The static energy of a quark-antiquark pair from Laplacian eigenmodes

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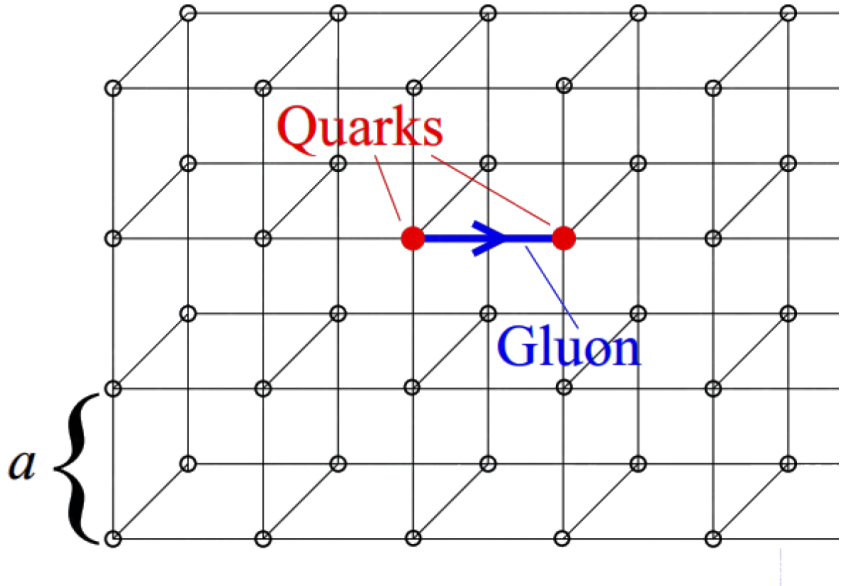
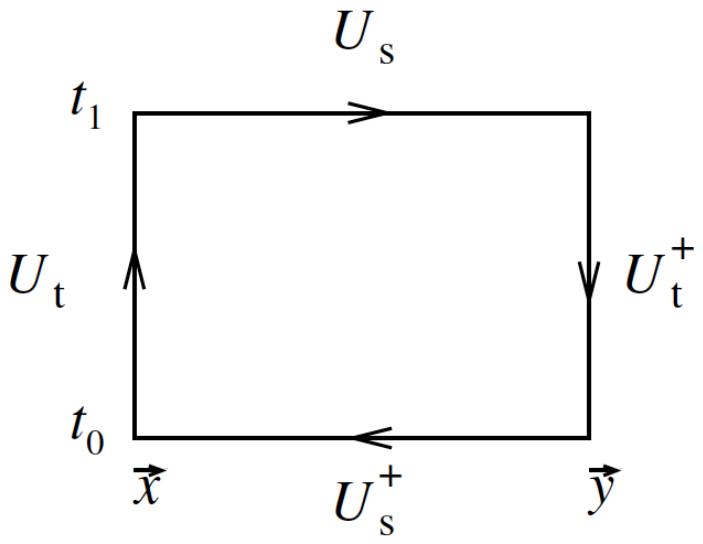
Lattice QCD





- link variables $U_\mu(x) = \exp(i \int_x^{x+a\hat{\mu}} A_\mu dx^\mu$
- Wilson line $U_s(\vec{x}, \vec{y}) = \exp(i \int_{\vec{x}}^{\vec{y}} A_\mu dx^\mu) = \prod U_\mu$
- path-ordered product of link variables, on-/off-axis
- plaquette, Wilson loop $W(R, T)$, static $Q\bar{Q}$ pair

Motivation

- calculate the static potential energy with high resolution
  - matching the lattice QCD potential with the perturbative potential to determine $\Lambda_{\overline{MS}}$ in Fourier space, *e.g.*, Karbstein et al. (2014)
  - observation of string breaking in QCD, *e.g.*, Bali et al. (2008), Bulava et al. (2019)

$\Rightarrow$ we have to work with off-axis separated quarks

- the spatial part of the Wilson loop has to go over stair-like paths through the lattice $\rightarrow$ not unique, computationally expensive

$\Rightarrow$ alternative operator which ensures gauge covariance of the quark-anti-quark $Q(\vec{x})\bar{Q}(\vec{y})$ trial state

- required gauge transformation behavior:

$$U_s'(\vec{x},\vec{y}) = G(\vec{x})U_s(\vec{x},\vec{y})G^\dagger(\vec{y})$$

Laplacian Eigenmodes

- Consider the 3D covariant lattice Laplace operator:

$$\begin{aligned}\Delta V \quad = \quad & \frac{1}{a^2}[U_x^\dagger(x-a,y,z)V(x-a,y,z) - 2V(\vec{x}) + U_x(\vec{x})V(x+a,y,z) \\ & + U_y^\dagger(x,y-a,z)V(x,y-a,z) - 2V(\vec{x}) + U_y(\vec{x})V(x,y+a,z) \\ & + U_z^\dagger(x,y,z-a)V(x,y,z-a) - 2V(\vec{x}) + U_z(\vec{x})V(x,y,z+a)]\end{aligned}$$

- transformation behavior: $\Delta' = G(\vec{x})\Delta G^\dagger(\vec{y})$
- consider $V(\vec{x})$ an eigenvector: $\Delta V(\vec{x}) = \lambda V(\vec{x})$

$$\begin{aligned}\Delta' V'(\vec{x}) &= \lambda V'(\vec{x}) \\ G(\vec{x})\Delta G^\dagger(\vec{x})V'(\vec{x}) &= \lambda V'(\vec{x}) \\ \Delta G^\dagger(\vec{x})V'(\vec{x}) &= \lambda G^\dagger(\vec{x})V'(\vec{x})\end{aligned}$$

- $V(\vec{x})$ and $G^\dagger(\vec{x})V'(\vec{x})$ are members of the same eigen-space

Static $Q\bar{Q}$ pair, Wilson loop alternative...

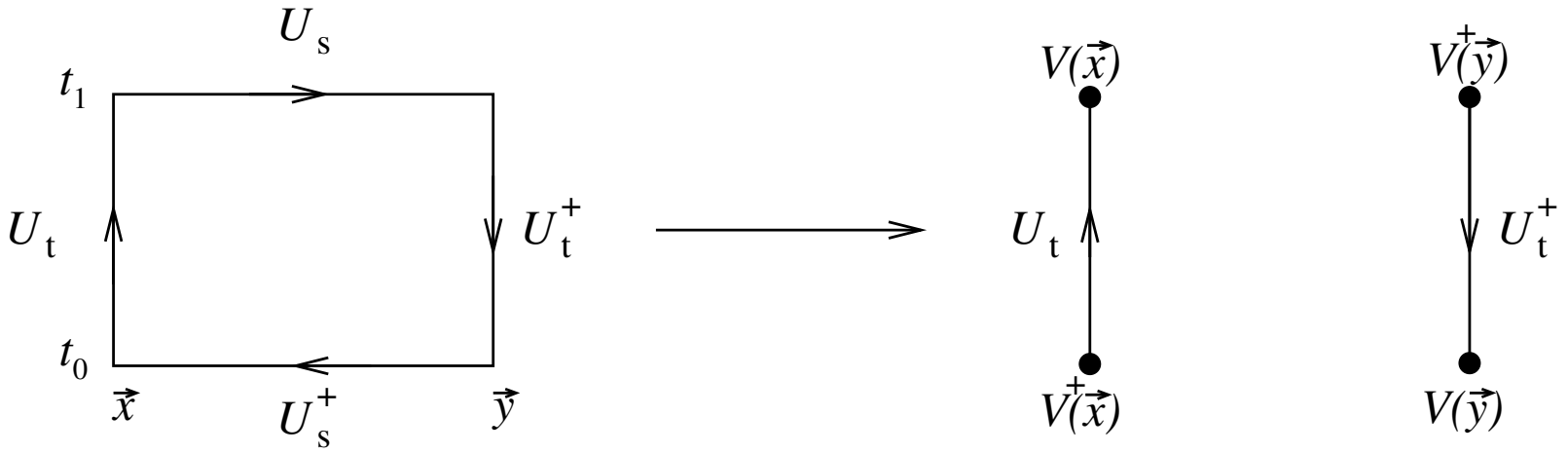
- idea taken from Neitzel et al. (2016) SU(2)
- SU(3): the eigenvalues are in general non-degenerate
- spatial Wilson line: $U'_s(\vec{x},\vec{y}) = G(\vec{x})U_s(\vec{x},\vec{y})G^\dagger(\vec{y})$

$$V'(\vec{x})V'^\dagger(\vec{y}) = G(\vec{x})V(\vec{x})V^\dagger(\vec{y})G^\dagger(\vec{y})$$

- Wilson loop of size $(R = |\vec{x}-\vec{y}|) \times (T = |t_1 - t_0|)$

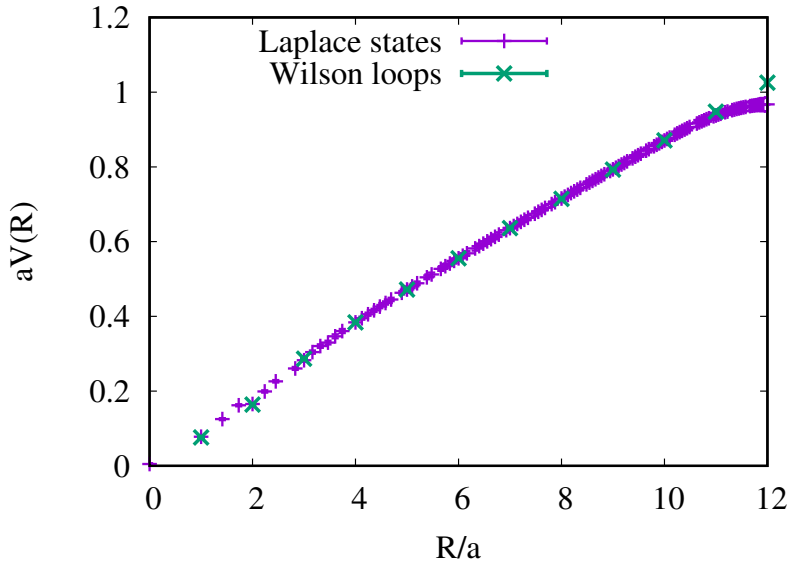
$$W(R,T) = U_t(\vec{x};t_0,t_1)U_s(\vec{x},\vec{y};t_1)U_t^\dagger(\vec{y};t_0,t_1)U_s^\dagger(\vec{x},\vec{y};t_0)$$

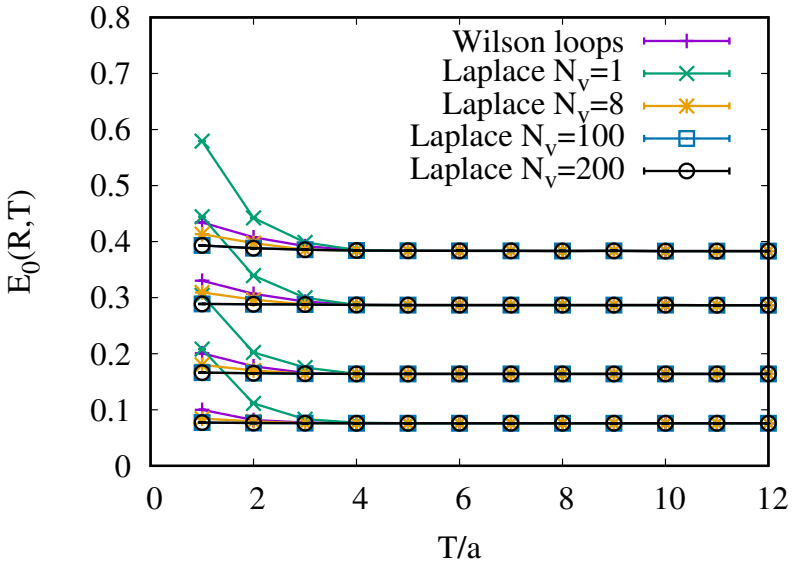
$$\rightarrow\ U_t(\vec{x};t_0,t_1)V_j(\vec{x},t_1)V_j^\dagger(\vec{y},t_1)U_t^\dagger(\vec{y};t_0,t_1)V_i(\vec{y},t_0)V_i^\dagger(\vec{x},t_0)$$



Simulations

- $24^3 \times 48$, $\beta = 5.3$, $N_f = 2$, $\kappa = 0.13270$, $a = 0.0658$ fm
- (on-axis) Wilson loops from 4646 measurements, 4D HYP
- Laplace states from 1161 measurements, 20 APE (0.5) for Lanczos, 1 4D HYP for temporal Wilson lines
- static potential $aV(R) = \lim_{T\to\infty} \log[W(R,T)/W(R,T+1)]$
- at $R = 12a$ the force between $Q\bar{Q}$ must vanish due to symmetry
- more eigenvectors gives earlier plateau and increase precision





Gaussian profiles

$$W_{kl}(R,T) = \sum_{i,j}^{N_v} N_{kl}(\lambda_i,\lambda_j) \sum_{\vec{x},t_0} \left\langle V_i^\dagger(\vec{x},t_0) U_t(\vec{x};t_0,t_1) V_j(\vec{x},t_1) V_j^\dagger(\vec{y},t_1) U_t^\dagger(\vec{y};t_0,t_1) V_i(\vec{y},t_0) \right\rangle$$

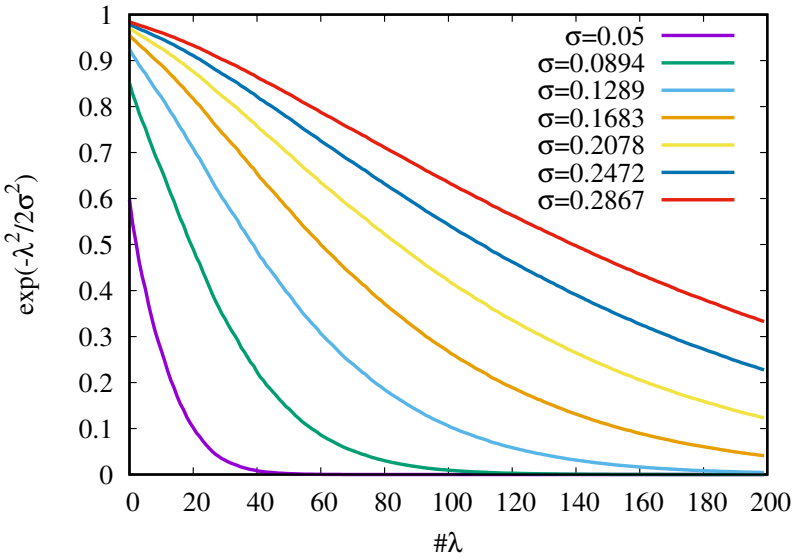
Gaussian profile functions:

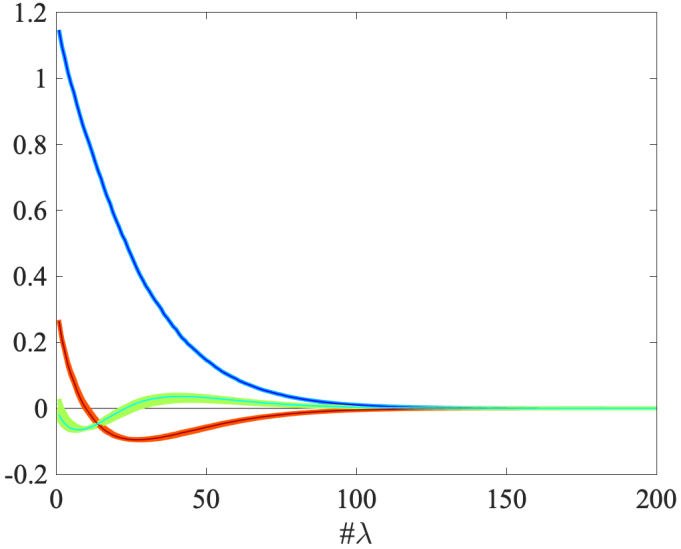
$N_{kl}(\lambda_i,\lambda_j) = \exp(-\lambda_i^2/2\sigma_k^2)\exp(-\lambda_j^2/2\sigma_l^2)$ using 7 different σ values

$\Rightarrow$ prune W_{kl} using 3 most significant singular vectors u_i at t_0

...improves stability and keeps useful operators

$\Rightarrow$ pruned ("optimal") profiles $u_{i,j}\exp(-\lambda^2/\sigma_j^2)$

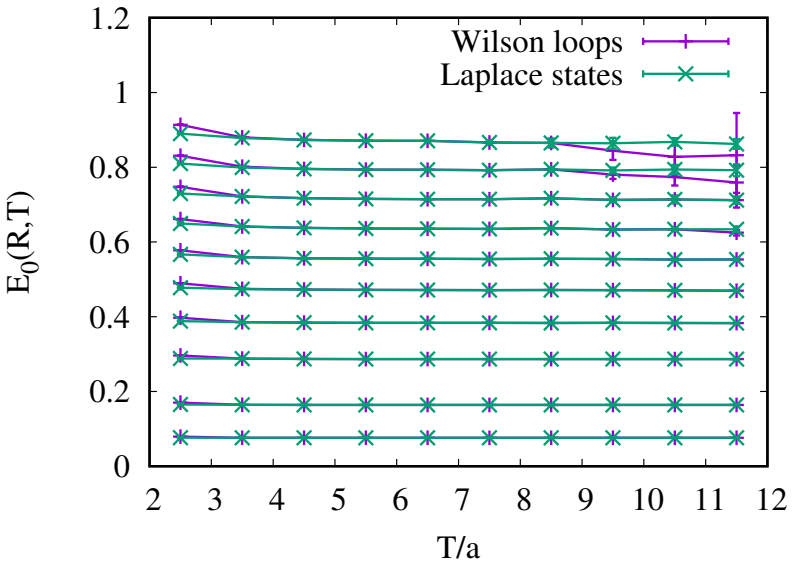


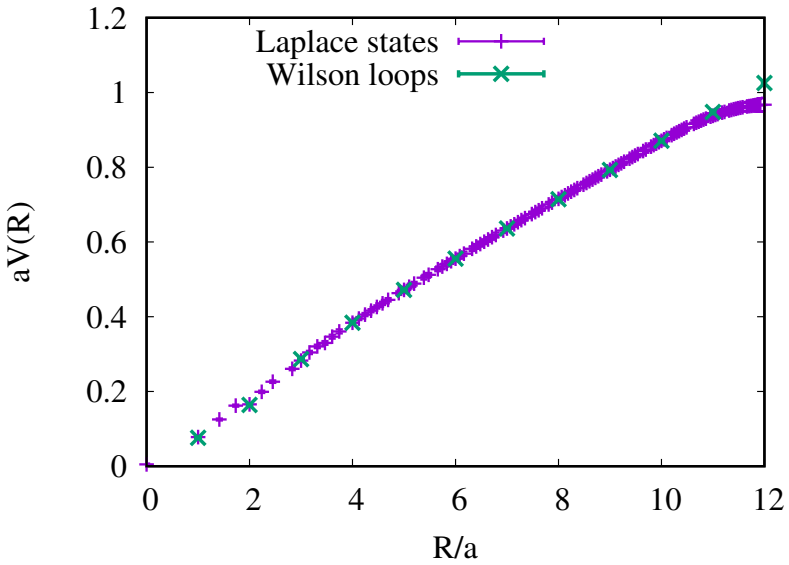


Improved Results

Solve generalized eigenvalue problems for

- correlation matrix of Wilson loops with 3 spatial smearing levels
- Laplace trial states W_{kl} (or pruned version $\bar{W}_{ij} = u_i^\dagger W_{kl} u_j$)





- on-axis Wilson loops for 3 spatial smearing levels (0,10,20HYP) are equally expensive as the calculation of 100 Laplacian eigenvectors and Laplace states with 3 Gaussian profiles including off-axis distances!

Fractional overlaps with the ground state

R	$\#\lambda = 1$	8	64	200	Gauss	Wloop	HYP2
1	0.773(3)	0.945(1)	0.970(1)	0.982(1)	0.993(1)	0.921(1)	0.983(1)
2	0.747(4)	0.929(2)	0.964(1)	0.987(1)	0.989(1)	0.891(1)	0.978(1)
3	0.723(4)	0.878(2)	0.984(2)	0.986(1)	0.988(1)	0.867(1)	0.972(2)
4	0.726(5)	0.874(3)	0.921(2)	0.984(2)	0.986(2)	0.841(2)	0.965(3)
5	0.637(6)	0.871(4)	0.979(3)	0.982(3)	0.983(3)	0.813(2)	0.956(5)
6	0.629(6)	0.869(4)	0.978(4)	0.980(3)	0.981(3)	0.793(3)	0.948(6)
7	0.619(7)	0.869(5)	0.977(4)	0.979(4)	0.987(4)	0.772(3)	0.934(7)
8	0.598(8)	0.862(6)	0.972(5)	0.970(4)	0.964(4)	0.745(4)	0.953(8)
9	0.572(8)	0.857(6)	0.960(5)	0.934(4)	0.963(3)	0.708(4)	0.947(9)
10	0.540(9)	0.840(7)	0.965(5)	0.931(5)	0.95(1)	0.671(5)	0.94(1)
11	0.426(9)	0.807(7)	0.943(6)	0.93(1)	0.956(9)	0.649(4)	0.93(1)
12	0.33(7)	0.79(2)	0.94(1)	0.92(1)	0.95(1)	0.64(2)	0.92(1)

t-average over mass-plateau region of

$$\frac{W(R,t)}{W(R,t_0)} \frac{\cosh\left(\left(\frac{T}{2} - t_0\right) m_0\right)}{\cosh\left(\left(\frac{T}{2} - t\right) m_0\right)}$$

Conclusions & Outlook

- ✓ alternative operator for a static quark-anti-quark pair based on Laplacian eigenmodes
- ✓ improved version (several eigenvectors weighted with Gaussian profiles) gives earlier plateau and better signal
- ✓ much higher resolution of the potential energy as off-axis distances basically come ”for free”
- ✓ implementation of static-light (charm) correlator using ”perambulators” $V(t_1)D^{-1}V(t_2)$ from distillation framework (quark field smearing via projection $\psi \to VV^\dagger\psi$) of our working group, see Knechtli et. al (2022), $\Rightarrow$ bonus slide
- putting together building blocks for observation of string breaking in QCD (mixing matrix of static and light quark propagators)
- also working on hybrid static potentials (hybrid meson masses), instead of ”gluonic handles” (excitations) use derivatives of V

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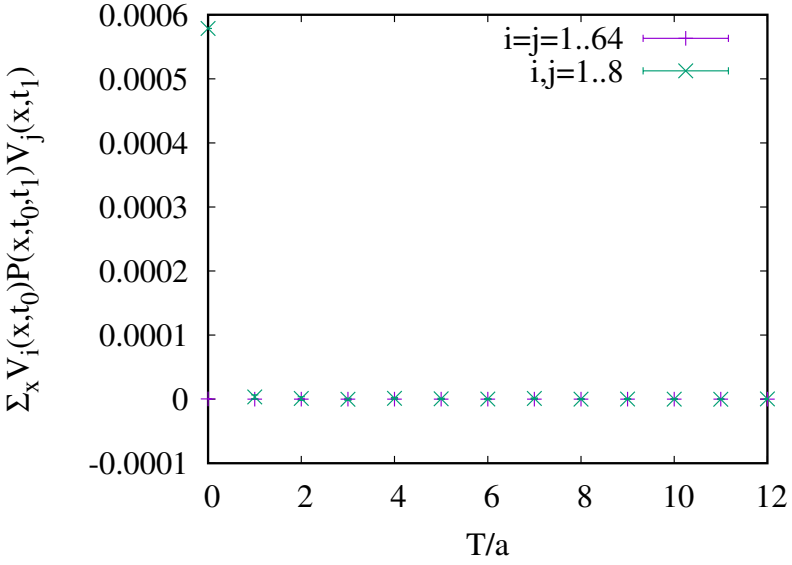
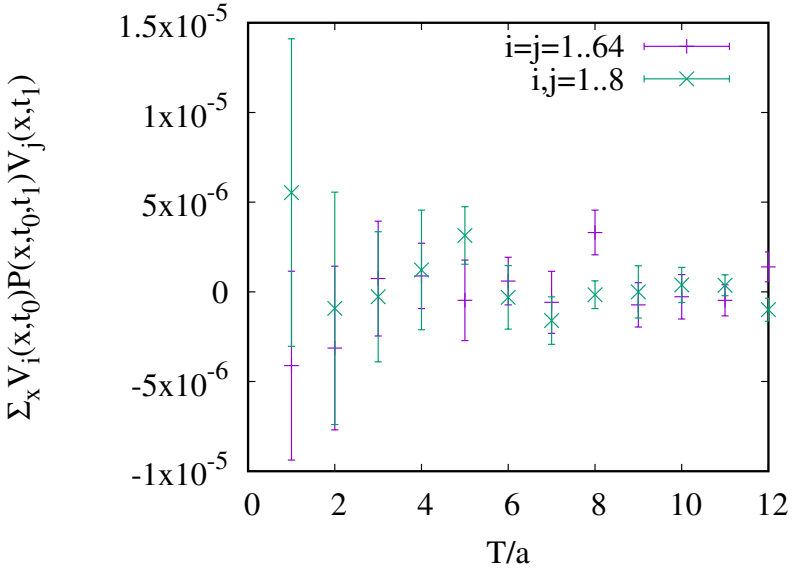
Extra: Static quark line, SVD, GEVP

$$Q(T) = \sum_{i,j}^{N_v} \sum_{\vec{x},t_0} \langle V_i^\dagger(\vec{x},t_0) U_t(\vec{x};t_0,t_1) V_j(\vec{x},t_1) \rangle$$





- SVD: $W = UDV$; U, V unitary, columns are orthonormal bases
- GEVP: $W(t)v_k = \rho_k W(t_0)v_k$, ρ_k give effective energies
- u_i or v_k can be used to get 'optimal' profiles for energy states

Bonus: Static-light (charm) meson

$$C(t) = -\sum_{t_0,i,j} \Big\langle \mathrm{tr}_d\{[V_i^\dagger D^{-1} V_j](t_0+t,t_0)P_+\} \sum_{\vec{x}} V_j^\dagger(\vec{x},t_0) U_t(\vec{x};t_0,t_0+t) V_i(\vec{x},t_0+t) \Big\rangle$$

- with light (charm) perambulators $V^\dagger D^{-1} V$
- HYP smearing of temporal links in U_t removes free energy