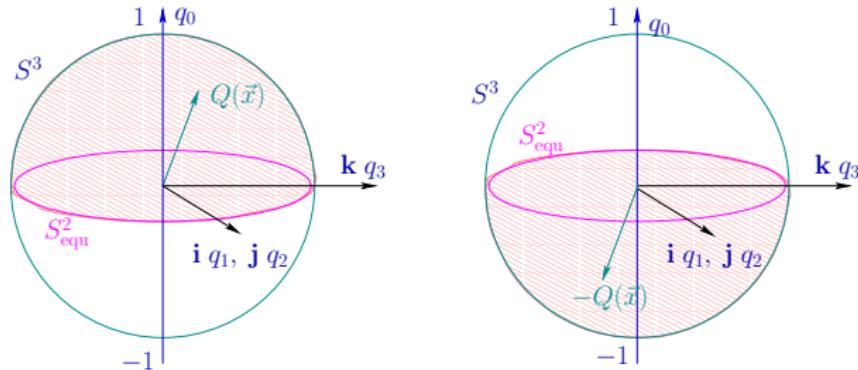
Use rotational group $D(x) \in SO(3)$

or simpler double covering group of $SO(3)$: $SU(2) \ni Q(x)$

$SO(3)$ versus $SU(2) \simeq \mathbb{S}^3$, $D(x) \leftrightarrow \pm Q(x)$

Field configurations $\pm\{Q(x)\}$ are identical

invariance under center-trafos z , exchanges of the hemispheres of \mathbb{S}^3



$$Q(\vec{x}) = q_0(\vec{x}) - i\vec{\sigma}\vec{q}(\vec{x}) = \cos \alpha(\vec{x}) - i\vec{\sigma}\vec{n}(\vec{x}) \sin \alpha(\vec{x}), \quad q_0^2 + \vec{q}^2 = 1$$

Caloron and monopoles

Manfried Faber

Imagine we have only Space-Time

What can we explain?

► Non-trivial metric: $g_{\mu\nu} \rightarrow$ Gravitation

► Rotating frames in $\mathbb{R}^3 : D(x) \in SO(3)$

Topological excitations \leftrightarrow Topological quantum numbers

$$\Pi_3(\mathbb{S}^3) = \mathbb{Z} \quad \leftrightarrow \text{ spin}$$

$$\Pi_3(\mathbb{S}^2) = \mathbb{Z} \quad \leftrightarrow \text{ charge}$$

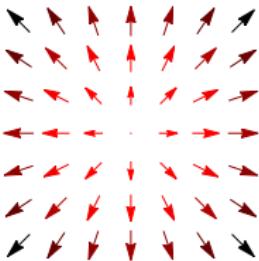
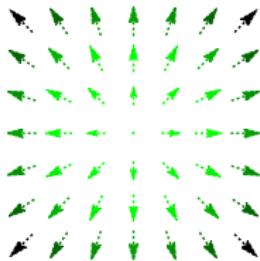
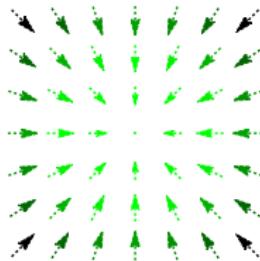
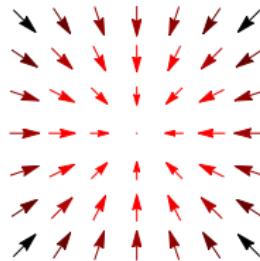
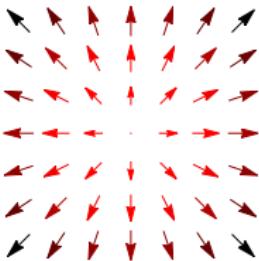
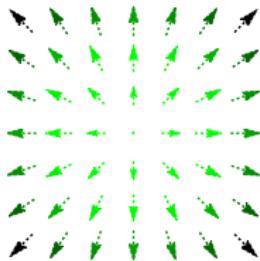
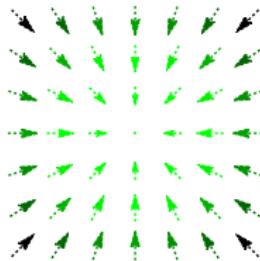
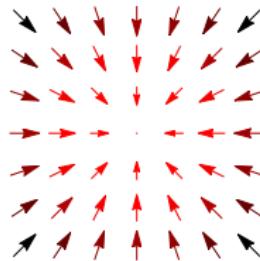
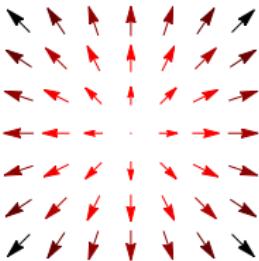
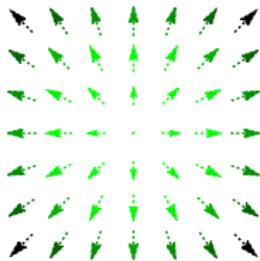
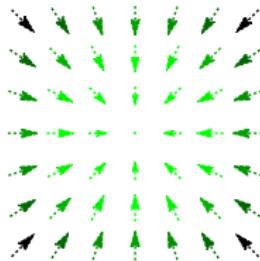
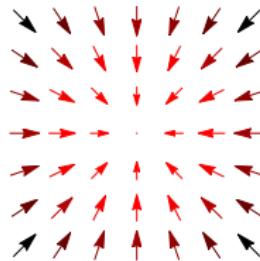
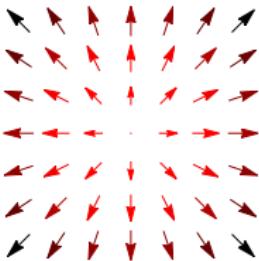
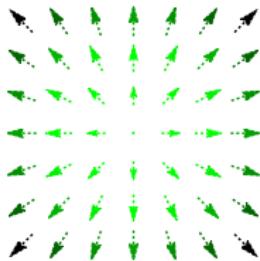
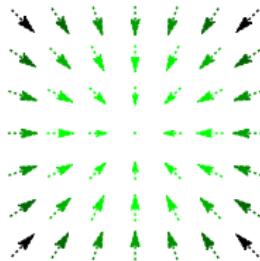
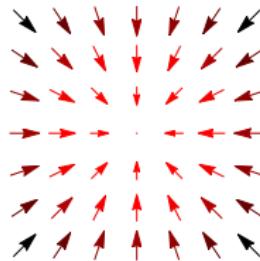
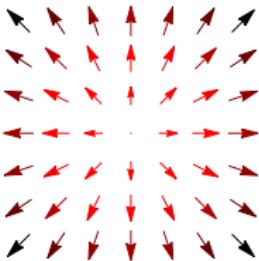
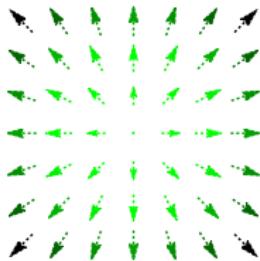
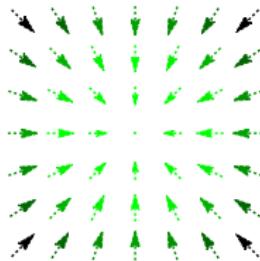
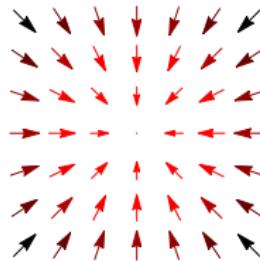
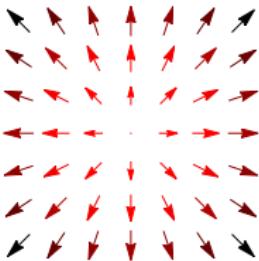
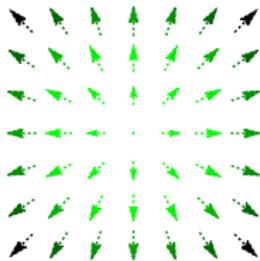
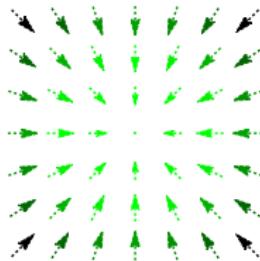
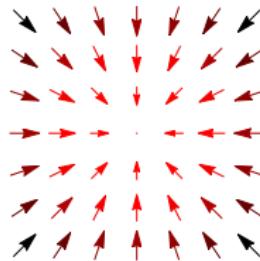
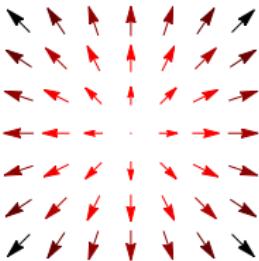
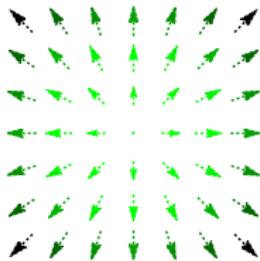
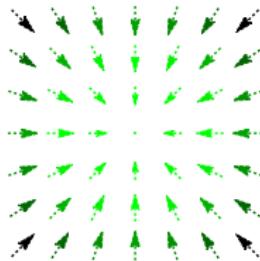
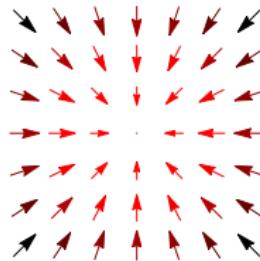
$$\Pi_2(\mathbb{S}^2) = \mathbb{Z} \quad \leftrightarrow \text{ photon number}$$

Non-topological excitations:

dark matter?

dark energy?

Four classes of solitons

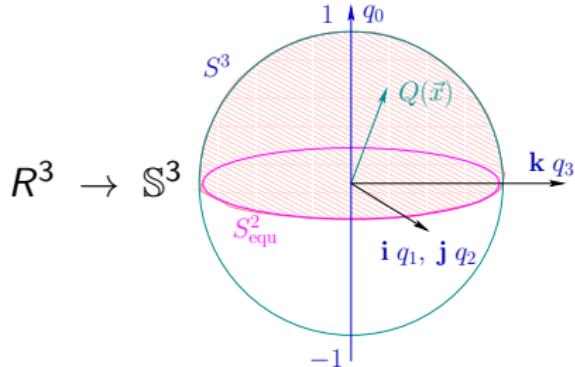
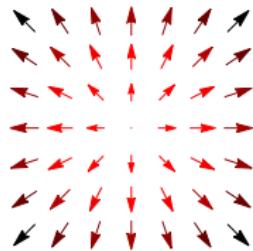
$\mathcal{T} = 1$	$\mathcal{T} = z\Pi$	$\mathcal{T} = z$	$\mathcal{T} = \Pi$
$Z = 1$	$Z = 1$	$Z = -1$	$Z = -1$
$\mathcal{Q} = \frac{1}{2}$	$\mathcal{Q} = -\frac{1}{2}$	$\mathcal{Q} = \frac{1}{2}$	$\mathcal{Q} = -\frac{1}{2}$
			
			
			
			
			
			
			

Corresponding to Dirac spinor $\psi = (e_-^\uparrow, e_-^\downarrow, e_+^\uparrow, e_+^\downarrow)$

Intrinsic parity operator: $\gamma_0 := \text{diag}(1, 1, -1, -1)$

$$H_0\psi_0 := m_0 c_0^2 \gamma_0 \psi_0$$

Spin, a topological quantum number



Field configuration $Q(\mathbf{r})$ of unit charge covers hemisphere of \mathbb{S}^3 , $s = \frac{1}{2}$.

Spin quantum number s

$$s := |\mathcal{Q}| = \left| \frac{1}{V(\mathbb{S}^3)} \int_0^\infty dr \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \vec{\Gamma}_r (\vec{\Gamma}_\vartheta \times \vec{\Gamma}_\varphi) \right|$$

Magnetic quantum numbers $m_s = \pm 1/2$: Upper and lower hemisphere

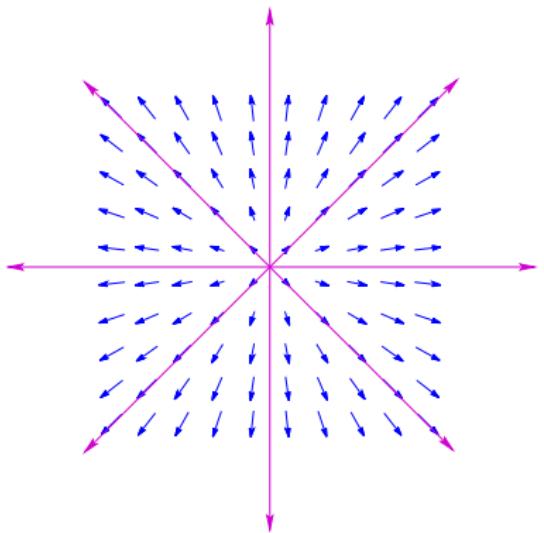
$$\mathcal{Q} = \chi \cdot S$$

Monopole is wired to surrounding space

flux lines \equiv lines of constant \vec{n} -field

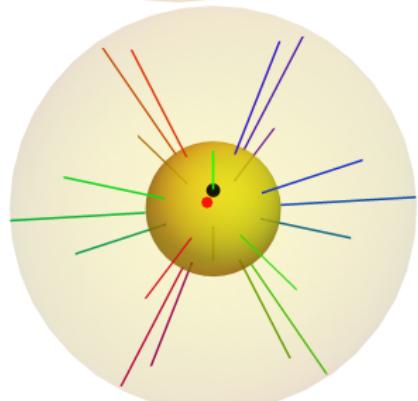
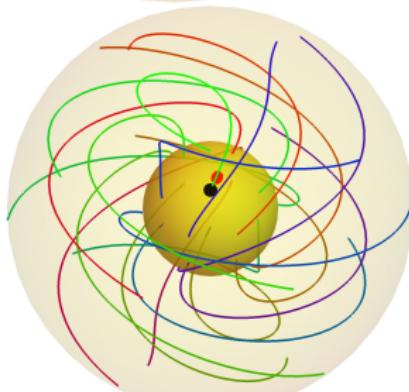
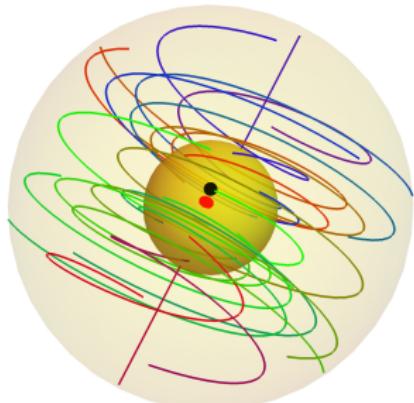
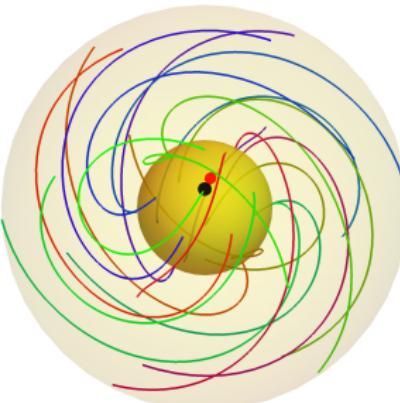
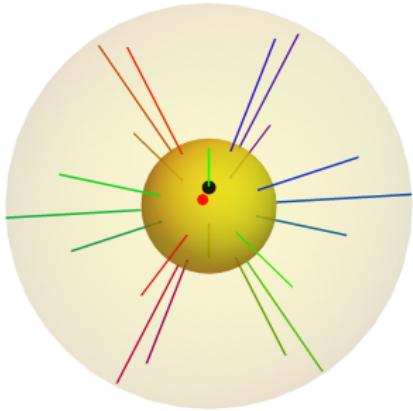
flux lines \equiv strings

they connect the soliton with the surrounding,
with other charges



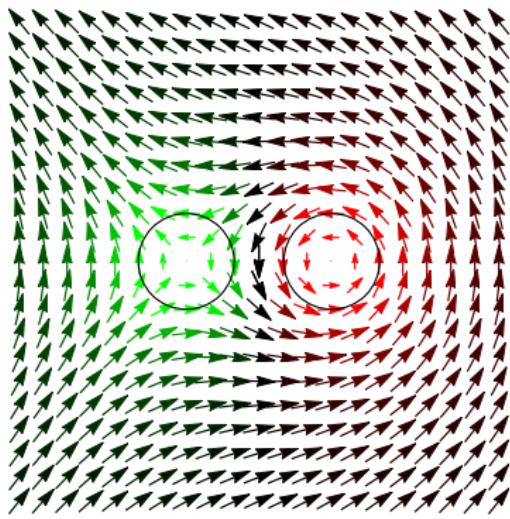
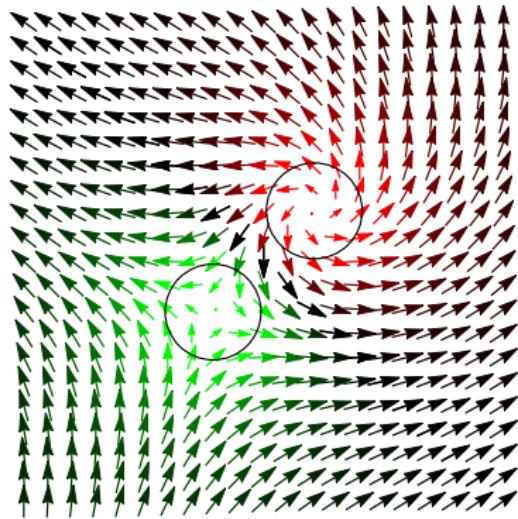
after 4π -rotation
soliton configuration is restored
a property of spin-1/2-particles

4π -rotations



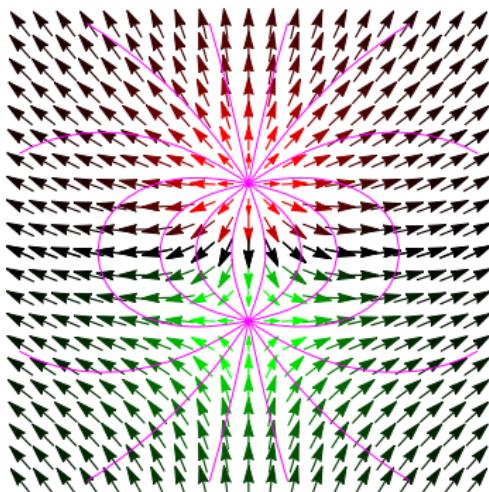
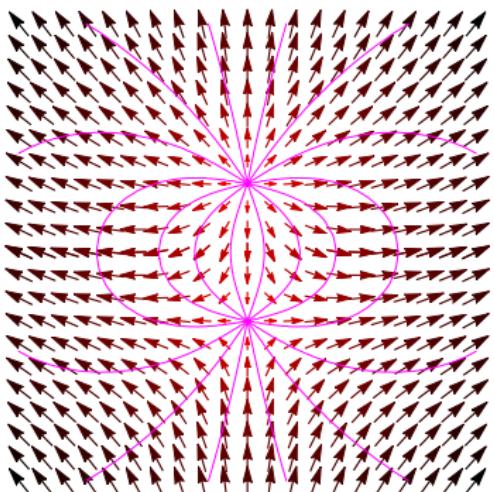
Spin, an angular momentum

Symmetry broken vacuum, $Q(\infty) = -i\sigma_3$, Field at infinity is constant
No rigid rotation possible



$S = 1$, Charge Zero

Field lines of dipol



Left: $S = 0$ configuration, Right: $S = 1$ configuration
field lines = lines of constant \vec{n} -field

Comparison to Maxwell's electrodynamics

1. The Lagrangian is Lorentz covariant, thus the laws of special relativity are respected.
2. Charges have Coulombic fields fulfilling Gaußes law.
3. Charges interact via $\frac{1}{r^2}$ electric fields, they feel Coulomb and Lorentz forces.
4. A local U(1) gauge invariance is respected.
5. There are two dofs of massless excitations for photons.

In distinction to Maxwell's electrodynamics

1. Electric charges are quantised, like the magnetic charges of Dirac monopoles. Charge is a topological quantum number.
2. By topological construction, mirror properties of particles and antiparticles.
3. The mass of solitons is completely due to field energy and finite.
4. The self-energy of charges is finite and does not need regularisation and renormalisation.
5. Charges and their fields are described by the same $SO(3)$ dofs.
6. $SO(3)$ dofs interpreted as orientations of spatial Dreibeins.
7. Gauge symmetry a geometrical phenomenon,basis changes on S^3 .
8. Spin has usual quantisation properties and combination rules.
9. 4 basic configurations of solitons, quantum numbers of Dirac spinors.

In distinction to Maxwell's electrodynamics

10. Solitons and antisolitons have opposite internal parity.
11. Solitons are characterised by a chirality quantum number which can be related to the sign of the magnetic quantum number.
12. Spin contributes to angular momentum due to internal rotations.
13. The canonical energy-momentum tensor is automatically symmetric.
14. Static charges are described by the spatial components of vector fields. Moving charges need time-dependent fields.
15. r -dependence of charge by finite size of solitons \rightarrow running coupling.
16. Local U(1) gauge invariance explained, bases choice on \mathbb{S}^2 .
17. Photon number \rightarrow Gaußian linking number of fibres on \mathbb{S}^2 .
18. Photon number changes by interaction with charges.

Rather unexpected

1. Spin and magnetic moment are dynamical properties only.
2. Electric and magnetic field vectors are perpendicular to each other
3. Existence of unquantised magnetic currents is allowed.
4. α -waves in $q_0 = \cos \alpha$ contribute to (dark) matter density.
5. α -waves lead to additional forces on particles and are a possible origin of quantum fluctuations.
6. Potential term allows mechanism of cosmic inflation
7. Potential term contributes to dark energy.

Aftermath

Physics is measurements of distances of objects and times of events.

This may indicate, that

Physics is geometry and not algebra.

Finally, one should use the algebra to describe the geometry.

General Relativity:

Wheeler: "Spacetime tells matter how to move;
matter tells spacetime how to curve."

My addition for Electrodynamics:

... Charges and electromagnetic fields tell space how to rotate.

Everything on earth is finite, besides ...

Thanks

References

[1, 2, 3, 4, 5, 6, 7, 8, 9]

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