Unitarizing infinite-range forces, $\pi\pi$ and graviton-graviton scattering (graviball)

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Related references

[1] D. Blas, J. Martín-Camalich,JAO, Phys.Lett.B827,136991(2022);

[2] arXiv:2010.12459[hep-th] *to appear in JHEP*;

[3] JAO, arXiv:2207.08784

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Outline

Introduction

Scattering amplitudes in the low-energy EFT of gravity; derivative expansion→ loops are higher orders Donoghue, PRL72,2996(1994); arXiv:1702.00319[hep-ph]

$$S_{\rm grav} = \int d^4x\sqrt{-g}\left\{\Lambda_{\rm cc} + \frac{2}{\kappa^2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \ldots\right\} \quad \kappa^2 = 32\pi G$$

Each R, $R_{\mu\nu}$, $R_{\mu\nu\alpha\beta} \sim p^2$ (two derivatives)
$i\partial_\alpha \sim p_\mu$ and so on

Each graviton: $G^{\frac{1}{2}}$ $\quad g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}$

Einstein theory (for pure gravity): $c_1 = c_2 = \ldots = 0$

$\Lambda_{\rm cc}$ is neglected (cosmological constant $-8\pi G\Lambda_{\rm cc}$)

EFT for $E \lesssim \Lambda$, e.g. $\Lambda_U \sim G^{-1/2} = 10^{19}$ GeV
(unitarity cutoff– more below)

Similarities with Chiral Perturbation Theory (ChPT), pion physics (QCD)

Massless pions (Chiral limit $m_q = 0$)

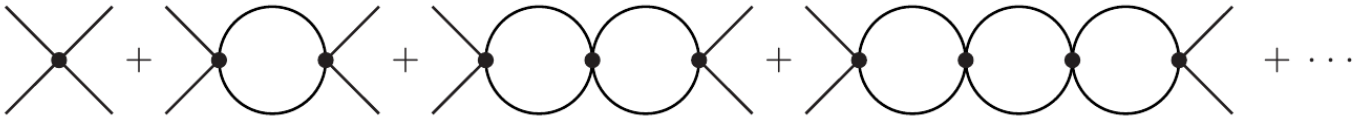
$i\partial_\mu \sim p_\mu$ Pure derivative expansion

EFT valid for $E < \Lambda$. Unitarity cutoff $\Lambda_U = 4\pi f_\pi \approx 1.2$ GeV
$\Lambda = M_\rho = 0.77$ GeV

σ of $f_0(500)$: Isoscalar Scalar $\pi\pi$ scattering is resonant

GKPY Equation García-Martín, Kaminski, Peláez, Ruiz de Elvira, PRL107,072001(2011)
$\sqrt{s_\sigma} = (457^{+14}_{-13} - i\,(297^{+11}_{-7}))$ MeV

One also generates the σ by unitarizing $I = J = 0$ $\pi\pi$ ChPT



General $\pi\pi$ interaction

NLO Unitarized ChPT, Albaladejo, JAO, PRD86,034003(2012)
$\sqrt{s_\sigma} = 458 \pm 14 - (261 \pm 17)\, i$ MeV

In an EFT the light degrees of freedom must be accounted for

$$\left| \frac{s_\sigma}{(4\pi f_\pi)^2} \right| = 0.22 \ll 1$$

This is an example of a Parametric enhancement JAO, Oset, PRD60,074023(1999)

LO 0^{++} $\pi\pi$ ChPT partial-wave amplitude (PWA), σ

$$T_{00}(s) = \frac{s - m_\pi^2/2}{f_\pi^2}$$

LO 1^{--} $\pi\pi$ ChPT PWA, $\rho(770)$

$$T_{11}(s) = \frac{s - 4m_\pi^2}{6 f_\pi^2}$$

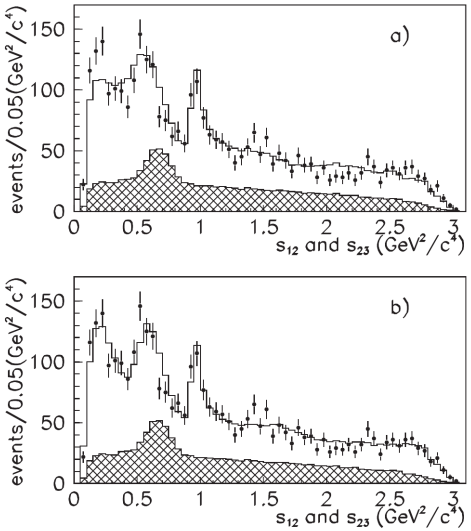


FIG. 2. s_{12} and s_{13} projections for data (error bars) and fast MC (solid line). The shaded area is the background distribution, (a) solution with the Fit 1, and (b) solution with Fit 2.

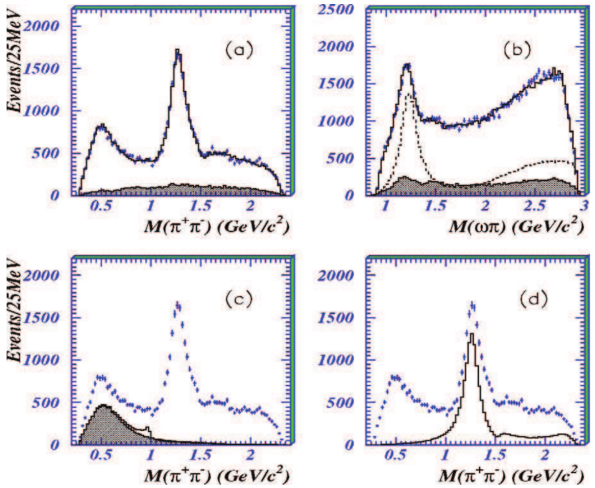


Fig. 2. Mass projections of data compared with the fit (histograms) using Eqs. (10)–(12) for the σ; the shaded region shows background estimated from sidebands. (a) and (b): $\pi\pi$ and $\omega\pi$ mass; the dashed curve in (b) shows the fitted $b_1(1235)$ signal (two charge combinations). (c) and (d): mass projections of 0^{++} and 2^{++} contributions to $\pi^+\pi^-$ from the fit; in (c), the shaded area shows the σ contribution alone, and the full histogram shows the coherent sum of σ and $f_0(980)$.

E791 PRD86,770(2001)

$$D^+ \to \pi^-\pi^+\pi^+$$

BES PLB598,149(2004)

$$J/\Psi \to \omega\pi^+\pi^+$$

The σ affects prominently low-energy scalar dynamics in QCD

$\star$ Vacuum, excitations of quark condensate: Scalar form factors of π
$\star$ π-nucleon σ term
$\star$ Large corrections to the current-algebraic prediction of 0^{++} $\pi\pi$ scattering lengths, and phase shifts in general
$\star$ Two-pion event distributions from heavy-meson decays
ETC Peláez, Phys.Rep.658,1(2016)

Is there a graviball (gravi-σ) in the QG EFT that could affect so much (relatively) low-energy gravitational physics?

$I = J = 0$ $\pi\pi$ are attractive
$J = 0$ graviton-graviton interactions are also attractive

The Graviball

At LO Unitarized EFT-QG the pole position of the graviball and the σ are very similar relative to the cutoff of the EFTs

[1] D. Blas, J. Martín-Camalich,JAO, Phys.Lett.B827,136991(2022)

$$\left|\frac{s_P}{\Lambda_{\rm G}^2}\right| \approx \left|\frac{s_\sigma}{\Lambda_{\rm Hadron}^2}\right| = 0.22 \ll 1$$

$$\Lambda_{\rm G} = \pi G^{-1} \ , \quad \Lambda_{\rm Hadron} = 4\pi f_\pi$$

For infinite-range interactions . . .

Infrared divergences, forward infinite peak in Coulomb scattering, non-existence of PWAs directly calculated from Feynman graphs

$$\int_{-1}^{+1} \frac{d\cos\theta}{|\mathrm{p}' - \mathrm{p}|^2} = \frac{1}{2p^2}\left\{\log 2 - \lim_{\theta\to 0}\log(1-\cos\theta)\right\} \to \infty$$

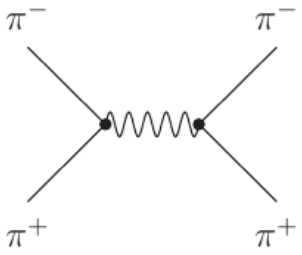
PWAs do not converge to $\langle \mathrm{p}', \lambda_1'\lambda_1' | T | \mathrm{p}, \lambda_1\lambda_2\rangle$
Giddings, Porto, PRD81,025002(2010)

For Coulomb scattering Kang,Brown,PR128,2828(1962)

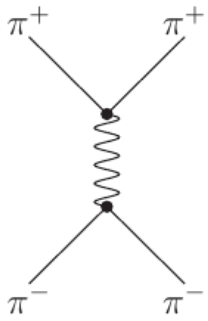
Lehmann ellipse

- Spectrum $[H, \mathrm{J}] = 0$
- Selection of PWAs:
- Interference with other forces, like Coulomb with strong interactions
- Final-state interactions in multi-graviton production processes

§2 Born terms. E.g. $\pi^+\pi^-$ Coulomb scattering



$s-$channel



$t-$channel

One-photon exchange $\pi^+\pi^- \to \pi^+\pi^-$ electromagnetic (EM) scattering amplitude

$$F(\pi^+\pi^- \to \pi^+\pi^-) = -\frac{e^2}{s}(s + 2t - 4m^2) - e^2\frac{s-u}{t}$$

Partial-wave amplitudes (PWAs)

$$\bar{T}_J(s) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta P_J(\cos\theta)\bar{T}(\cos\theta)$$

Unitarity in PWAs:

$$\Im\bar{T}_J(s) = \frac{p}{8\pi\sqrt{s}}|\bar{T}_J(s)|^2$$

Infrared (IR) divergences

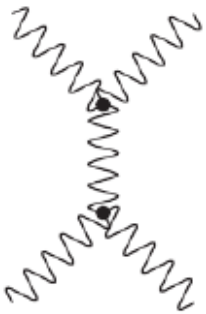
So far so good . . . but PWAs are IR divergent

$J = 0$ partial-wave projection of the Born term

$$-\int_{-1}^{+1} \frac{dcos\theta}{t} = \frac{1}{2p^2}\left[\log 2 - \lim_{\theta\to 0}\log(1-\cos\theta)\right]$$

$$t = (\mathrm{p}-\mathrm{p}')^2 = -2\mathrm{p}^2(1-\cos\theta)$$

This is due to the exchange of a *virtual soft* photon(graviton) ($t \to 0$) in between two *external on-shell* lines
Classification of Weinberg in PR140,B516(1965)



$t-$channel

soft: $|t| << s$

Phase divergences

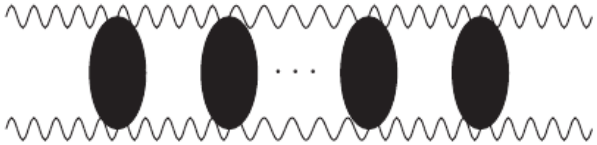
Dalitz,Proc.Roy.Soc.(London)206,509(1951)

Phase conjectured by Dalitz when studying the Born series up to 2nd order (e^2) for the scattering of a Dirac electron by a Yukawa/Coulomb potential

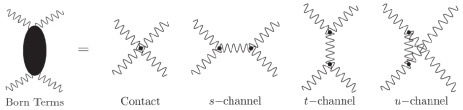
$$V(r) = \frac{e_1 e_2}{4\pi r} e^{-\mu r} \qquad \text{Phase} - \text{factor} = \exp\left\{ \frac{i e_1 e_2}{2\pi \beta_{12}} \log \mu \right\}$$

Lorentz invariant relative velocity between particles a and b

$$\beta_{ab} = \frac{\left[(p_a p_b)^2 - (m_a m_b)^2\right]^{1/2}}{p_a p_b}$$



Born Series



Extended up to 3rd order (e^3) for the non-relativistic case

$$f(\mathrm{p}') = \frac{e^2}{4p^2 \sin^2 \frac{1}{2}\theta} \left\{ 1 - i \frac{m\alpha}{p} \left(\log \sin^2 \frac{1}{2}\theta + \log \frac{4p^2}{\mu^2} \right) \right.$$
$$\left. + \left(\frac{m\alpha}{p} \right)^2 \left(-\frac{3}{4} \left(\log \frac{4p^2}{\mu^2} \right)^2 + [\log(\alpha\mu)]^2 \right) + \mathcal{O}(\alpha^3) \right\}$$

Phase divergences

This phase was demonstrated by Weinberg in *Infrared Photons and Gravitons*, PR140,B516(1965)

"the full effect of virtual infrared photons is to contribute to the S matrix for any process $\alpha \to \beta$ a factor"

$$\frac{S_{\beta\alpha}}{S^0_{\beta\alpha}(\mathcal{L})} = \exp\left\{\frac{1}{2}\int_\mu^{\mathcal{L}} A(q)\right\}$$

The real part of $A(q)$ generates the IR divergence $(\mu/\mathcal{L})^{A/2}$. This is cancelled by real soft-photon emission

Its imaginary part generates the Dalitz phase: *"so each different pair of particles in the initial or final state contributes to the S matrix a phase factor which for $\mu \ll \mathcal{L}$ may be written"*

$$\exp\left\{\frac{i}{2\pi}\frac{e_n e_m}{\beta_{nm}}\log\frac{\mu^2}{\mathcal{L}^2}\right\}, \quad \mathcal{L}^2 = \frac{4p^2}{a^2} \to \text{Left-hand cut (LC)}$$

Redefinition of the S matrix in PWAs

Comparing with Weinberg:

$$S_{\alpha\beta} \to \bar{S}_J \text{ and } S^0_{\alpha\beta} \to S_J$$

$$S_J = S_c^{-1}\bar{S}_J = \exp\left\{2iGs\log\frac{\mu}{\mathcal{L}}\right\}\bar{S}_J$$

$$S_J = S_c^{-1}\bar{S}_J = \exp\left\{\frac{2i\alpha}{\beta}\log\frac{\mu}{\mathcal{L}}\right\}\bar{S}_J$$

Note: $\underline{S_J \text{ is unitary}}$ because only a phase factor has been introduced

$$S_J S_J^\dagger = S_J^\dagger S_J = I$$

$$\bar{S}_J = 1 + i\frac{m_r p}{\pi}\bar{T}_J \qquad\qquad S_J = 1 + i\frac{m_r p}{\pi}T_J$$

IR divergent

IR finite

IR-safe PWAs up to $\mathcal{O}(\alpha^2)$

JAO arXiv:2207.08784

Including the Weinberg phase $S_c = \exp\left(2i\gamma \log \frac{\mathcal{L}}{\lambda}\right)$

$$S_J = \left[1 + i\frac{mp}{\pi}(F_J^{(1)} + F_J^{(2)})\right]\left[1 - 2i\gamma \log\frac{\mathcal{L}}{\lambda} - 2\gamma^2(\log\frac{\mathcal{L}}{\mu})^2\right] + \mathcal{O}(\alpha^3)$$

$$= 1 - 2i\gamma\log\frac{\mathcal{L}}{\mu} - 2\gamma^2(\log\frac{\mathcal{L}}{\mu})^2 + i\frac{mp}{\pi}\left[F_J^{(1)}\left[1 - 2i\gamma\log\frac{\mathcal{L}}{\mu}\right] + F_J^{(2)}\right]$$

$$+ \mathcal{O}(\alpha^3)\ , \qquad \gamma = \frac{m\alpha}{p}$$

$$S_J = 1 + \frac{mp}{4\pi}T_J$$

$$T_J^{(1)}(p) = F_J^{(1)}(p) - \frac{e^2}{2p^2}\log\frac{\mathcal{L}}{\mu} = \frac{e^2}{2p^2}\log\frac{2p}{\mu} - \frac{e^2}{2p^2}\log\frac{\mathcal{L}}{\mu} = \frac{e^2}{2p^2}\log a$$

$$T_J^{(2)}(p) = F_J^{(2)}(p) - iF_J^{(1)}(p)\frac{me^2}{2\pi p}\log\frac{\mathcal{L}}{\mu} + i\frac{me^4}{8\pi p^3}\left(\log\frac{\mathcal{L}}{\mu}\right)^2$$

One-loop calculation, JAO arXiv:2207.08784

$$\bar{T}^{(2)}(\mathrm{p}',\mathrm{p}) = \frac{me^4}{4\pi^3}\int \frac{d^3q}{[\mu^2+(\mathrm{p}'-\mathrm{q})^2][\mu^2+(\mathrm{p}-\mathrm{q})^2][q^2-p^2-i\epsilon]}$$

S-wave projection

$$F_0^{(2)}(p) = \frac{me^4}{16\pi^2p^2}\int_0^\infty \frac{dq}{q^2-p^2}\left[\log\frac{\mu^2+(p+q)^2}{\mu^2+(p-q)^2}\right]^2$$

It satisfies perturbative unitarity

$$\Im F_0^{(2)}(p) = \frac{mp}{2\pi}F_0^{(1)^2} = \frac{me^2}{8\pi p^3}\left(\log\frac{2p}{\mu}\right)^2$$

Its real part is zero

$$\Re F_0^{(2)}(p) = \lim_{\mu\to 0}\frac{me^4}{16\pi^2p^2}\mathrm{PV}\int_0^\infty \frac{dq}{q^2-p^2}\left[\log\frac{\mu^2+(p+q)^2}{\mu^2+(p-q)^2}\right]^2 = 0$$

Up to and including one loop

$$F_0(p) = \frac{e^2}{2p^2}\log\frac{2p}{\mu} + i\frac{me^4}{8\pi p^3}\left(\log\frac{2p}{\mu}\right)^2 + \mathcal{O}(\alpha^3)$$

IR-safe PWAs

$$T_0^{(1)}(p) = \frac{e^2}{2p^2}\log\frac{2p}{\mathcal{L}} = \frac{e^2}{2p^2}\log a\ ,$$

$$T_0^{(2)}(p) = i\frac{me^4}{8\pi p^3}\left(\log\frac{2p}{\mathcal{L}}\right)^2 = i\frac{me^4}{8\pi p^3}(\log a)^2$$

Matching with the unitarization formula

$$T_0^{(1)} + T_0^{(2)} = \left[\frac{1}{V_0^{(1)} + V_0^{(2)}} - i\frac{mp}{2\pi}\right]^{-1} = V_0^{(1)} + V_0^{(2)} + i\frac{mp}{2\pi}V_0^{(1)^2} + \mathcal{O}(\alpha^3)$$

$$V_0^{(1)}(p) = T_0^{(1)}(p) = \frac{e^2}{2p^2}\log a$$

$$V_0^{(2)}(p) = 0$$

In agreement with the exact solution

Considering higher orders

$$S_J = \frac{\Gamma\left(1+J-i\gamma\right)}{\Gamma\left(1+J+i\gamma\right)} = 1 + i\frac{m_r p}{\pi} T_J$$

$$T_J = \left[V_J^{-1} - i\frac{m_r p}{2\pi}\right]^{-1}$$

$$V_J = \frac{2i\pi}{m_r p}\frac{\Gamma(1+J+i\gamma)+\Gamma(1+J-i\gamma)}{\Gamma(1+J+i\gamma)-\Gamma(1+J-i\gamma)}$$

$$V_J = -\frac{2\pi}{m_r p} v_J$$

$$v_J(p) = \psi_0(1+J)\gamma + \frac{1}{6}[2\psi_0(1+J)^3 - \psi_2(1+J)]\gamma^3$$
$$+ \frac{1}{120}[16\psi_0(1+J)^5 - 20\psi_0(1+J)^2\psi_2(1+J) + \psi_4(1+J)]\gamma^5$$
$$+ \mathcal{O}(\alpha^7)$$

$$V_0 = \underbrace{-\frac{2\pi}{mp}\psi_0(1)\gamma}_{V_0^{(1)}} \underbrace{+\mathcal{O}(\alpha^3)}_{V_0^{(2)}=0} = \frac{2\pi\alpha\gamma_E}{p^2} + \mathcal{O}(\alpha^3)$$

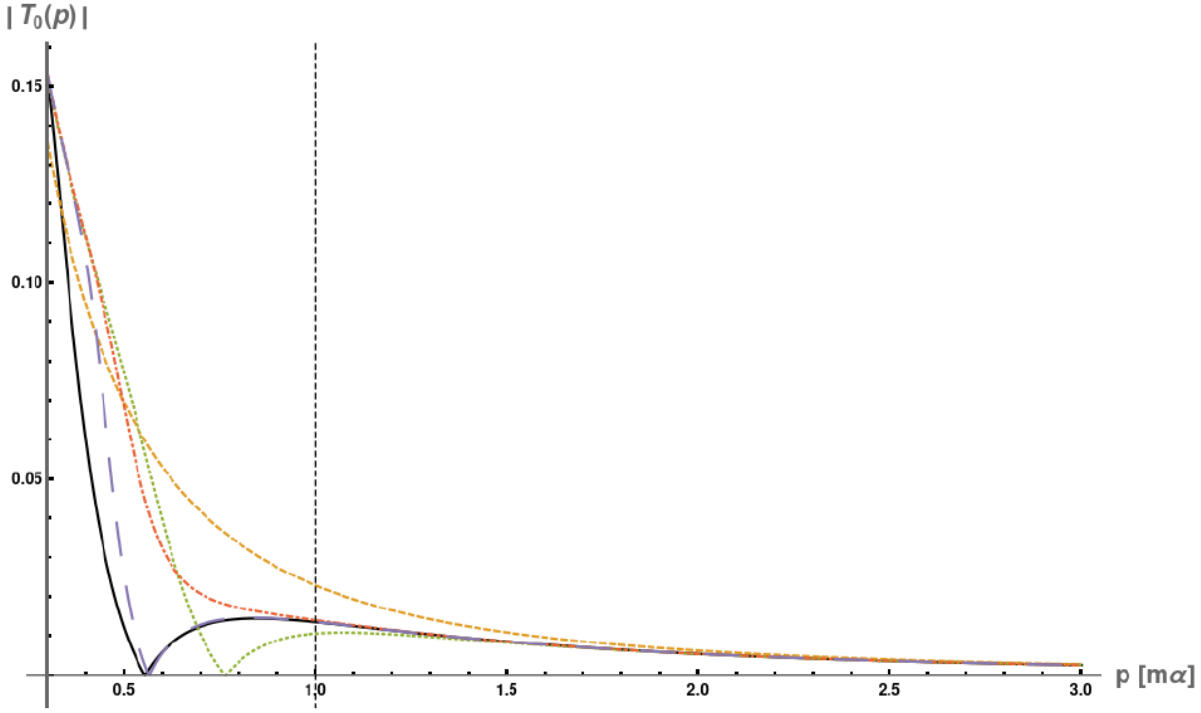
$$\psi_0(1) = -\gamma_E$$

$$\log a = \gamma_E = 0.577$$

Pole position of the ground state, $p_{\rm exact} = i m_r \alpha$

n	1	3	5	7
$p^{(n)}/p_{\rm exact}$	0.58	0.95	1.00	1.00

Comparison with the exact Coulomb PWA



n = 1(orange), 3(green), 5(red), 7(magenta)

Screened Coulomb potential

$$\frac{\alpha}{r} \to \frac{\alpha}{r}\theta(R-r)\ , \quad R \to \infty$$

IR-safe $V_0^C(s)$ up to $\mathcal{O}(\gamma)$ from the partial-wave projected Born term

$$F = \frac{4\pi\alpha}{q^2}(1-\cos qR)$$

$$F_0 = \frac{2\pi\alpha}{p^2}(\gamma_E + \log 2pR) + \mathcal{O}(R^{-2})$$

$$V_0 = F_0 - \underbrace{\delta F_c}_{\frac{2\pi\alpha}{p^2}\log 2pR} = \frac{2\pi\alpha\gamma_E}{p^2}$$

IT IS THE SAME AS THE LO TERM IN THE EXPANSION OF $v_0(p)$ $\quad V_0 = -\frac{2\pi}{mp}\psi_0(1)\gamma + \mathcal{O}(\alpha^3) = \frac{2\pi\alpha\gamma_E}{p^2} + \mathcal{O}(\alpha^3), \psi_0(1) = -\gamma_E$

QED

Bazhanov *et al* Theor.Math.Phys.33,982(1977) up to 2nd order in the expansion of small t (one order more than in Eikonal approximation)

$$T(s,t)=\frac{8\pi\alpha Z_1Z_2(s-m_1^2-m_2^2)}{t+i0}\left(\frac{-t-i0}{\lambda^2}\right)^{\gamma/2}e^{-\gamma C}\frac{\Gamma(1-\gamma/2)}{\Gamma(1+\gamma/2)}\left\{1-\frac{\pi\alpha Z_1Z_2(m_1+m_2)}{2(s-m_1^2-m_2^2)}D(s)(-t-i0)^{1/2}+o(\sqrt{t})\right\},$$

$$D(s)=\frac{\Gamma(1+\gamma/2)\Gamma(1/2-\gamma/2)}{\Gamma(1-\gamma/2)\Gamma(1/2+\gamma/2)},$$

$$\gamma=-2i\alpha Z_1Z_2/v,$$

The extra phase ($\lambda \equiv \mu$, $C \equiv \gamma_E$)

$$\exp\left[2i\alpha\frac{Z_1Z_2}{v}\left(\gamma_E+\log\frac{\lambda}{2p}\right)\right]$$

$$\mathcal{L}=\frac{2p}{a}\to\log a=\gamma_E$$

and this is a calculation to all orders in α

In the static limit (scattering in an external Coulomb field)

$$T = T_B \left(\frac{-t - i0}{\lambda^2} \right)^{\gamma/2} e^{-\gamma C} \frac{\Gamma(1-\gamma/2)}{\Gamma(1+\gamma/2)} \left[1 - \frac{\pi \alpha Z_1 Z_2 v}{2} D \sin \frac{\theta}{2} \right].$$

Versus the nonrelativistic amplitude for Coulomb

$$f_{\mathrm{N.R.}}(\theta) = \frac{\gamma}{2p} \left(\sin \frac{\theta}{2} \right)^{-2+2i\gamma} \frac{\Gamma(1-i\gamma)}{\Gamma(1+i\gamma)}$$

The difference is the phase factor $\exp\left(2i\gamma \frac{\mathcal{L}}{\lambda}\right)$ with $\mathcal{L} = 2p/e^{\gamma_E}$, or $\log a = \gamma_E$

Delgado,Dobado,Espriu,arXiv:2207.06070: Keeping $\mu \to 0$

Talk by Delgado on Monday, 16h, Session G.

They keep the IR divergent $F_J^{(n)}$. E.g. for non-relativistic Coulomb scattering

They do not take into account the Weinberg phase

$$F_0^{(1)}(p) = \frac{e^2}{2p^2} \log \frac{2p}{\mu}$$

$$F_0^{(2)}(p) = i \frac{me^4}{8\pi p^3} \left(\log \frac{2p}{\mu} \right)^2$$

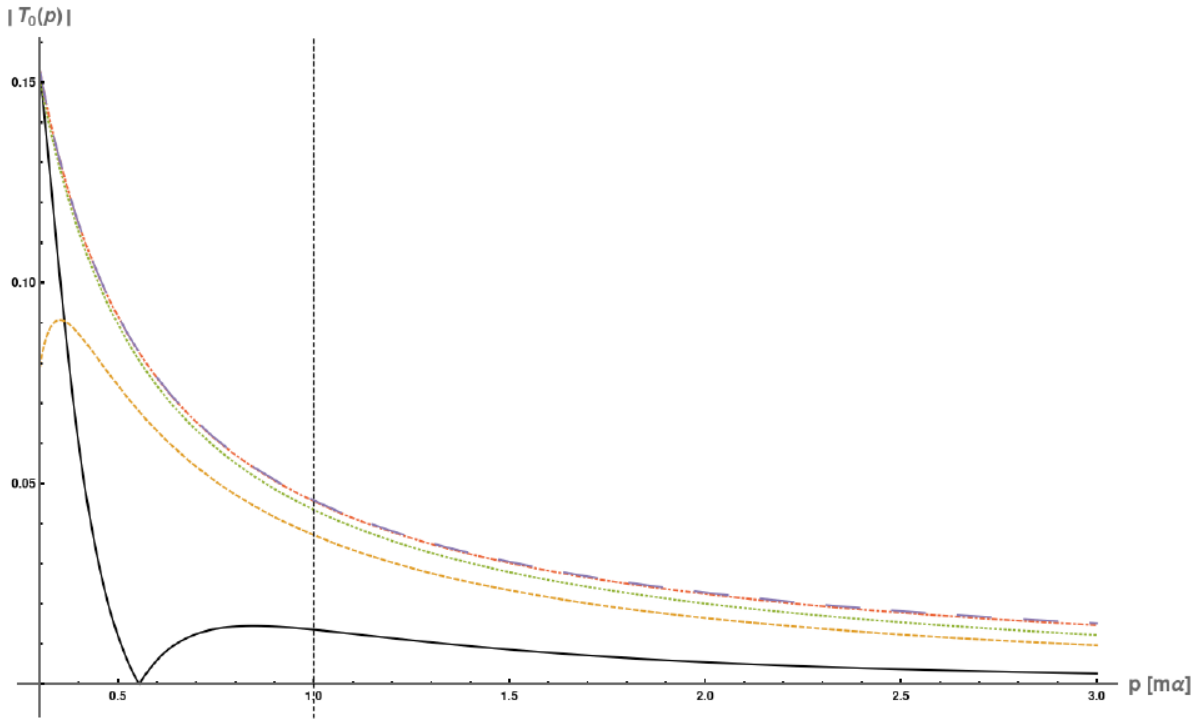
The Inverse Amplitude Method (IAM) is applied with the IR divergent perturbative PWAs $F_J^{(1)}$, $F_J^{(2)}$

$$T_0(p) = \frac{F_0^{(1)}(p)^2}{F_0^{(1)}(p) - F_0^{(2)}(p)} \xrightarrow[\mu \to 0]{} \frac{i2\pi}{mp}$$

The method of arXiv:2207.06070 clearly fails to reproduce Coulomb PWAs

μ $[2m\alpha]$	0.5	10^{-1}	10^{-2}	10^{-4}
p/μ	$0.79 - i\,0.32$	$0.99 - i\,0.10$	$1.00 - i0.01$	$1.00 - i0.00$

Pole position of the ground state, $p \to 0$ for $\mu \to 0$



$\mu = 0.5$(orange),10^{-1}(green),10^{-4}(red),10^{-12}(magenta)

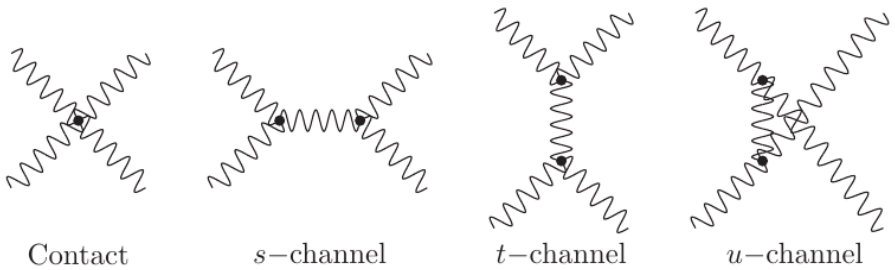
Delgado,Dobado,Espriu,arXiv:2207.06070 is not a suitable unitarization method for infinite-range interactions

§3 Prediction of the graviball

Graviton-graviton Born terms

$$|p_1, \lambda_1\rangle |p_2, \lambda_2\rangle \to |p_3, \lambda_3\rangle |p_4, \lambda_4\rangle$$

$$\lambda_i = \pm 2\,, \quad p = p_1 = -p_2\,, \quad p' = p_3 = -p_4$$



Born terms. Grisaru,van Nieuwenhuizen,Wu,PRD12,397(1975)

$$F_{22,22}(s,t,u) = F_{-2-2,-2-2}(s,t,u) = \frac{\kappa^2}{4}\frac{s^4}{stu}\,,$$

$$F_{-22,-22}(s,t,u) = F_{2-2,2-2}(s,t,u) = \frac{\kappa^2}{4}\frac{u^4}{stu}\,,$$

$$F_{2-2,-22}(s,t,u) = F_{2-2,2-2}(s,u,t) = F_{-22,2-2}(s,t,u) = \frac{\kappa^2}{4}\frac{t^4}{stu}$$

Related by parity and Bose-Einstein symmetry

Blas,Martin Camalich,JAO,PLB827,136991(2022)

$$V^{(0)}_{22,22}(s) = \frac{8Gs}{\pi}\log a$$

$$T^{(0)}_{22,22;II}(s) = \left[\frac{\pi}{8Gs\log a} + \frac{1}{8}\log\frac{-s}{\Lambda^2} - i\frac{\pi}{4}\right]^{-1}$$

$$\omega = \frac{\Lambda^2}{\Lambda_U^2} = \Lambda^2\frac{G\log a}{\pi}\ ,\ \Lambda_U = \frac{\pi}{G\log a}\ ,\ \log a \simeq 1$$

Secular equation

$$\frac{1}{x} + \log(-x) - i2\pi = 0\ ,\ x = \frac{s_P}{\Lambda^2}$$

$$x \simeq -i\frac{2}{3\pi} = -i0.20 \qquad\qquad x \sim \frac{1}{\omega}$$

$$x = 0.07 - i\,0.20\ ,\ s_P = 0.22 - i\,0.63\ G^{-1}$$

Estimated 20% uncertainty

Postdiction of the σ

Chiral limit

$$V^{(0)}_{\pi\pi} = \frac{s}{f_\pi^2}$$

$$T^{(0)}_{\pi\pi;II} = \left[\frac{f_\pi^2}{s} + \frac{1}{(4\pi)^2}\log\frac{-s}{\Lambda^2} - i\frac{1}{8\pi}\right]^{-1}$$

$$\Lambda = 4\pi f_\pi$$

$$x_\sigma = \frac{s_\sigma}{\Lambda^2}$$

Secular equation

$$\frac{1}{x_\sigma} + \log(-x_\sigma) - i2\pi = 0$$

$$x_\sigma \simeq -i\frac{2}{3\pi} = -i\,0.20$$

Numerically, $x_\sigma = 0.07 - i\,0.20$

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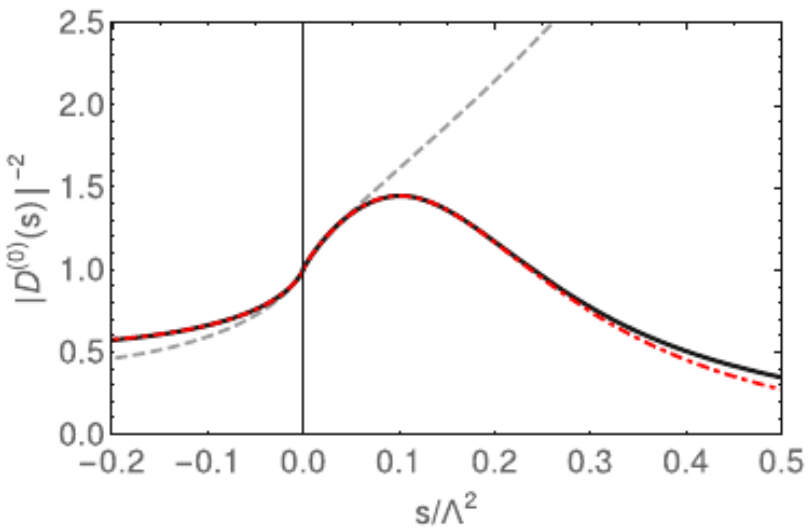
Physical π mass, GKPY: $x_\sigma = 0.09 - i\,0.20$

Not bad! Back envelope calculation 

$x = 0.07 - i\,0.20 \rightarrow$ Resonant shape peaks at surprisingly low values of s

Omnès function $\Omega^{(J)}(s) = 1/D^{(J)}(s)$

$$\Omega^{(J)}(s) = \left[1 + V^{(J)}(s)g(s)\right]^{-1} = T^{(J)}(s)/V^{(J)}(s)$$



At $s \simeq 0.2$ then $1 \sim V^{(0)}g$ (graviball)
Dashed line: Perturbative $|1 - V^{(0)}(s)g(s)|^2$
Solid line: $|D^{(0)}(s)|^{-2}$

Analogous to $|\Omega^{(0)}_{\pi\pi}(s)|^2$ driving e.g. final-state interactions in $D^+ \rightarrow \pi^+\pi^+\pi^-$ [E791]

Include light degrees of freedom in EFT

We have a parametric enhancement in the EFT both for the σ and graviball

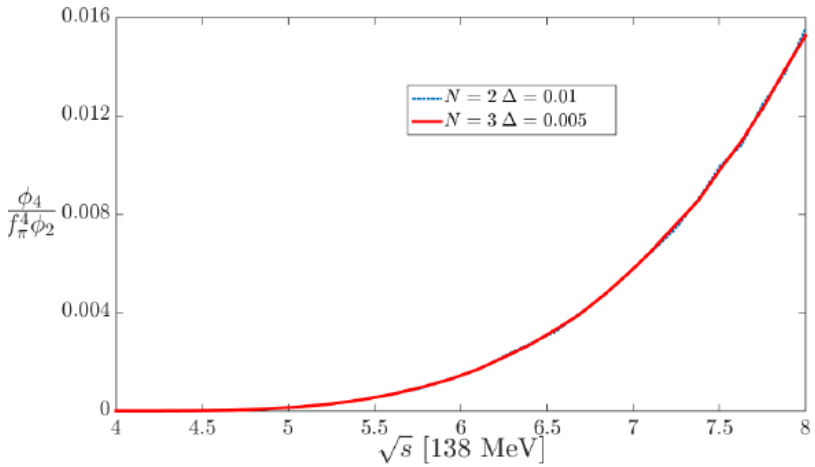
One has to account for such light resonances $|s/\Lambda^2| \approx 0.2$

Unitarized EFT is a way to accomplish this by resumming $(s/\Lambda^2)^n$ to account for two-body unitarity along the RC: Unitarity and analyticity

Suppression of phase space for massless multi-particle states Salas-Bernández, Llanes-Estrada, JAO, Escudero-Pedrosa,SciPost Phys.11,020(2021)

Phase space of n massless particles pions

$$\phi_n = \frac{s^{n-2}}{2(4\pi)^{2n-3}(n-1)!(n-2)!}$$



$\sqrt{s} \in [0.55, 1.1]$ GeV , 4π & 2π

Making lighter the graviball

By increasing the number N of fields with $m^2 \ll G^{-2}$
The number of channels in the intermediate increases as $\sim N$
The unitarity loop function $g(s) \to Ng(s)$

$$T^{(0)}_{22,22}(s) \approx \left[\frac{\pi}{8Gs\log a} + \frac{N}{8}\log\frac{-s}{\Lambda^2}\right]^{-1}$$

Secular equation

$$\frac{1}{x_N} + \log(-x_N) - i2\pi = 0$$

Solution

$$x_N \approx -i\frac{2}{3\pi N} + i\frac{\log N}{10\pi N}$$

Gravity interactions between fields are attractive

Scenarios with a large number N of light fields:

Dvali,Fortschr.Phys.58,528(2010); Dvali,Redi,PRD77,045027(2008);

Arkani-Hamed,Cohen,D'Agnolo,Hook,Kim,Pinner,PRL117,251801(2016);

Extra Dimensions Arkani-Hamed,Dimopoulos,Dvali,PLB429,263(1998);

Antoniadis,Arkani-Hamed,Dimopoulos,Dvali,PLB436,257(1998);

Csaki,arXiv:0806.3801[hep-th]

Monomials involving three or four Riemannian tensors

$\{R^3\}$: Six derivatives; $\{R^4\}$: Eight derivatives

$\{R^3\}$ gives vanishing contributions to $F_{22,22}$

van Nieuwenhuizen,Wu, J.Math.Phys.18,182(1977)

$\{R^4\}$ contributions to $F_{22,22}$ evaluated in Huber, Brandhuber, De Angelis, Travaglini, PRD102,046014(2020) with spinor formalism

$$F_{R^4;22,22}(s,t,u) = \frac{\widetilde{\beta}\kappa^2}{\pi} s^4$$

$$V^{(0)}_{R^4;22,22}(s) = \frac{8\widetilde{\beta}}{\pi^2} s^4 \ , \quad \widetilde{\beta} \sim \Lambda^{-6}$$

$$V^{(0)}_{R^4;22,22}(s) = \frac{32 s^4}{\pi \Lambda^8 \log a}$$

Interaction kernel in the unitarization formula

$$R^{(0)}_{22,22}(s) = \frac{8s}{\Lambda^2} + \frac{32s^4}{\pi\Lambda^8} \ , \quad \log a = 1$$

Secular equation $s = s_P/\Lambda^2$

$$(x + 4x^4/\pi)^{-1} + \log(-x) - i\,2\pi = 0$$
$$x = 0.07 - i\,0.21 \ , \quad 3\% \text{ of deviation}$$

In agreement with the estimate $|x|^3 \sim 1\%$ for a N^{3}LO correction (s^4&s)

Expected leading corrections are NLO loop ones $|x| \sim 0.2$ (there are no counterterms at NLO in pure gravity)

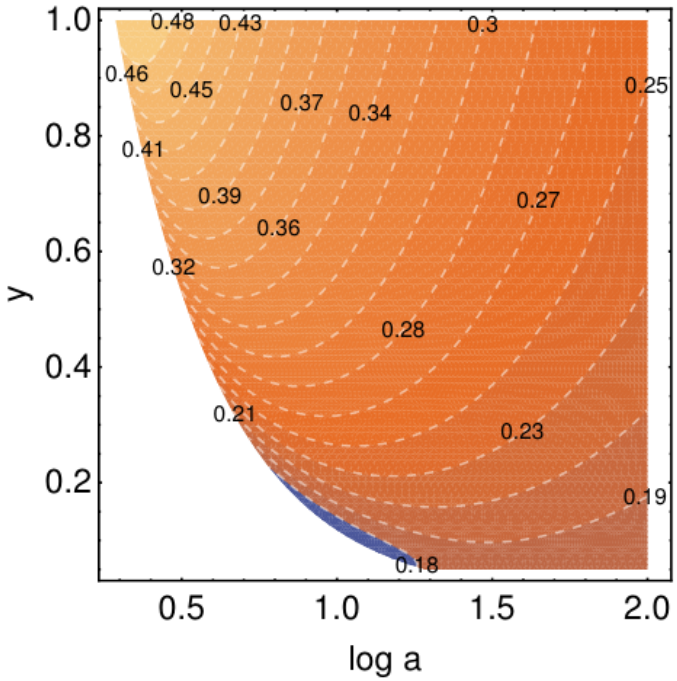
The graviball persists in $d > 4$

Maximal-stability estimate of $\log a$

$$r(\log a, y) = \frac{|\Lambda_{d_c}^2 - \Lambda^2|}{\Lambda^2}$$

which should be minimized to enhance smoothness in the transition from $d = 5$ to $d = 4$

Graviball: $\log a \approx 1$



§4 Summary and outlook

1. Formalism for unitarizing forces of infinite range
2. Removing of a global phase factor S_c in the S-matrix. Dalitz-Weinberg phase
3. IR-safe PWAs
4. Standard unitarization techniques from hadron physics satisfying two-body unitarity and analyticity
5. The exactly solvable Coulomb scattering is a check and good example for our formalism
6. Coulomb PWAs cannot be reproduced by the method of arXiv:2207.06070

- Prediction of the graviball (or gravi-σ)
  $s_P = (\varkappa - i\,\frac{2}{3\pi})\Lambda^2$, $\Lambda^2 \simeq \pi G^{-1}$ and $\varkappa \ll 1$
- Its resonance effects peak at $s \ll \Lambda^2$
- Large corrections to S-wave graviton-graviton scattering calculated in perturbation theory from EFT
- Suppression of multi-graviton[pion] intermediate states for $s < G^{-1}[(4\pi f_\pi)^2]$
- Close analogy with the σ of $f_0(500)$ resonance
- Many more applications should be pursued in hadron physics

8. *Guerrieri, Penedones, Vieira, PRL127,081601(2021):* S-matrix **Bootstrap**. $d = 10$ maximal supergravity. Unitarity bound demanded$\rightarrow$ derive lower bound for leading Wilson coefficient of UV completeness (R^4). Connect with the $J^{PC} = 0^{++}$ graviball, which dominates a spectral function

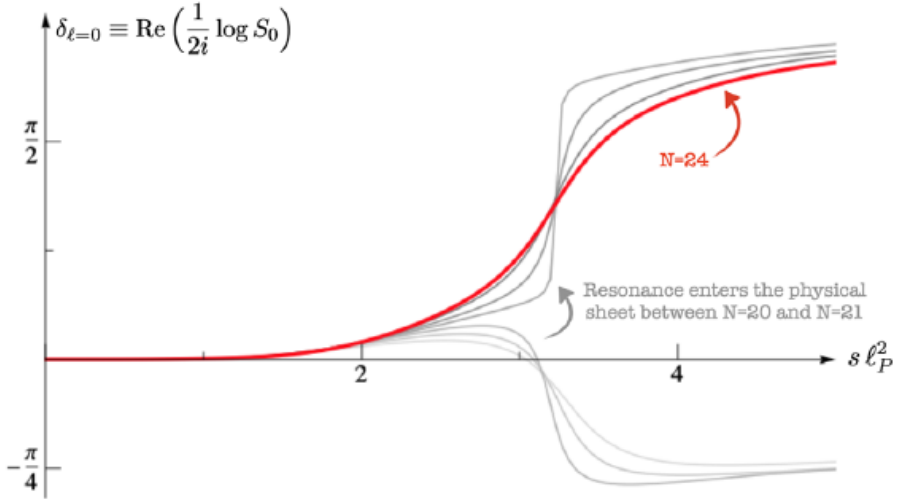


Figure: $s_P \simeq 3.2 + 0.3\,i$ GeV^{-1} ($d = 10$) "It is therefore tempting to identify the graviball as the first excited string state."

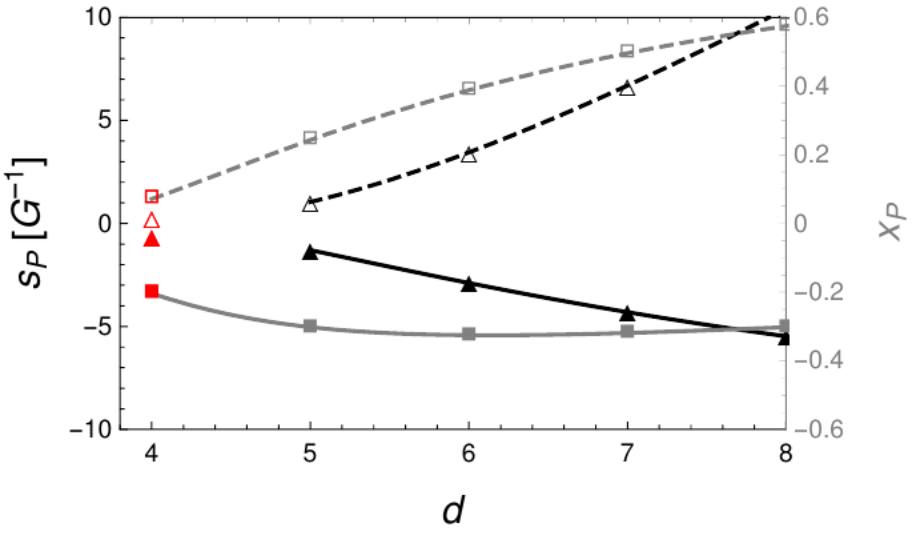


Figure: Graviball pole s_p and x for $d \geq 4$