Localisation of Dirac eigenmodes and confinement in gauge theories: the Roberge-Weiss transition

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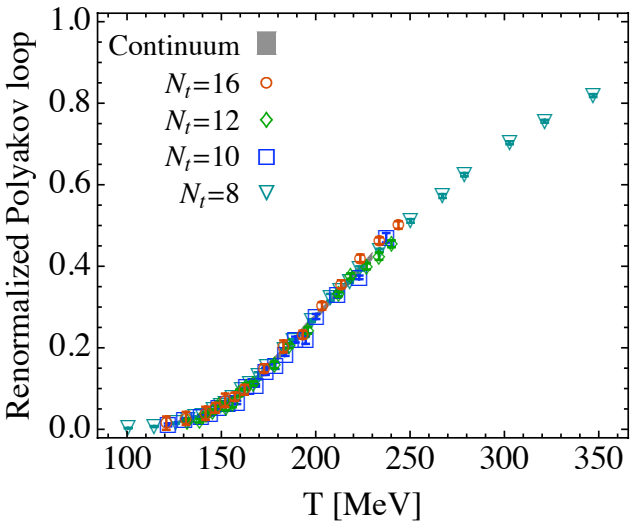
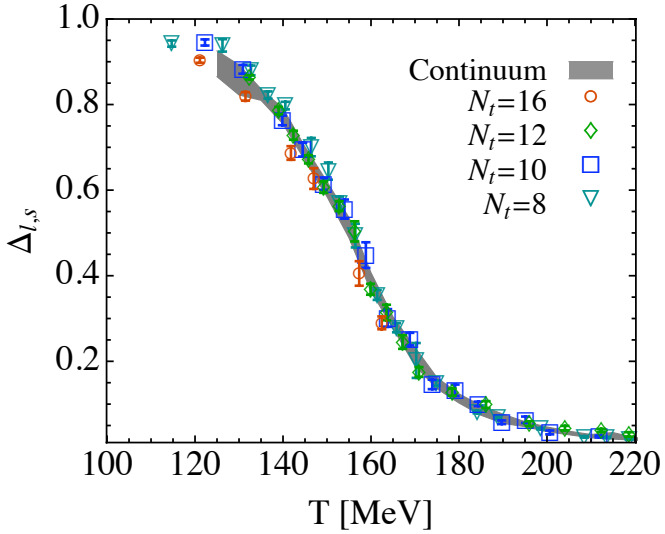
Based on M. Cardinali, M. D'Elia, F. Garosi, MG,
PRD 105 (2022) 014506 [arXiv:2110.10029]

QCD at Finite Temperature

Analytic crossover in the range $T \simeq 145 - 165$ MeV

Deconfinement, chiral symmetry restoration in the same temperature range





Renormalised Polyakov loop and chiral condensate
Figures from [Borsányi *et al.* (2010)]

In other QCD-like gauge theories with genuine phase transitions

- deconfinement improves chiral symmetry properties if single transition (e.g., pure gauge theory, $N_f = 3$ staggered fermions on coarse lattices)
- $T_{\rm dec} < T_\chi$ if two transitions are present (e.g., adjoint fermions)

Relation between the two phenomena still not fully clear

Finite-Temperature Gauge Theory on the Lattice

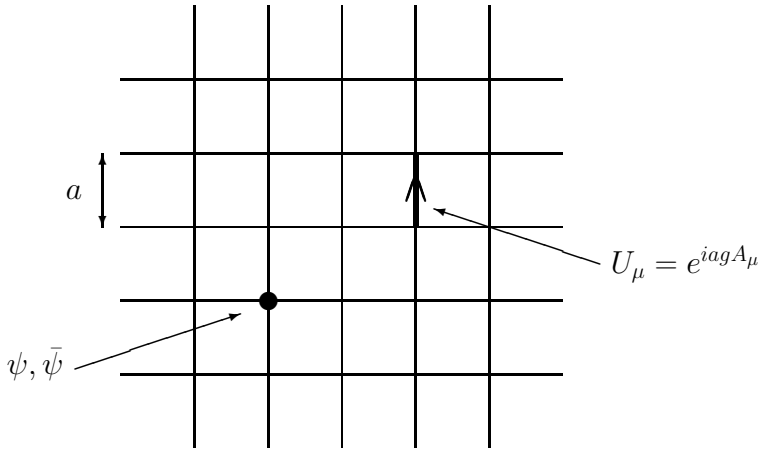
Partition function at finite T (imaginary time formulation)

$$Z = \int [DA] \det(\not{D}[A] + m) e^{-S_G[A]} \to \int [DU] \det(\not{D}_{\rm lat}[U] + m) e^{-S_G^{\rm lat}[U]}$$

- Euclidean fields, compact temporal direction of size $1/T$
- periodic/antiperiodic temporal b.c. for gauge/fermion fields

Lattice approach: Euclidean continuum replaced by discrete finite lattice

- Fields associated with lattice elements: fermions $\to$ sites, gauge fields $\to$ edges
- Hypercubic $N_t \times N_s^3$ lattice w/ periodic spatial b.c., temporal b.c. as above
- Thermodynamic ($V \to \infty$) and continuum ($a \to 0$) limits taken eventually at fixed $T = (aN_t)^{-1}$



Deconfinement and χSB from spontaneous SB

Deconfinement and χSB from SSB in opposite quark-mass limits

Quark mass $m \to \infty$ (pure gauge theory)

- Exact $\mathbb{Z}_3$ centre symmetry
- Spontaneously broken above $T_c \approx 290\,\mathrm{MeV}$ [G. Boyd *et al.* (1996)]

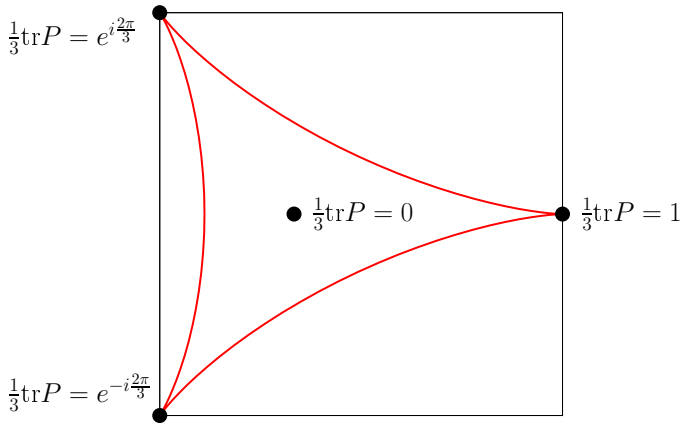
Quark free energy from Polyakov loop

$$\langle \mathrm{tr}\, P \rangle \propto e^{-F_q/T}$$

$T < T_c$: $\langle \mathrm{tr}\, P \rangle = 0 \Rightarrow F_q = \infty$

$T > T_c$: $\langle \mathrm{tr}\, P \rangle \neq 0 \Rightarrow F_q < \infty$

Deconfinement = PL ordering



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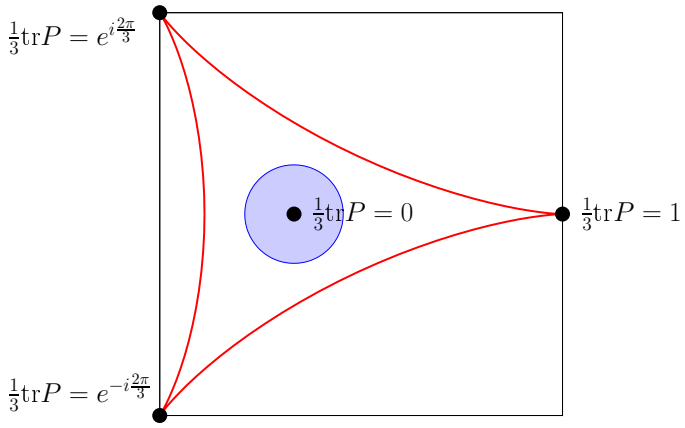

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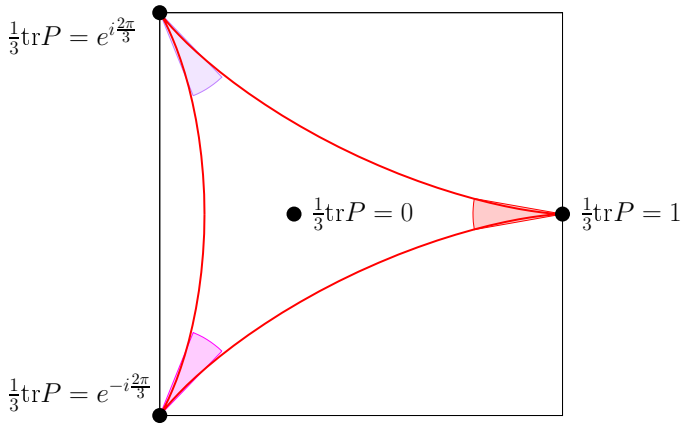


Deconfinement and χSB from spontaneous SB

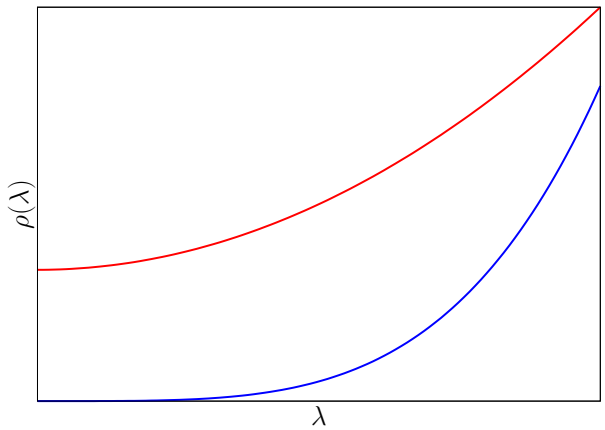
Quark mass $m \to 0$

- Exact $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ chiral symmetry
- Spontaneously broken below $T_c(N_f = 2) \approx 132\,\mathrm{MeV}$ [Ding *et al.* (2019)]

$$|\langle\bar{\psi}\psi\rangle| = \int_0^\infty d\lambda\, \frac{2m\rho(\lambda)}{\lambda^2 + m^2} \underset{m\to 0}{\to} \pi\rho(0^+)$$

$$\rho(\lambda) = \lim_{V\to\infty} \frac{T}{V}\Big\langle \sum_n \delta(\lambda - \lambda_n)\Big\rangle$$

χSB = accumulation of Dirac modes at $\lambda = 0$



Different symmetries, approximate @m_{phys}: how do they affect each other?

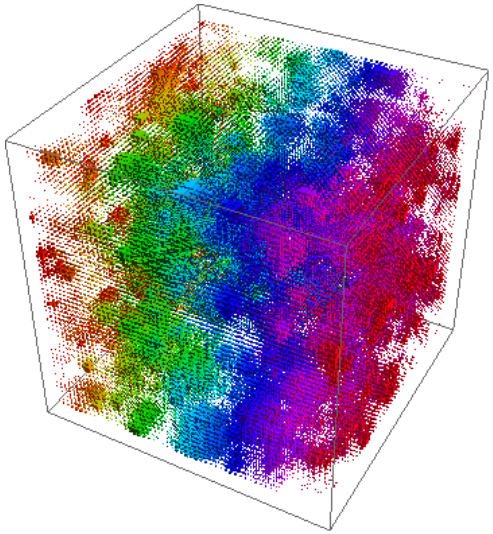
Localisation of Dirac eigenmodes

Low Dirac modes become localised at the QCD transition

<u>Delocalised</u> mode

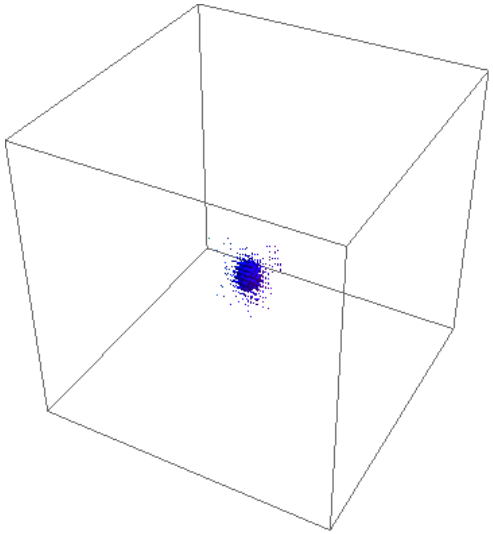
- extends throughout system
- $\|\psi(x)\|^2 \sim 1/L^\alpha$ with $0 < \alpha \leq d$

cond-mat: delocalised if $\alpha = d$, critical if $0 < \alpha < d$



<u>Localised</u> mode

- confined in finite region
- $\|\psi(x)\|^2 \sim 1/L^0$ inside, negligible outside



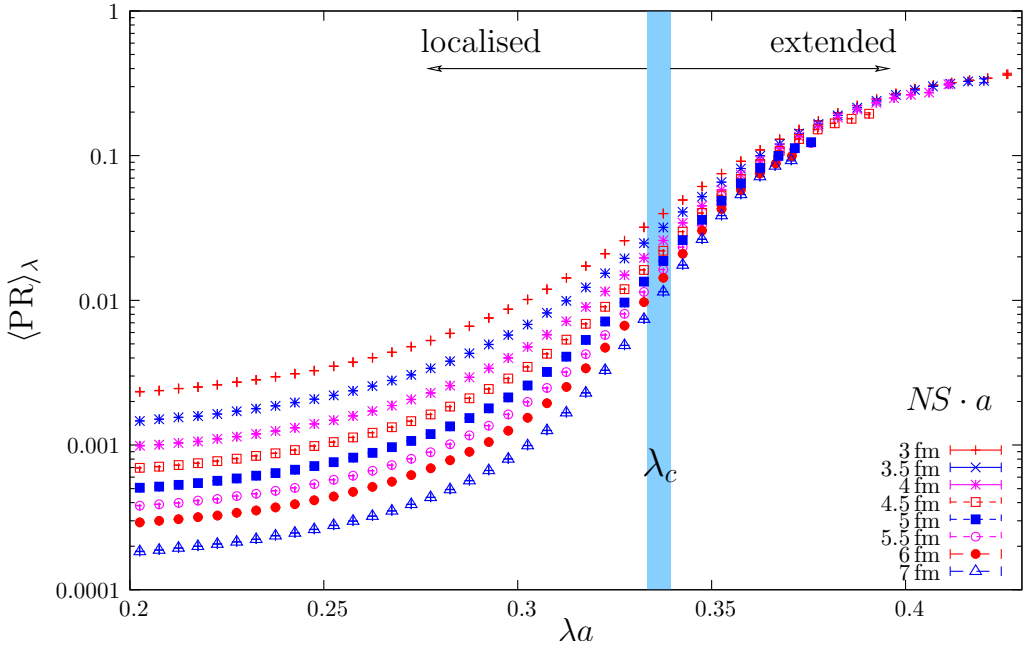
$$\int_0^\beta dt \int d^3x \, \|\psi_n(t,\vec{x})\|^2 = 1$$

$$\|\psi_n(t,\vec{x})\|^2 \equiv \sum_{c,\eta} |\psi_{n\,c,\eta}(t,\vec{x})|^2$$

Figures from [Ujfalusi *et al.* (2015)]

Localisation of Dirac eigenmodes in QCD

Participation ratio ≈ fraction of system occupied by a mode



Data for $T \simeq 2.6\,T_c$ from [MG *et al.* (2014)]

$$\mathrm{IPR}_n = \int_0^{\frac{1}{T}} dt \int d^d x \, \|\psi_n(t,\vec{x})\|^4 \sim L^{-\alpha}$$

$$\langle \mathrm{PR} \rangle_\lambda = \frac{T}{L^d} \left\langle \sum_n \delta(\lambda - \lambda_n) \mathrm{IPR}_n^{-1} \right\rangle \sim L^{-(d-\alpha)}$$

- Mobility edge λ_c separates localised and delocalised modes ($T > T_c$)
- Second-order (Anderson) transition at λ_c

Localisation and spectral statistics

Localisation of eigenmodes reflects on statistical properties of eigenvalues

- delocalised modes easily mixed by fluctuations $\rightarrow$ RMT- type statistics
- localised modes fluctuate independently $\rightarrow$ Poisson statistics

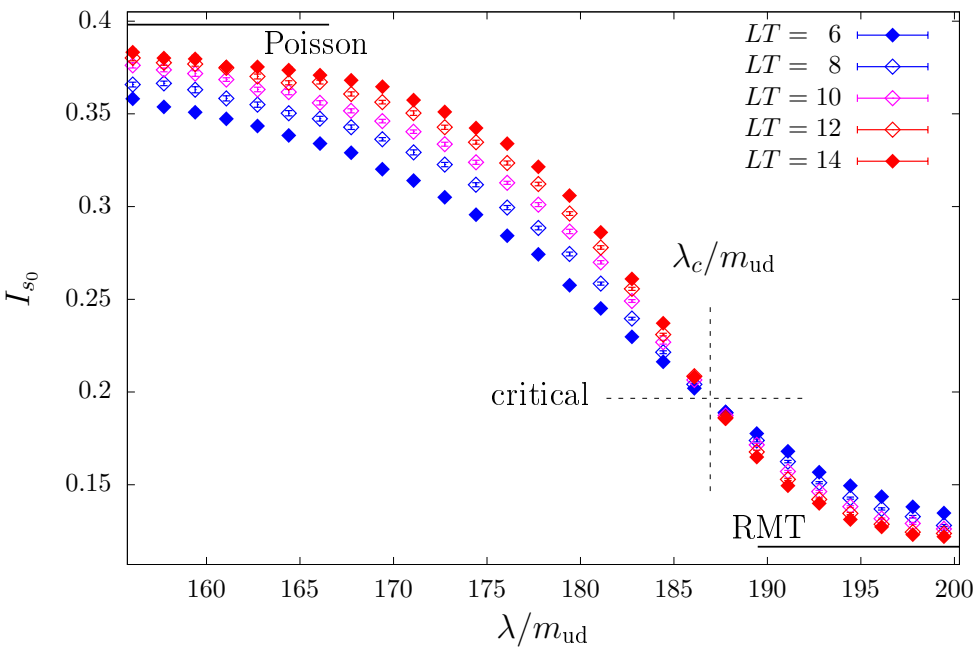
Universal expectations for <u>unfolded level spacings</u> $s_n = \frac{\lambda_{n+1}-\lambda_n}{\langle \lambda_{n+1}-\lambda_n \rangle_\lambda}$

- local spacing distribution $p_\lambda(s, L)$

$$I_{s_0}(\lambda) = \int_0^{s_0} ds\, p_\lambda(s)$$

$s_0 \approx 0.508$ maximises RMT/ Poisson difference

- $p_\lambda \rightarrow p_{\rm Poisson}$ (localised) or $p_\lambda \rightarrow p_{\rm RMT}$ (delocalised) as system size $L \rightarrow \infty$
- λ_c = <u>scale-invariant</u> point, critical statistics $p_{\rm crit}$



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Mobility edge in QCD

Mobility edge extrapolates to 0 in the crossover region [Kovács, Pittler (2012)]
No localised modes in the confined/chirally broken phase at low T

$\lambda_c/m_{\rm ud}$ is RG-invariant
[Kovács, Pittler (2012), MG (2022)]

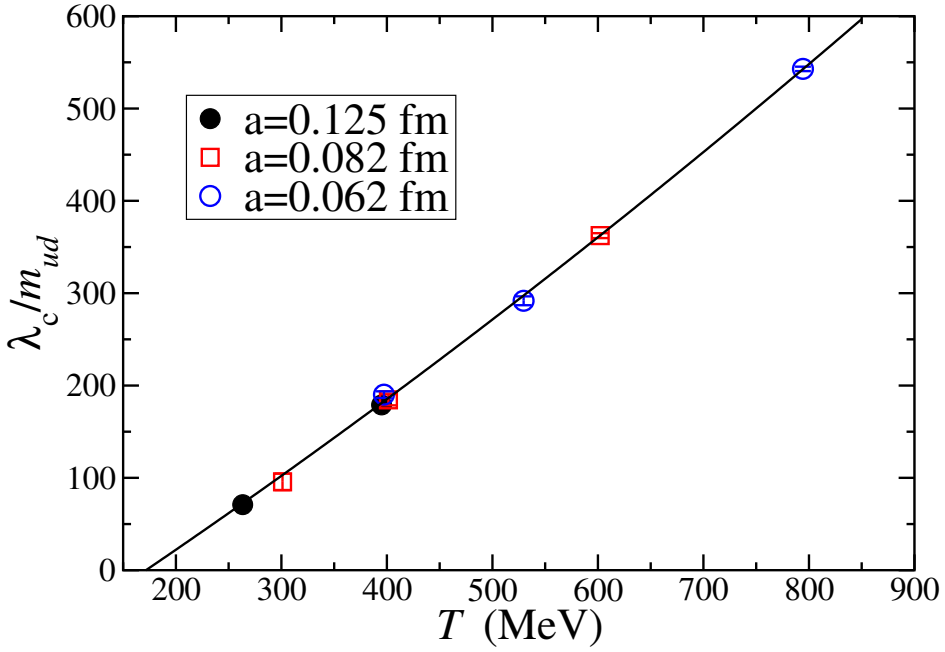


Figure from [Kovács, Pittler (2012)]

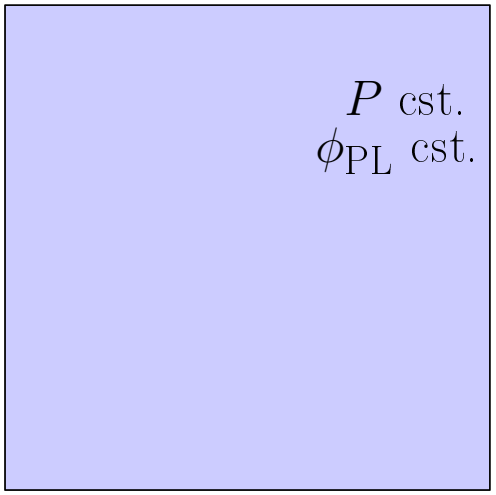
- Numerical evidence from the lattice [MG, Kovács (2021)]
- Localised modes seen with various fermion discretisations, survive continuum limit
  $\Rightarrow$ not a lattice artefact
- Same critical features as 3D unitary Anderson model
  [MG *et al.* (2014), Ujfalusi *et al.* (2015), Nishigaki *et al.* (2014)]

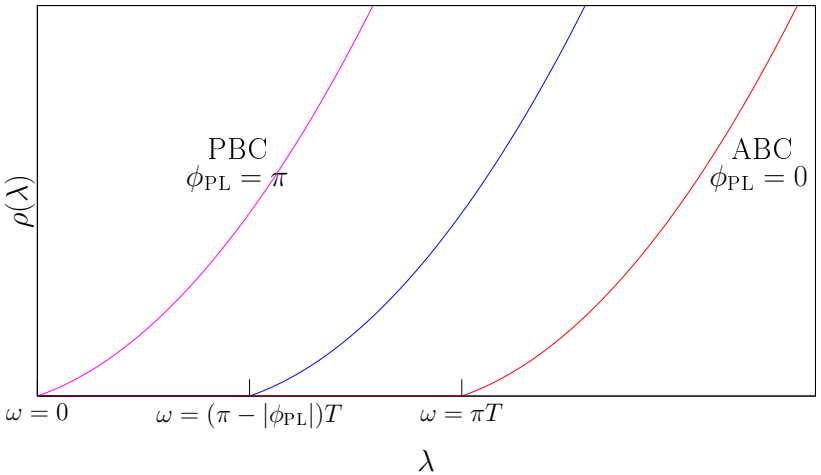
$\not{D} \sim$ disordered Hamiltonian, localisation expected - but why at the origin?

Sea/islands picture

Ordered phase $\approx$ configs. fluctuate around $\vec{A} = 0$, $P(\vec{x}) = \mathrm{diag}(e^{i\phi_a})$

Temporal "twist" from ABC $\Rightarrow$ gapped spectrum $|\lambda| \geq \omega = (\pi - \phi_{\mathrm{PL}})T$





$$\phi_{\mathrm{PL}} = \max_{a, \phi_a \in (-\pi, \pi]} |\phi_a|$$

- "Sea" of $\phi_{\mathrm{PL}} = 0$ selected because of largest twist/spectral gap
- "Islands" $|\phi(\vec{x})| < \phi_{\mathrm{PL}}$ reduce twist/λ, support localised modes $< \omega$

[Bruckmann *et al.* (2011); MG *et al.* (2015, 2016)]

Too simplistic, requires refinement [Baranka, MG (2022)]

Prediction: localisation in the deconfined phase of a generic gauge theory

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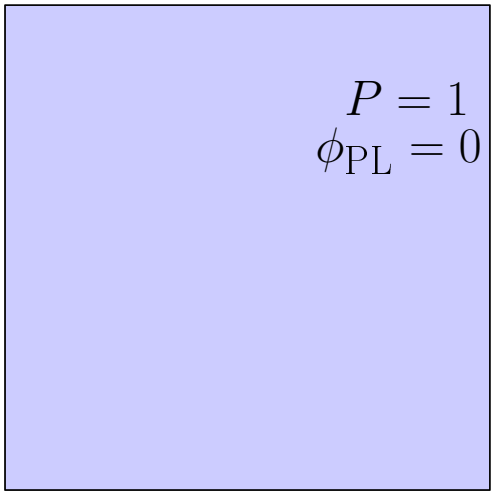

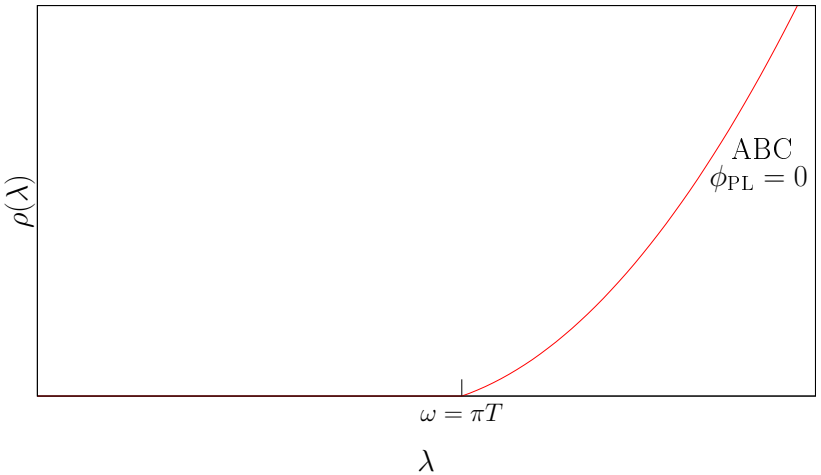

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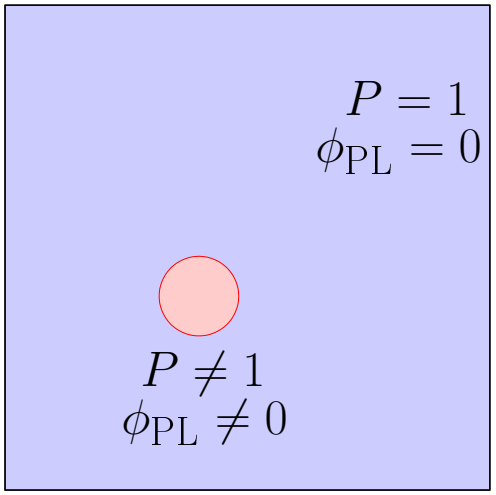

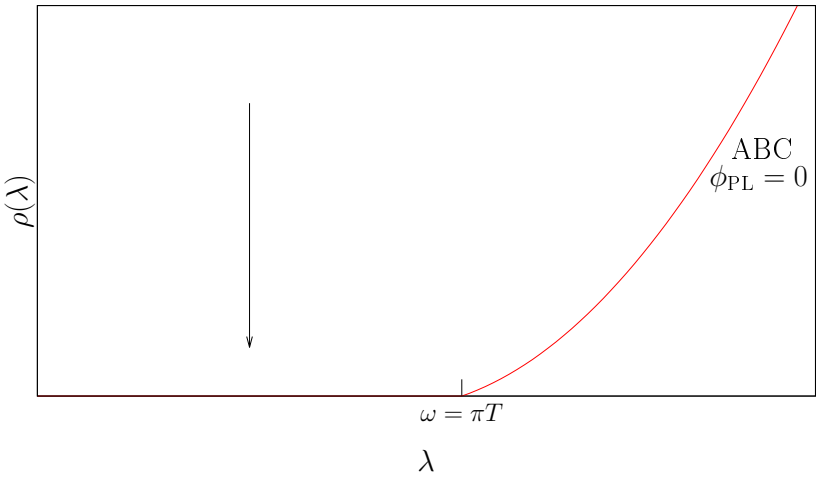

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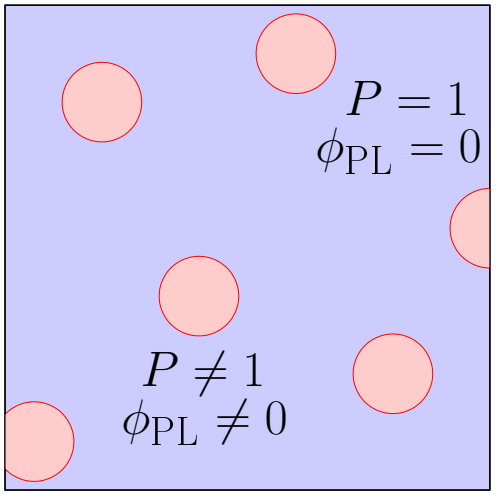

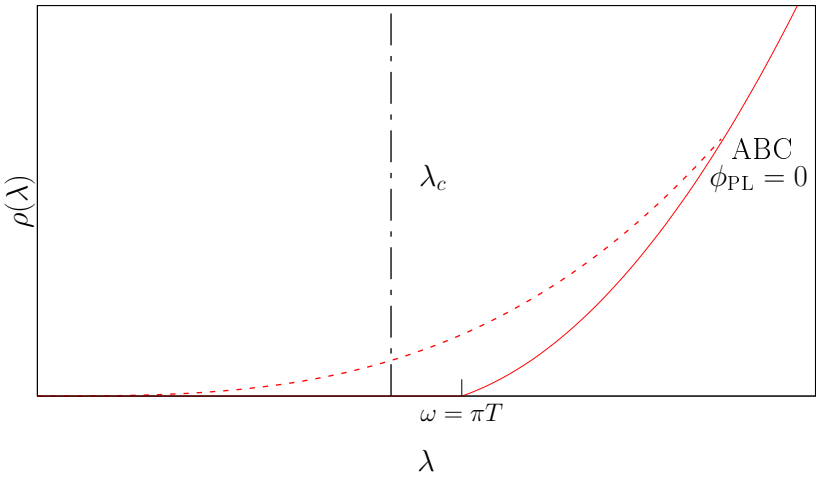

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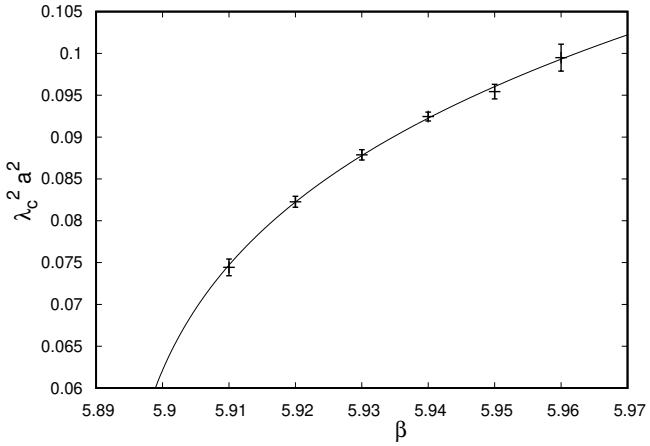
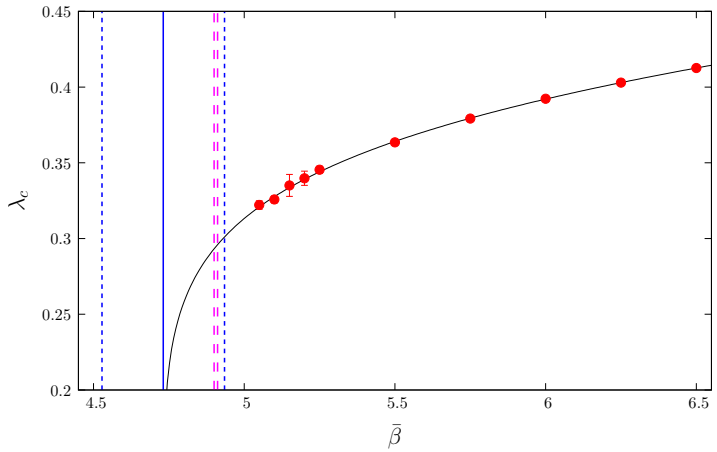
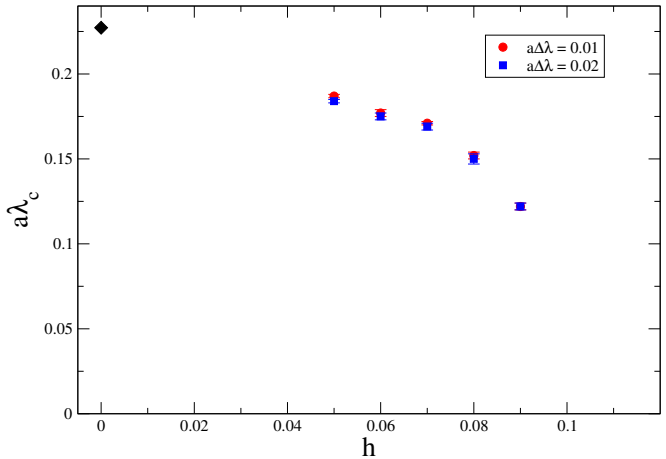
Localisation and deconfinement – pure gauge theory

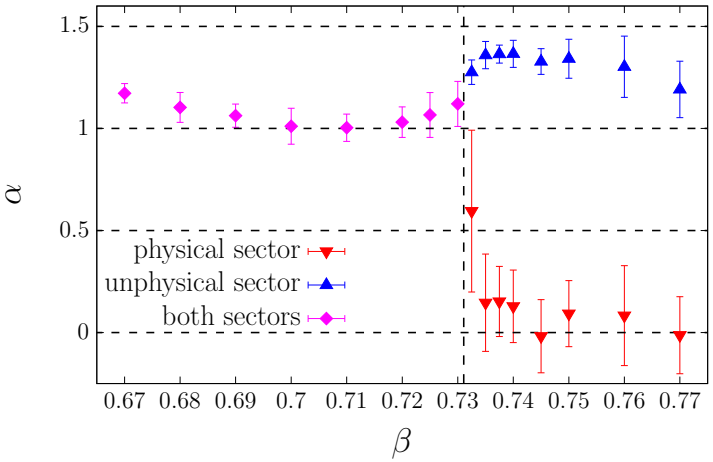
Localisation observed in the deconfined phase in many pure gauge systems

- SU(3) in 3+1D [Kovács, Vig (2018, 2020)] and 2+1D [MG (2019)]
- SU(3) + trace deformation [Bonati *et al.* (2021)]
- $\mathbb{Z}_2$ [Baranka, MG (2021)] and $\mathbb{Z}_3$ [Baranka, MG (2022)] in 2+1D





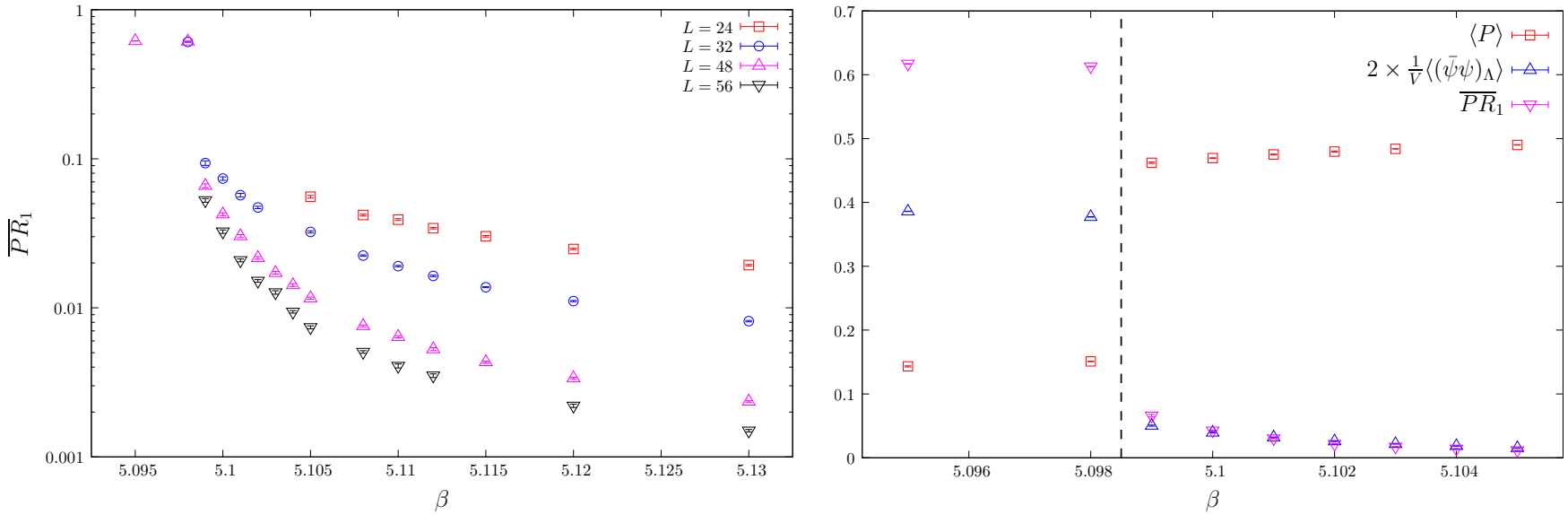




Localisation and deconfinement – fermions

Localisation in a system with fermions displaying genuine phase transition: SU(3) + $N_f = 3$ unimproved rooted staggered fermions on $N_t = 4$ lattices

[De Forcrand, Philipsen (2003)]



Figures from [MG *et al.* (2017)]

- First-order, (partially) deconfining and (partially) chirally restoring p.t.
- Localised modes appear at the transition
- Transition is a lattice artefact, does not survive continuum limit

Imaginary chemical potential and Roberge-Weiss symmetry

System with physical transition? Roberge-Weiss transition at imaginary μ

- imaginary quark chemical potential μ_I enters like gA_4
- effective temporal boundary condition $e^{i(\pi+\phi_{\rm PL})} \to e^{i(\pi+\phi_{\rm PL}+\hat{\mu}_I)}$
- $\hat{\mu}_I \to \hat{\mu}_I \pm \frac{2\pi}{3}$ reabsorbed by centre transformation

$$\boxed{\text{Partition function periodic } Z(\hat{\mu}_I) \text{ under } \hat{\mu}_I \to \hat{\mu}_I \pm \frac{2\pi}{3}}$$

[Roberge, Weiss (1986)]

Periodicity realised differently at low/high T:

- low T: $Z(\hat{\mu}_I)$ smooth periodic function
- high T: lines of first-order phase transitions at $\hat{\mu}_I = \frac{\pi}{3} + \frac{2\pi}{3}n$, $n \in \mathbb{Z}$
- first-order lines end at second-order Ising point at $T_{\rm RW} = 208(5)\,{\rm MeV}$

[C. Bonati *et al.* (2016)]

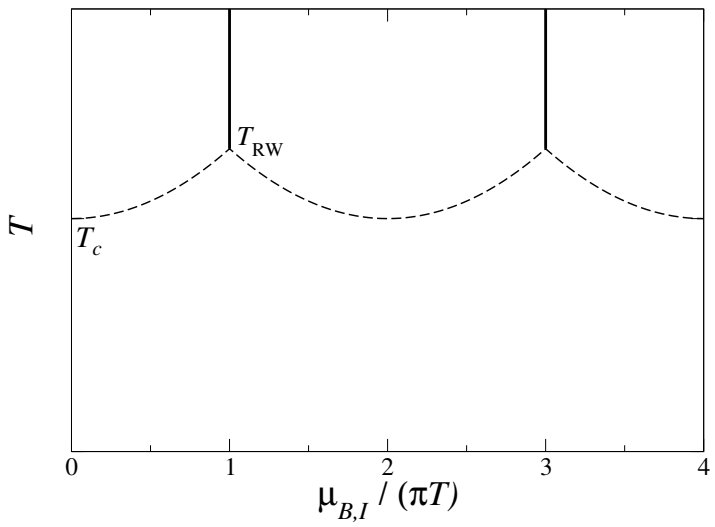
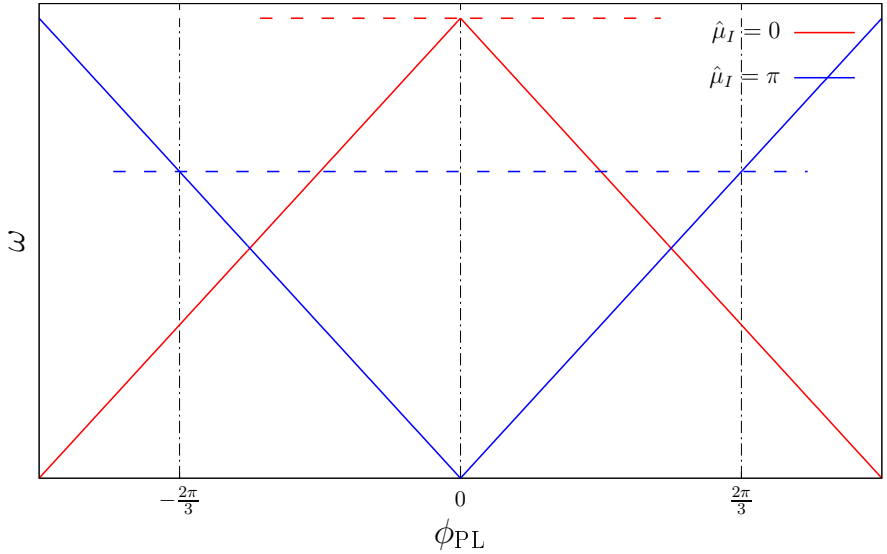


Figure from [C. Bonati *et al.* (2018)]

Roberge-Weiss transition and sea/islands picture



- gap in uniform config. modified, favoured sector changes with $\hat{\mu}_I$

$$\hat{\mu}_I = 0\colon\ \omega = (\pi - |\phi_{\rm PL}|)\,T \qquad\qquad \hat{\mu}_I = \pi\colon\ \omega = |\phi_{\rm PL}|\,T$$

- $\hat{\mu}_I = \frac{\pi}{3} + \frac{2\pi}{3}n$: exact $\mathbb{Z}_2$ centre symmetry (two centre sectors equally favoured), breaks spontaneously for $T > T_{\rm RW}$
- $\hat{\mu}_I = \pi$: complex sectors chosen, $\omega = \frac{2\pi}{3}T$ for either sector
- fluctuations away from $e^{\pm i\frac{2\pi}{3}}$ reduce eigenvalue, lead to localisation

$\Longrightarrow$ expect low modes to turn from delocalised to localised at $T_{\rm RW}$

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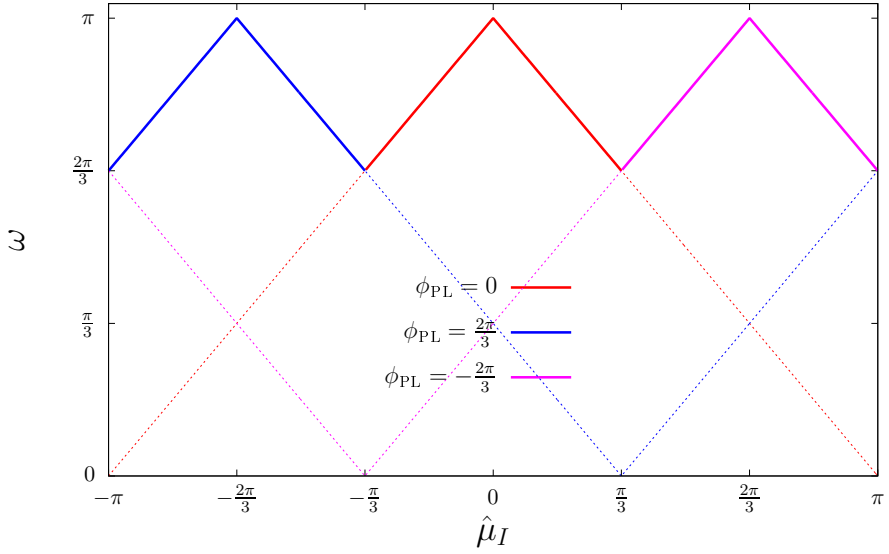

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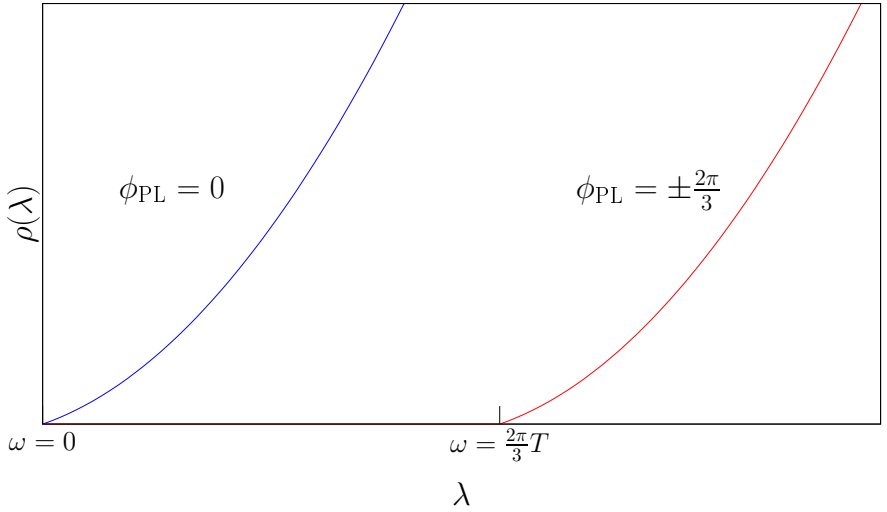

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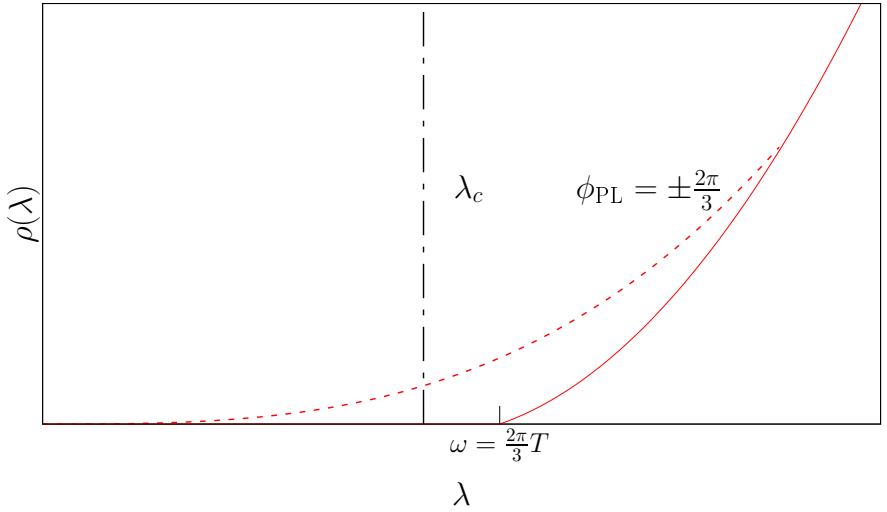
$\Longrightarrow$ expect low modes to turn from delocalised to localised at $T_{\rm RW}$

Roberge-Weiss transition and sea/islands picture



- gap in uniform config. modified, favoured sector changes with $\hat{\mu}_I$

$$\hat{\mu}_I = 0\colon\ \omega = (\pi - |\phi_{\rm PL}|)\,T \qquad\qquad \hat{\mu}_I = \pi\colon\ \omega = |\phi_{\rm PL}|\,T$$

- $\hat{\mu}_I = \frac{\pi}{3} + \frac{2\pi}{3}n$: exact $\mathbb{Z}_2$ centre symmetry (two centre sectors equally favoured), breaks spontaneously for $T > T_{\rm RW}$
- $\hat{\mu}_I = \pi$: complex sectors chosen, $\omega = \frac{2\pi}{3}T$ for either sector
- fluctuations away from $e^{\pm i\frac{2\pi}{3}}$ reduce eigenvalue, lead to localisation

$\Longrightarrow$ expect low modes to turn from delocalised to localised at $T_{\rm RW}$

Numerical setup

Lattice discretisation of $N_f = 2 + 1$ QCD

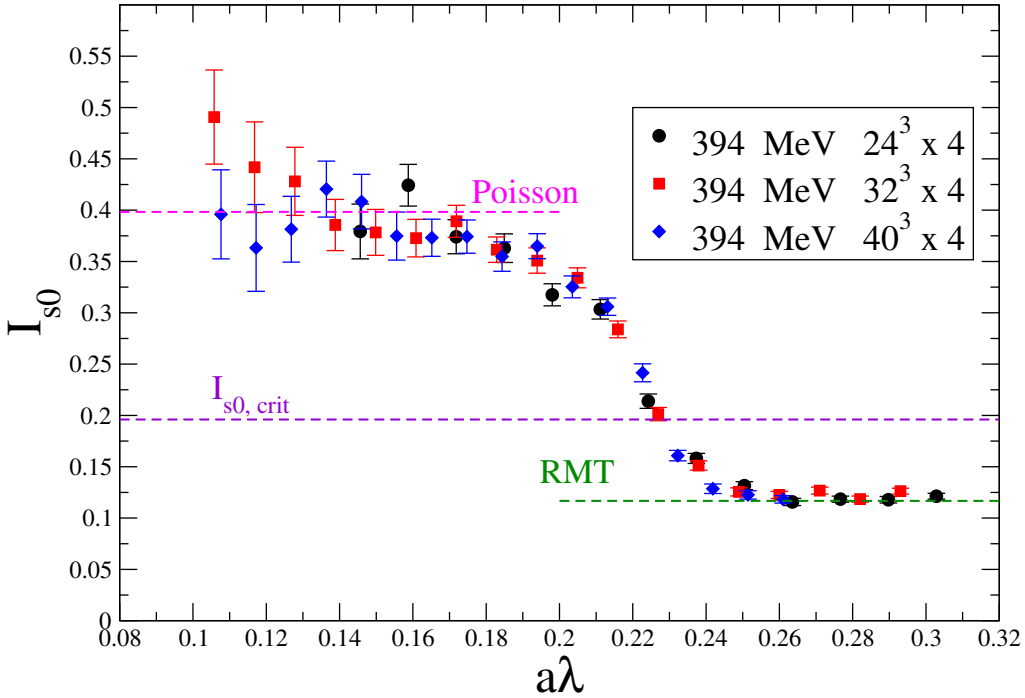
- physical quark masses
- finite temperature T
- imaginary chemical potential $\hat{\mu}_I = \pi$ (= PBC in temporal direction)

Details for the practitioners:

- 2-stout improved rooted staggered fermions
- tree-level improved Symanzik gauge action
- $N_t = 4, 6, 8$ with aspect ratio 6 (also 8 for $N_t = 4$)

Scan in temperature for $T > T_{\mathrm{RW}}$, study localisation using statistical properties of the Dirac spectrum

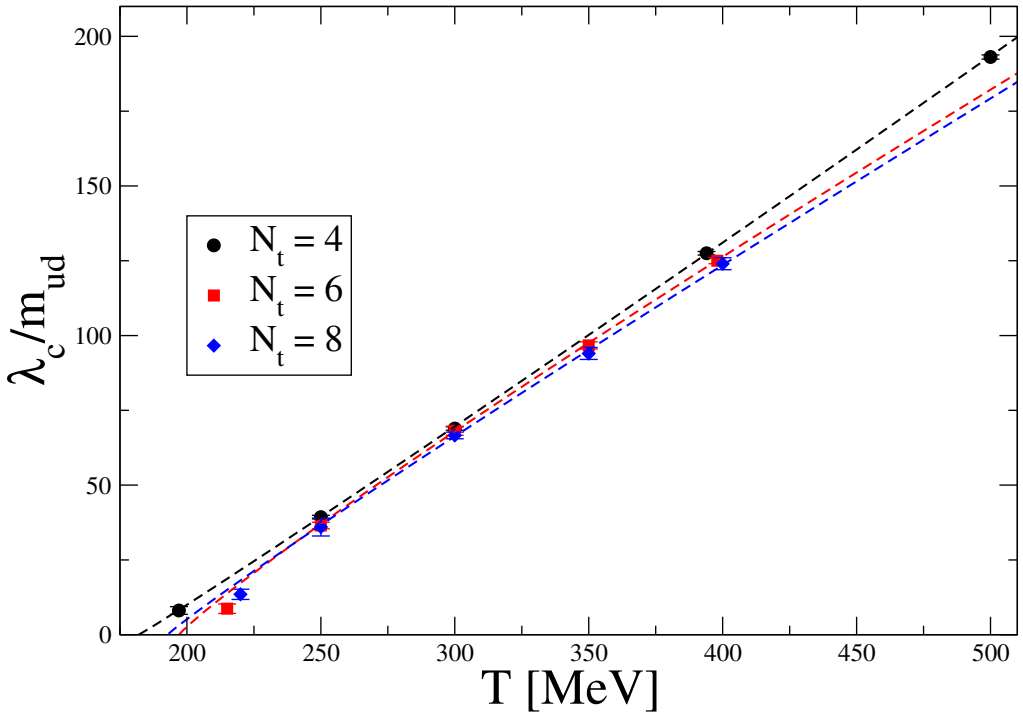
Localisation above T_{RW}



$$I_{s_0}(\lambda) = \int_0^{s_0} ds\, p_\lambda(s)$$

Critical value $I_{s_0}^{\mathrm{crit}}$ known [MG *et al.* (2014)], use to find λ_c via $I_{s_0}(\lambda_c) = I_{s_0}^{\mathrm{crit}}$

Mobility edge

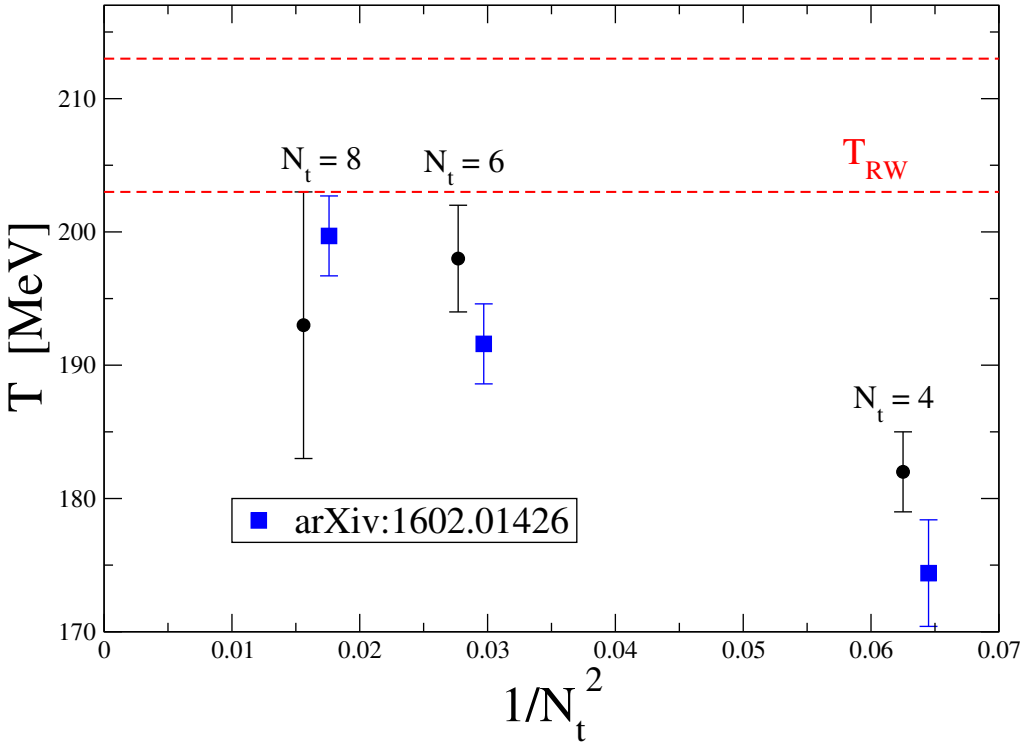


Find $T_{\rm loc}(N_t)$ where $\lambda_c = 0$ for each lattice spacing $a = (TN_t)^{-1}$ fitting to

$$\frac{\lambda_c}{m_{\rm ud}} = A(N_t)[T - T_{\rm loc}(N_t)]^{B(N_T)}$$

Lowest T for each N_t excluded from fit,
large finite-size effects for $N_t = 6, 8$

Localisation temperature



Within errors $T_{\rm loc}(N_t) = T_{\rm RW}(N_t) \Longrightarrow$ strongly supports $T_{\rm loc} = T_{\rm RW}$

Localised modes appear right at a genuine deconfinement transition (Roberge-Weiss transition) also in the presence of fermions

Summary and outlook

We studied localisation in QCD at imaginary chemical potential $\hat{\mu}_I = \pi$ above the Roberge-Weiss temperature T_{RW}, finding that:

- localised low Dirac modes are present
  $\Longrightarrow$ confirms expectations of "sea/islands" picture
- localisation appears at T_{RW}
  $\Longrightarrow$ confirms strong connection with deconfinement

Open issues:

- many clues connecting localisation and deconfinement, but something still missing
  - no studies yet in models with a trivial centre
- physical meaning of localisation still unclear
  - in the chiral limit it affects (possibly kills) Goldstone excitations [MG (2021), MG (2022)]
- is localisation how deconfinement improves chiral symmetry properties?

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