Semiclassics with ’t Hooft flux and confinement of 4d gauge theories

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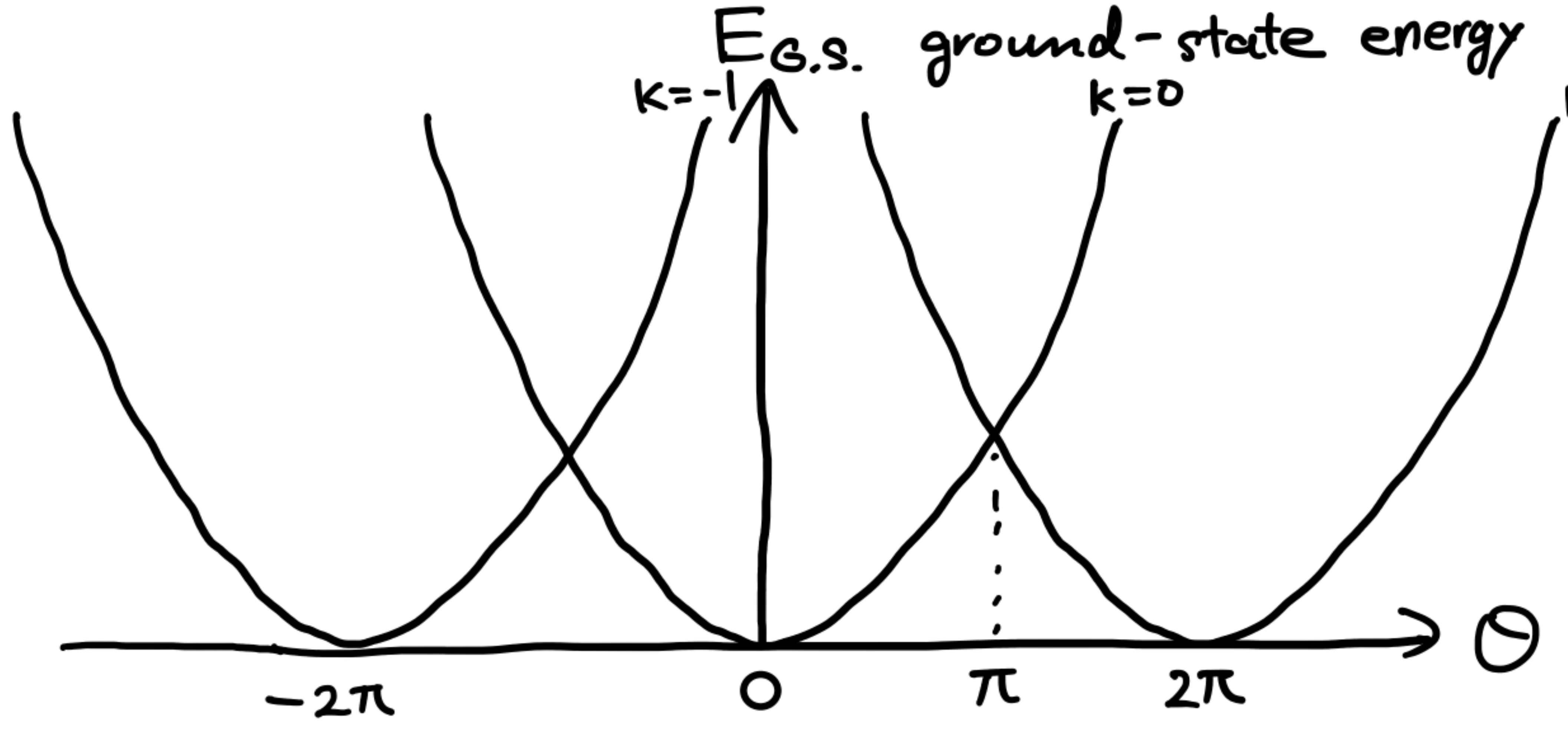
θ-vacua, multi-branch confining vacua, and SPT

Although the θ-term, $\frac{\theta}{8\pi^2}\mathrm{tr}(F\wedge F)$, is a topological term, it significantly affects the local dynamics.



In the large-N (g^2N fixed, $N\to\infty$),

$$E(\theta) \sim \min_{k\in\mathbb{Z}} \Lambda^4 (\theta-2\pi k)^2$$

if we assume confinement.
(Witten '80, ...)

In the modern understandings, these vacua are distinct as symmetry-protected topological phases with $\mathbb{Z}_N^{[1]}$. (Gaiotto, Kapustin, Komargodski, Seiberg '17
Kapustin, Thorngren '13)

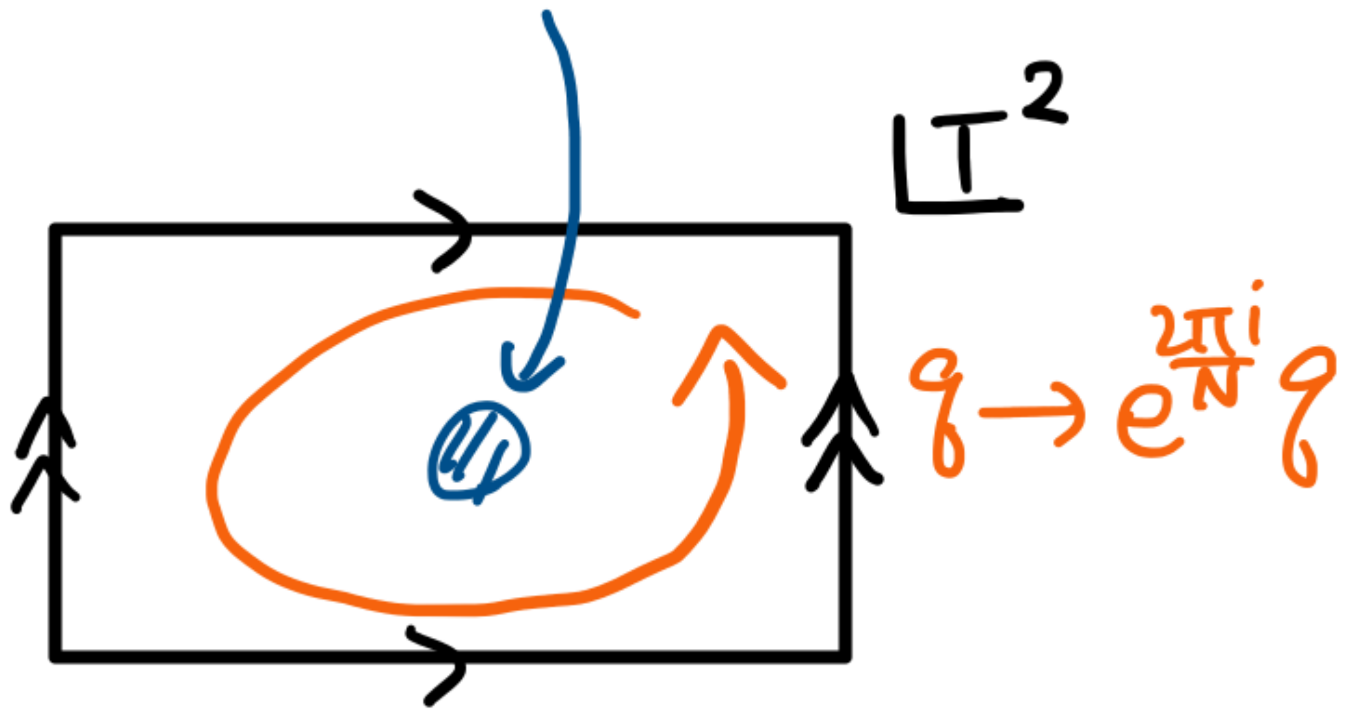
$$Z_\theta[B] = |Z_\theta[B]| \cdot e^{i\frac{kN}{4\pi}\int_M B\wedge B} \qquad (|\theta-2\pi k|<\pi)$$

($B\in H^2(M;\mathbb{Z}_N)$ $\mathbb{Z}_N$ 2-form gauge field)

Conjecture (YT, Ünsal, '22)

YM, QCD on $\mathbb{R}^2 \times T^2$

w/ $\boxed{\text{'t Hooft flux}}$



Weak-coupling description

via $\boxed{\text{Center Vortices}}$

$\Longleftrightarrow$

Adiabatic Continuity

YM, QCD on $\mathbb{R}^4$

Strong-couplings

Supporting evidence

For small $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux,

we can use the Dilute Gas Approximation w/ center vortices.

It predicts

- $\boxed{\text{(YM theory)}}$ $\quad E_k(\theta) \sim -\Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$ (Multi-branch vacua)

- ($\mathcal{N}=1$ SYM) $\quad \langle \mathrm{tr}(\lambda\lambda) \rangle \sim \Lambda^3 e^{i\frac{\theta-2\pi k}{N}}$

  (Discrete chiral SSB)

- (QCD w/ non-commuting flavor twist ($N_c = N_f = N$)) $\quad \langle \mathrm{tr}_{cf}(\bar\Psi)\, \mathrm{tr}_{cf}(\Psi) \rangle \sim \Lambda^3 e^{i\frac{\theta-2\pi k}{N}}$.

- $\boxed{\text{(QCD w/ } U(1)_B \text{ monopole flux)}}$ $\quad S_{eff} \sim \int \left(|dU|^2 + \frac{1}{12\pi} \mathrm{tr}((U^\dagger dU)^3) + \underbrace{\chi_{top} (i \ln \det U - \theta)^2}_{} \right)$

  η' mass consistent with Witten-Veneziano formula

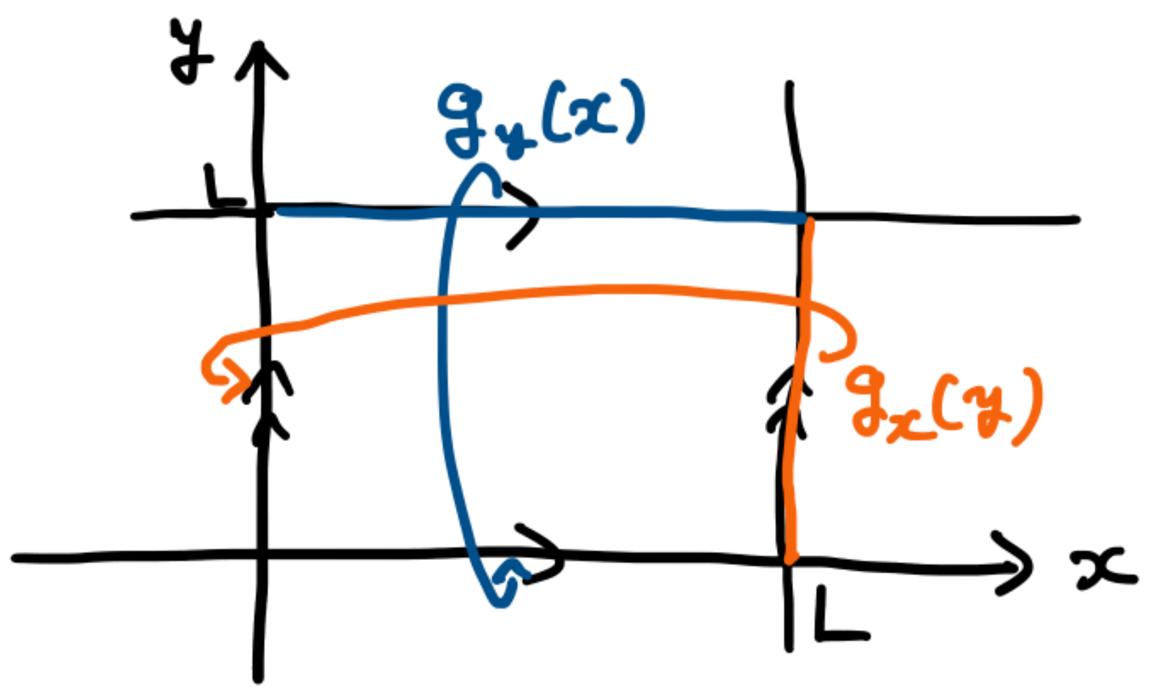
Plan

- Introduction
  - Confinement & adiabatic continuity
  - Our conjecture
- T^2 compactification with 't Hooft flux
  - Center vortex & confinement
  - Multi-branch structure of confinement vacua

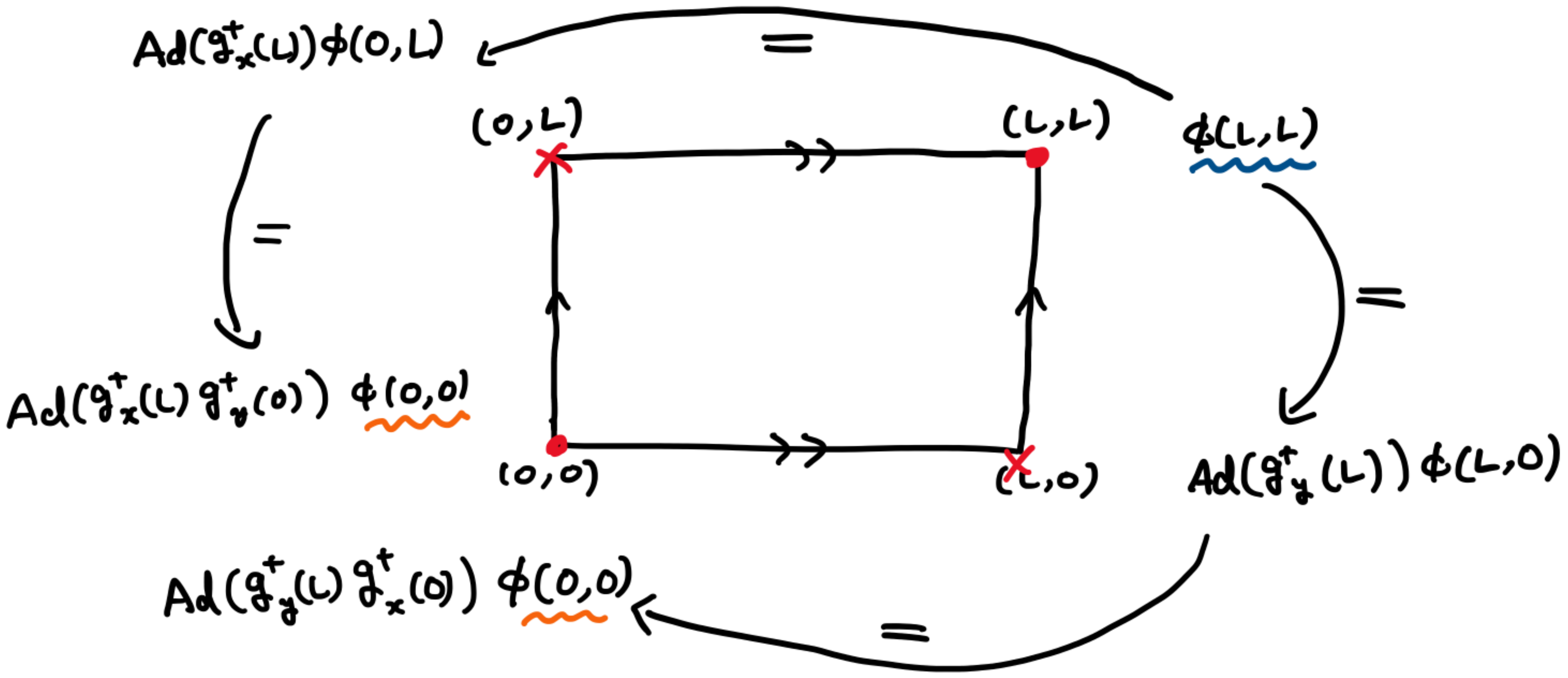
YM theory

- 't Hooft flux for QCD
  - $U(1)_B$ magnetic flux
  - Derivation of chiral effective Lagrangian

't Hooft twisted boundary condition on $T^2 \ni (x,y)$



$$\begin{pmatrix} x \sim x+L \\ y \sim y+L \end{pmatrix}$$



$\phi(x,y)$: adjoint matter field

$$\phi(L,y) = \mathrm{Ad}(g_x^\dagger(y))\, \phi(0,y)$$

$$\phi(x,L) = \mathrm{Ad}(g_y^\dagger(x))\, \phi(x,0)$$

Uniqueness of the matter wavefunction requires

$$g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0)\, e^{\frac{2\pi i}{N} n_{xy}} .$$

$\uparrow$ 't Hooft flux.

(cf When fundamental matters exist, the condition becomes

$$g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0) .$$
)

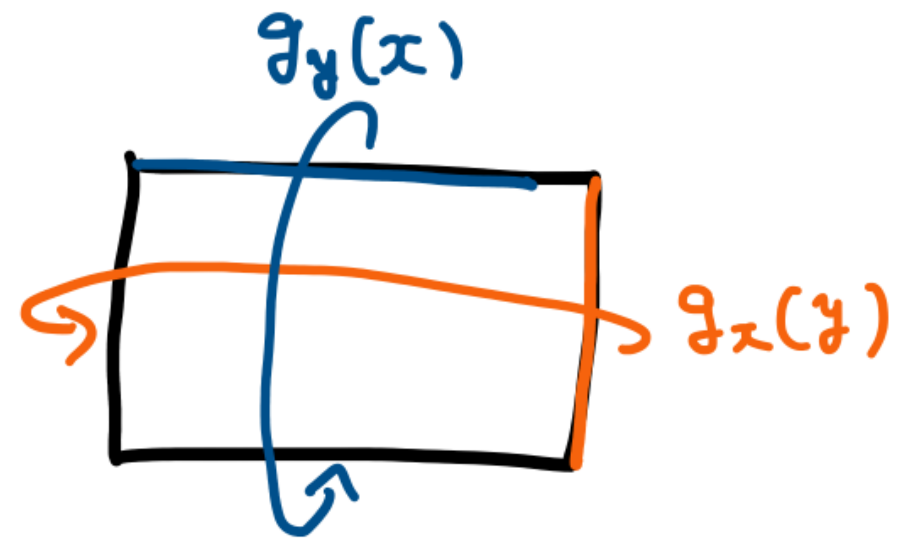
Classical configuration and unbroken center symmetry

$F = 0$ gives the classical vacuum on T^2.

$\Rightarrow$ We can take a gauge so that $\boxed{a_\mu = 0.}$

However, Polyakov loops are forced to be nontrivial for the 't Hooft twist : $(n_{xy} = 1)$

$$g_x(y) \equiv S \propto \begin{pmatrix} 0 & 1 & & \mathbb{0} \\ & 0 & 1 & \\ & & \ddots & 1 \\ 1 & \mathbb{0} & & 0 \end{pmatrix}$$



$$g_y(x) \equiv C \propto \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} \quad \left(\omega = e^{\frac{2\pi i}{N}}\right)$$

$$\left(S, C \in SU(N) \text{ and } SC = CS\, e^{\frac{2\pi i}{N}} \right).$$

$$\Rightarrow \quad P_x = S, \quad P_y = C.$$

This is the $\boxed{\text{center-symmetric}}$ configuration. $(\mathrm{tr}(P_x) = 0$ etc.$)$

Perturbative analysis of $SU(N)$ YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

- $\mathbb{Z}_N \times \mathbb{Z}_N$ center symmetry is unbroken.

- 2d gluons are gapped.

  $\Leftarrow$ Polyakov loops along T^2 are adjoint Higgs fields for $\mathbb{R}^2$.

  $P_3 = S, \ P_4 = C$ gives

  $$SU(N) \xrightarrow{\text{Higgsing}} \mathbb{Z}_N.$$

  Weak-coupling analysis is free from IR divergences.

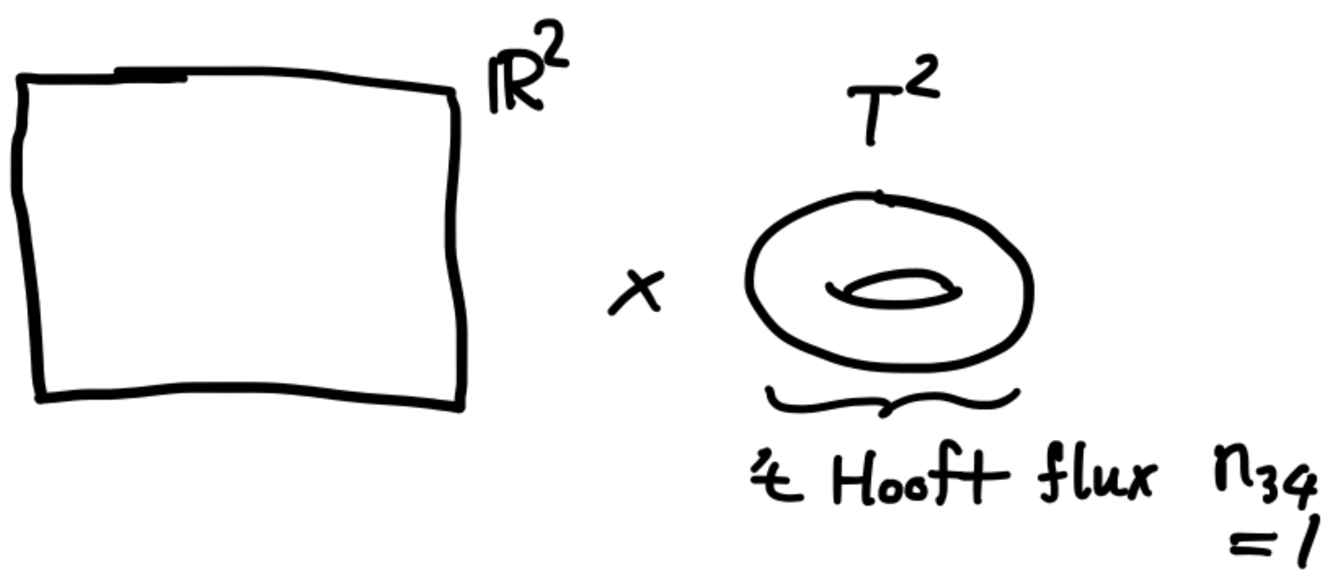
- However, Wilson loops inside $\mathbb{R}^2$ obey perimeter laws.

$$\Downarrow$$

We have to resolve this problem to achieve adiabatic continuity. $\Longrightarrow$ Center vortex

Center vortex as a fractional instanton on $\mathbb{R}^2 \times T^2$

In this setup, the minimal topological charge is given by



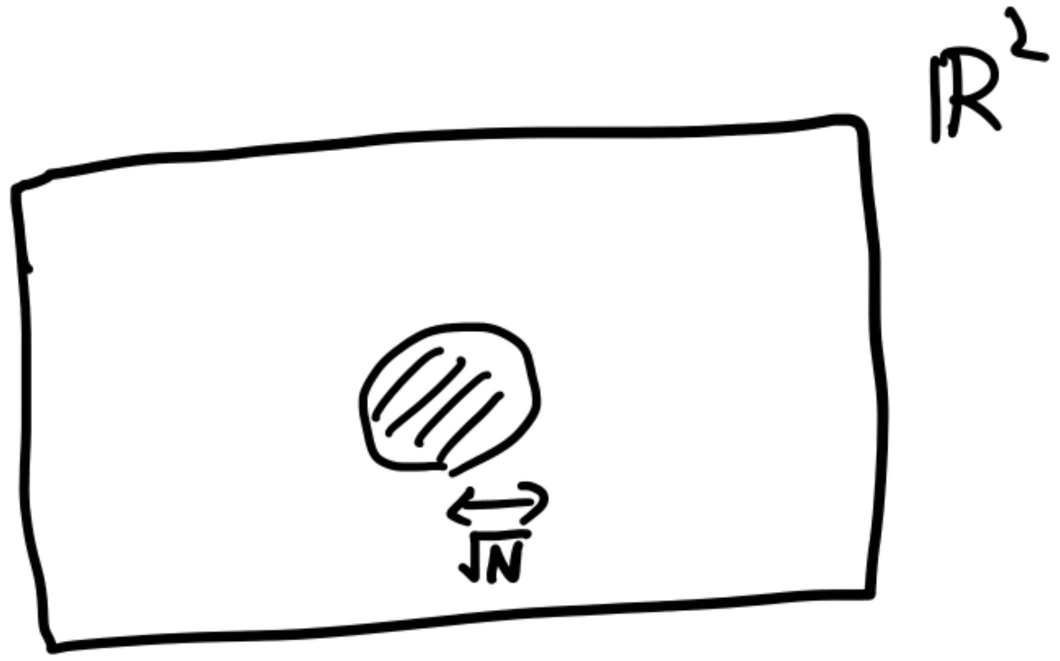
$$\boxed{Q_{top} = \frac{1}{8\pi^2}\int \mathrm{tr}(F \wedge F) = \frac{1}{N}}$$

(More precisely, $Q_{top} \in \frac{1}{N}\left(\frac{-\varepsilon_{\mu\nu\rho\sigma} n_{\mu\nu} n_{\rho\sigma}}{8}\right) + \mathbb{Z}$ (van Baal '82))

If there exists a self-dual configuration, its Yang-Mills action becomes

$$\boxed{S_{YM} = \frac{8\pi^2}{g^2}|Q_{top}| = \frac{8\pi^2}{g^2 \cdot N}}$$

Gonzalez-Arroyo, Montero '98, Montero '99 numerically confirmed such a classical solution exists:
center vortex or fractional instanton.



$$\left(\begin{array}{l} Q_{top} = \frac{1}{N} \\ S_{YM} = \frac{8\pi^2}{g^2 N} \end{array}\right)$$

(cf. Garcia Perez, Gonzalez-Arroyo, '92, Itou '18)

Partition funcion on $M_2 \times T^2$ & θ-dependence

$\rightarrow \mathbb{R}^2$

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

Using the 1-loop vertex of the center vortex

$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

we have

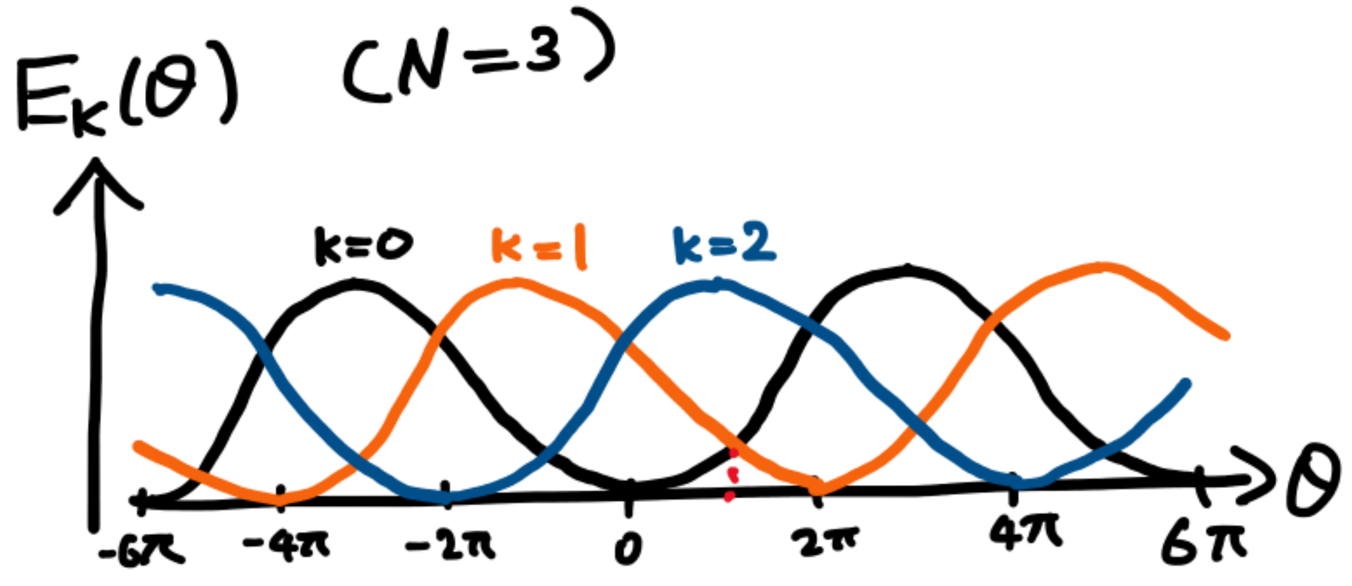
vortex   anti-vortex

$$Z(\theta) = \sum_{n,\bar{n} \geq 0} \frac{\delta_{n-\bar{n} \in N\mathbb{Z}}}{n!\ \bar{n}!} \left(\underline{V \cdot K e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}}}\right)^n \left(\underline{V \cdot K e^{-\frac{8\pi^2}{g^2 N} - i\frac{\theta}{N}}}\right)^{\bar{n}}$$

$$= \sum_{k=0}^{N-1} \exp\left[-V\left(\underbrace{-2K e^{-\frac{8\pi^2}{g^2 N}} \cos\left(\frac{\theta - 2\pi k}{N}\right)}_{E_k(\theta)}\right)\right]$$

$E_k(\theta)$: Ground-state energy densities

$\Rightarrow \begin{cases} \bullet \text{ N-branch structure of ground states.} \\ \bullet \text{ Each branch has a fractional } \theta\text{-dependence.} \end{cases}$



Anomaly matching in 4d and 2d

4d SU(N) YM theory has an 't Hooft anomaly: (Gaiotto, Kapustin, Komargodski, Seiberg '17)

$$Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B\wedge B} Z_\theta[B]$$

$\Longrightarrow$ Confinement implies the multi-branch structure of vacua.

With T^2-compactification, $(\mathbb{Z}_N^{[1]})_{4d}$ splits into $(\mathbb{Z}_N^{[1]})_{2d} \times \mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$.

When 't Hooft flux n_{34} (mod N) is introduced, the 4d anomaly becomes

$$Z_{\theta+2\pi}[B_{2d}, A, A'] = e^{i\left(n_{34}\int B - \frac{N}{2\pi}\int A\wedge A'\right)} Z_\theta[B, A, A'].$$

$\Longrightarrow$ When $\gcd(n_{34}, N)=1$ (especially when $n_{34}=1$),

confinement implies the multi-branch structure. (cf. YT, Misumi, Sakai, '17. Yamazaki '17.)

Both properties are obtained by the center vortex.

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QCD

't Hooft flux with fundamental quarks

We have seen that

Semiclassical theory w/ center vortices

$\Rightarrow$ Confinement vacua of YM theory.

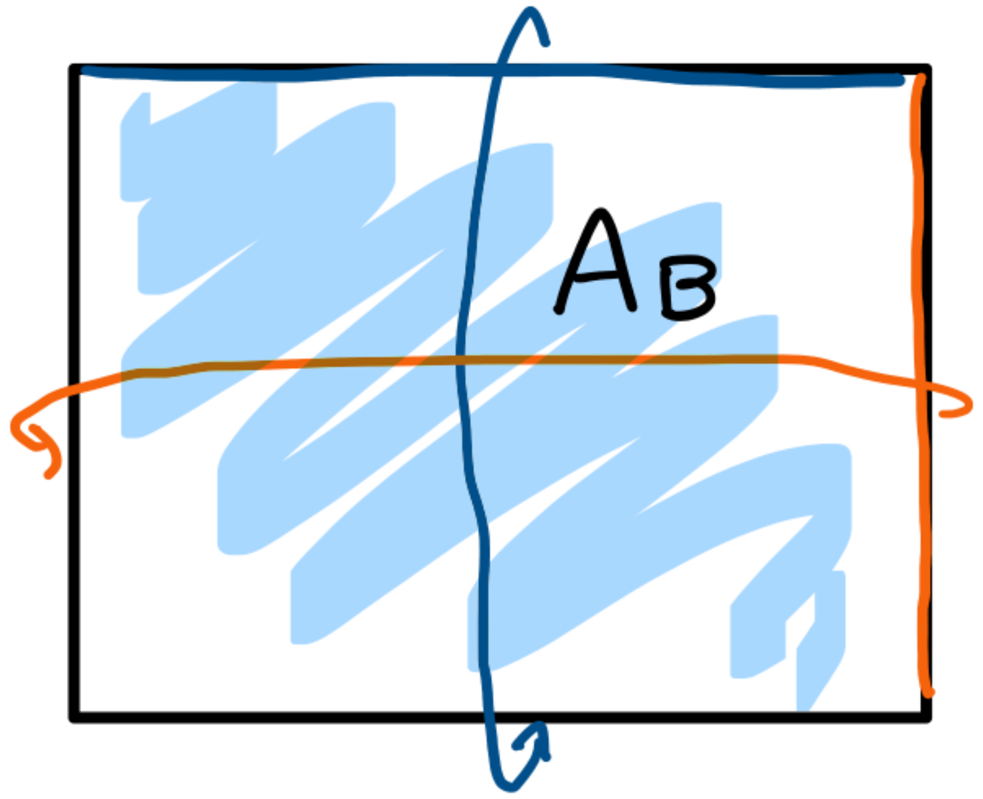
Next, we would like to apply this idea for

$\boxed{\text{Chiral effective Lagrangian of QCD}}$.

Obstacle: We cannot introduce the 't Hooft flux

when fundamental matters exist.

$U(1)_B$ monopole flux & 't Hooft flux on T^2



quark field

$$\begin{cases} \psi(L,y) = g_x^\dagger(y)\, e^{-i\frac{\phi_x(y)}{N}}\, \psi(0,y) \\ \psi(x,L) = \underbrace{g_y^\dagger(x)}_{\text{color-transition functions}}\, \underbrace{e^{-i\frac{\phi_y(x)}{N}}}_{U(1)_q\text{-transition functions}}\, \psi(x,0) \end{cases}$$

Cocycle condition

$$g_x^\dagger(L)\, g_y^\dagger(0)\, e^{-i\frac{1}{N}(\phi_x(L)+\phi_y(0))} = g_y^\dagger(L)\, g_x^\dagger(0)\, e^{-i\frac{1}{N}(\phi_y(L)+\phi_x(0))}.$$

$U(1)_B$ monopole flux

$$2\pi = \int_{T^2} dA_B = (\phi_x(L)-\phi_x(0)) - (\phi_y(L)-\phi_y(0))$$

$$\Longrightarrow \quad g_x^\dagger(L)\, g_y^\dagger(0) = g_y^\dagger(L)\, g_x^\dagger(0) \boxed{e^{\frac{2\pi i}{N}}} \qquad \text{(cf. YT, '18)}$$

't Hooft flux !!

Perturbative spectrum

Gauge fields : Same with $SU(N)$ YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux.

$\Rightarrow$ $\mathbb{Z}_N$ gauge field.

Quark fields : Solve the Dirac zero-mode eq.

$$\left[\gamma_3\left(\partial_3 + i\frac{1}{N}A_{B,3}\right) + \gamma_4\left(\partial_4 + i\frac{1}{N}A_{B,4}\right)\right]\psi = 0.$$

$\Rightarrow$ For each 4d fundamental Dirac fermion,

there is a 2d massless Dirac fermion.

For N_f - flavor massless QCD, we have 2d N_f Dirac fermions.

$\xleftrightarrow[\text{non-Abelian bosonization}]{}$ $U(N_f)_1$ WZW model $\quad \frac{1}{8\pi}\int_{M_2} \mathrm{tr}(dU^\dagger \wedge * dU) + \frac{1}{12\pi}\int_{M_3} \mathrm{tr}\left[(U^\dagger dU)^3\right]$

w/ $U(N_f)$-valued field $U : M_2 \to U(N_f)$.

Center-vortex induced mass for η'

Let us construct the center-vortex vertex $\sim e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}$.

$U(1)$ axial anomaly requires the spurious symmetry

$$\psi \to e^{i\alpha\gamma_5}\psi, \quad \bar\psi \to \bar\psi e^{i\alpha\gamma_5}, \quad \theta \to \theta + 2N_f\alpha.$$

In the bosonized description

$$U \to e^{-2i\alpha} U.$$

$$\Rightarrow \quad e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}} \cdot (\det U)^{\frac{1}{N}} \quad \text{satisfies}$$

$$\left\{ \begin{array}{l} \cdot\ SU(N_f)_L \times SU(N_f)_R \text{ chiral symmetry}, \\ \cdot\ U(1) \text{ axial anomaly relation}. \end{array} \right.$$

$$\Rightarrow \quad \Delta S_{\text{center-vortex}} \sim \underbrace{\Lambda^2 (\Lambda L)^{\frac{5N-2N_f}{3N}}}_{\eta' \text{ mass}} \cdot \cos\left(\frac{i\ln(\det U) - \theta}{N}\right)$$

QCD for large $\mathbb{R}^2 \times T^2$ w/ $U(1)_B$ monopole flux

So far, we explicitly solved QCD for small $\mathbb{R}^2 \times T^2$ w/ $U(1)_B$ flux.

What happens for large $\mathbb{R}^2 \times T^2$?

Chiral effective Lagrangian is now available :

$$\int_{M_4} f_\pi^2 \, |dU|^2 + \int_{M_4} A_{B\mu} J_B^\mu .$$

$$* J_B = * J_{\text{skyrmion}} = \frac{1}{24\pi^2} \operatorname{tr}[(U^\dagger dU)^3] .$$

$$\Downarrow \; T^2\text{- compactification w/ } \int_{T^2} dA_B = 2\pi$$

$$\int_{M_2} (L^2 f_\pi^2) |dU|^2 + \frac{1}{12\pi} \int_{M_3} \operatorname{tr}[(U^\dagger dU)^3] .$$

$\Rightarrow$ $SU(N_f)_1$ WZW model when $L^2 f_\pi^2 = \frac{1}{8\pi}$.

Consistent result with the explicit QCD computation.

SUMMARY

YM, QCD on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux $\Longleftrightarrow$ YM, QCD on $\mathbb{R}^4$

$\underbrace{\text{YM, QCD on } \mathbb{R}^2\times T^2 \text{ w/ 't Hooft flux}}_{\text{Semiclassics w/ center vortices}}$ — Adiabatic Continuity — $\underbrace{\text{YM, QCD on } \mathbb{R}^4}_{\text{Strong-couplings}}$

- $\boxed{\text{(YM theory)}}$ $\quad E_k(\theta) \sim -\Lambda^2 (\Lambda L)^{\frac{5}{3}} \cos\left(\frac{\theta - 2\pi k}{N}\right)$ (Multi-branch vacua)
- ($\mathcal{N}=1$ SYM) $\quad \langle \mathrm{tr}(\lambda\lambda)\rangle \sim \Lambda^3 e^{i\frac{\theta-2\pi k}{N}}$
- (QCD w/ non-commuting flavor twist ($N_c = N_f = N$)) $\quad \langle \mathrm{tr}_{cf}(\bar\Psi)\,\mathrm{tr}_{cf}(\Psi)\rangle \sim \Lambda^3 e^{i\frac{\theta-2\pi k}{N}}$

(Discrete chiral SSB)

- $\boxed{\text{(QCD w/ } U(1)_B \text{ monopole flux)}}$ $\quad S_{\text{eff}} \sim \int \left(|dU|^2 + \frac{1}{12\pi}\mathrm{tr}((U^\dagger dU)^3) + \underbrace{\chi_{\text{top}}\,(i \ln\det U - \theta)^2}_{\eta' \text{ mass that is Witten-Veneziano formula}} \right)$

$\Rightarrow$ Application to other 4d gauge theories!! (2-index quarks, chiral theories ...)