Fractional Topological Charge

RDP & V.P. Nair, 2206.11284

In SU(N) gauge theories with OUT dynamical quarks, at large N, need objects not $\bar{c}$ $Q_{top} = \pm 1, \pm 2 \ldots$, but

$$Q_{top} = \pm \frac{1}{N}, \pm \frac{2}{N} \ldots$$

Arise immediately $\bar{c}$ $Z(N)$ twisted b.c.'s

't Hooft '80 + ...

On "femto-slab": width $L \ll \Lambda_{QCD}^{-1}$

Unsal 2007.03880; Poppitz 2111.10423

Today: "quantum" instantons $\bar{c}$ size $\sim 1/\Lambda_{QCD}^{-1}$

measurable on lattice <u>without</u> cooling

Instantons

Topological charge $\quad Q \sim \int d^4x \, \mathrm{tr}\, G_{\mu\nu} \tilde{G}^{\mu\nu}$ $\quad \varepsilon^{\mu\nu\alpha\beta} G^{\alpha\beta}$

Instantons – solutions to classical equations of motion

self-dual, $G_{\mu\nu} = \pm \tilde{G}_{\mu\nu}$

As classical eqs. scale invariant, instantons

come in all scale sizes, $\rho : 0 \to \infty$

For $Q = n =$ integer, $\quad S = \int \mathrm{tr}\, G_{\mu\nu}^2 = \frac{8\pi^2}{g^2} n$

Construction of all instantons known –

collective coor.'s (moduli space) involved

Atiyah, Hitchin, Drinfeld, Manin Phys. Lett. A65 '78

Instantons @ $T \neq 0$

Inst.'s valid semi-classically, when $g^2 \ll 1$.

E.g., temperature $T \gg \Lambda_{QCD}$.

By asymptotic freedom, $g^2(T) \sim \#/\ln T$

$$\Rightarrow Z_{inst.} \sim e^{-8\pi^2/g^2(T)} \sim \#'/T^c \qquad c = \frac{11N_c - 2N_f}{3}$$

$T \gg \Lambda_{QCD}$ $\sim \Lambda_{QCD}$!

Lattice, $N_c = 3$:

Instantons dominate down to $T \sim 300$ MeV

$\#' \sim 10 * 1$-loop result - <u>need</u> 2-loop!

Borsanyi + ... 1606.07494

Petreczky + ... 1606.03145 ...

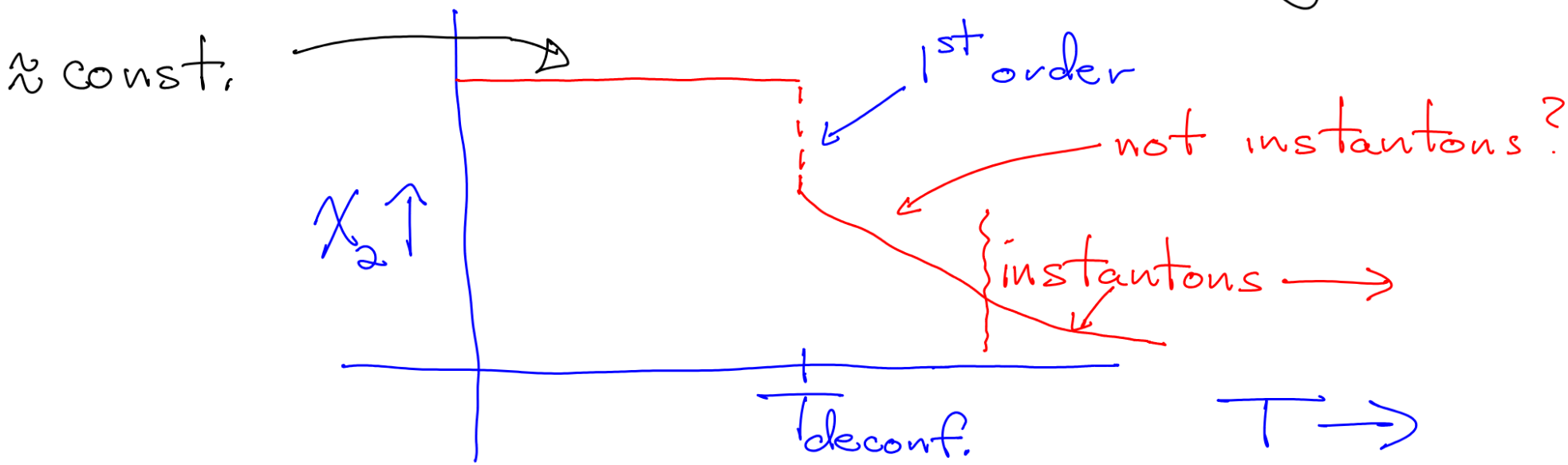
Lattice - pure glue

Compute top. susceptibility:

$$S_\theta = S_0 + i\theta Q \qquad \chi_2 = \frac{\partial^2 \ln Z}{\partial \theta^2} \sim \langle Q^2 \rangle$$

For pure SU(N) glue, for $N \geq 3$, weak dependence on N. First order transition @ $T_{deconfinement} \sim 270$ MeV

E.g., $T_c/\sqrt{\sigma} \sim$ const. $\bar{c}$ N ($\sigma = T=0$ string tension)



Lattice – $\bar{c}$ quarks

$T > 300$ MeV – instantons

$300 > T > 155$ $(= T_\chi)$ – not instantons, slower fall off

$155 > T$ – $\chi_2 \approx$ const.

Also:

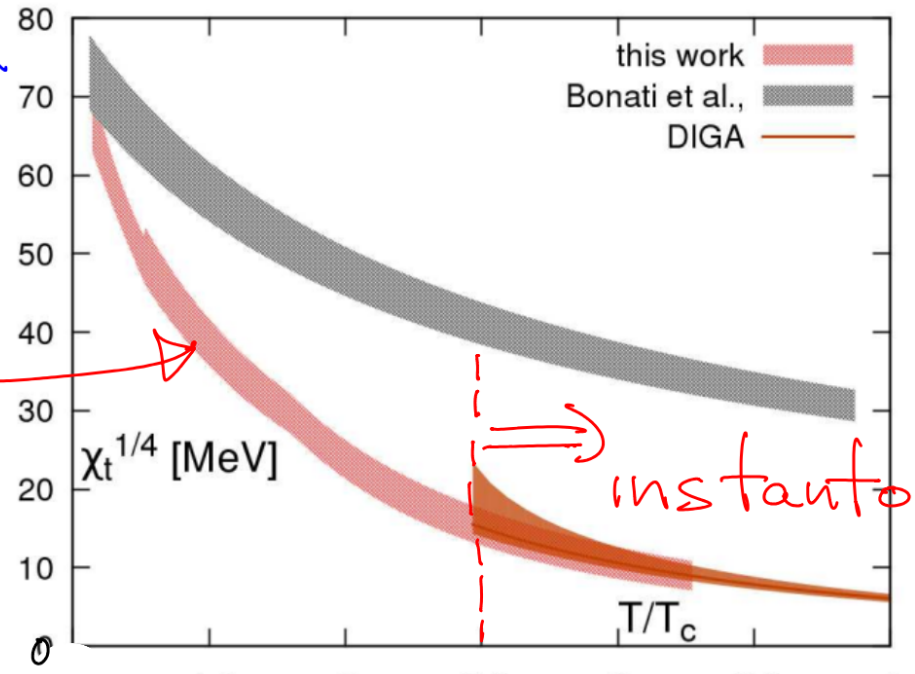
Bonati + ...

D'Angelo + ...

Nogradi + ...

$\chi^{1/4} \uparrow$

NOT instantons



instantons

Petreczky + ...
1606.03145

T_χ

$T/T_\chi \rightarrow$

Instantons @ large N

Veneziano '79, Witten '79

As $N \to \infty$, hold $g^2 N$ fixed as $N \to \infty$

$$\chi_2 \sim e^{-8\pi^2/g^2} \sim e^{\left(-8\pi^2/g^2 N\right) N}$$

Even so, argued $\chi_2 \sim 1$ @ $T=0$, $N=\infty$

Instanton liquid?

At $N=\infty$, instantons of <u>one</u> scale will dominate

$$S(g^2) = 8\pi^2 N \left(\frac{1}{g^2 N} + n \ln g^2 N + O(g^2 N) \right)$$

1 loop 2 loop

$$\left.\frac{\partial S}{\partial g^2}\right|_{g_*^2} = 0 \Rightarrow g_*^2 N \sim \frac{1}{n}$$

But $S(g_*^2) \sim N \neq 0$

Instantons of one size dominate as $N \to \infty$, but <u>need</u> the action $\sim N = 0$!

N.B. - Liu, Shuryak, Zahed 1802.00540

"Fractional" instantons

IF there are objects $\bar{c}$ $Q_{top} \sim 1/N$, then no problem!

$$\chi_2 \sim e^{-8\pi^2/(g^2 N)} \sim 1 \quad \text{as} \quad N \to \infty$$

't Hooft '80, van Baal '82, Sedlacek CMP 86 '82

$Q_{top} \sim 1/N$ $\bar{c}$ $Z(N)$ twisted boundary conditions

manifestly finite volume

only analytic soln's for finite box

$\bar{c}$ sizes in certain ratio, etc.

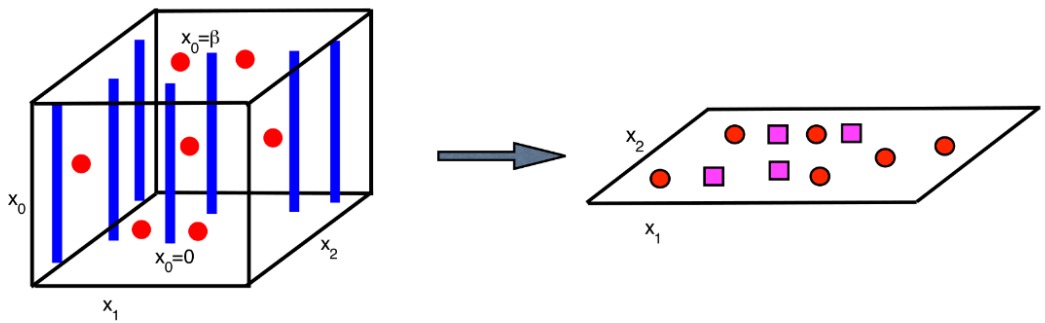
Do they persist in infinite volume?

Femto-slab

Unsal, Poppitz, Anber ....

Take <u>one</u> spatial dimension, $L \ll \Lambda_{QCD}^{-1}$, so $g^2(L\Lambda_{QCD}) \ll 1$. Using results in 2+1 dim.'s, over large distances, confining thy $\bar{c}$ "monopole-instantons", $Q_{top} \sim 1/N$

Poppitz, 2111.10423



Size of monopole-inst.'s $\sim L$

What happens as $L \sim \Lambda_{QCD}^{-1}$?

Also: Andrew Cox
Yuya Tanizaki

Punch line

Size of monopole-instanton gets stuck at $L \sim \Lambda_{QCD}^{-1}$

Dominant configurations $\approx$ one size

Measurable on lattice $\bar{c}$ <u>adjoint</u> (not fund.) quark prop. as <u>external</u> probe

<u>not</u> dynamical

CP^{N-1} model

In 1+1 dim.'s: N component field z^i, $\bar{z}^i z^i = 1$, inv.:

$$z^i(x) \to e^{i\alpha(x)} z^i(x)$$

local $U(1)$

$$\mathcal{L} = \frac{1}{g^2}\int d^2x \, |D_\mu z^i|^2 \qquad D_\mu = \partial_\mu - iA_\mu$$

Global sym.: $z^i \to U^{ij} z^j \Rightarrow SU(N)$

But: if $U = \omega_1 = e^{2\pi i/N}$, $z^i \to \underbrace{e^{2\pi i/N}}_{\text{part of } U(1)} z^i$

$\Rightarrow$ global sym. $SU(N)/Z(N)$

g^2 asymptotically free, soluble as $N \to \infty$

Witten '79 d'Adda, Lüscher, Di Vecchia '78

CP^{N-1} inst.'s

Topological chg. $$Q_{top} = \frac{1}{2\pi}\int d^2x\, \varepsilon^{\mu\nu}\partial_\mu A_\nu$$

<u>All</u> classical soln's known; self-dual, $$z^i \sim \frac{(x+iy)\, v^i}{\sqrt{x^2+y^2+\rho^2}}$$

$$D_\mu z = \varepsilon^{\mu\nu} D_\nu z$$

Can compute 1-loop fluc's about all instantons

$$S_{top} \sim \frac{1}{g^2} \sim \Big(\underbrace{\frac{1}{g^2 N}}_{\text{fixed}}\Big) N \Rightarrow e^{-S_{top}} \sim e^{-\# N}$$

But: large N soln. shows that

$$\chi \sim \langle Q_{top}^2 \rangle \sim \frac{1}{N} \quad \underline{\text{not}} \quad e^{-N} \; !$$

Frac. inst.'s in CP^{N-1}

Consider

$$z^1(r,\theta) = e^{i\varphi/N} h(r) \qquad z^{2\ldots} = 0$$

Not single-valued:

$$z^1(r,2\pi) = e^{2\pi i/N} h(r) \sim z^1(r,0)$$

by $Z(N)$ sym. Then the corresponding

$$A_\varphi \sim \frac{1}{rN}, \quad r \to \infty \quad \Rightarrow \quad Q_{top} = \frac{1}{N}$$

To obtain frac. Q, require multi-valued soln.'s allowed by $Z(N)$

CP^N @ $N \to \infty$

Introduce a constraint field $\lambda(|z|^2 - 1)$, integrate out z's

$$S_{eff} = N \, \mathrm{tr} \ln(-D_\mu^2 + i\lambda) - i \int \frac{\lambda}{g^2} d^2x$$

Vacuum: $i\lambda = m^2$ (dim. trans.), $A_\mu = 0$

Frac. inst.: action non-local, only limiting behavior

But: scale sym. of classical action lost

frac. inst. has <u>one</u> size $\sim 1/m$ = confinement distance

SU(N) gauge, NO quarks

Back to 3+1 dim.'s, <u>no</u> quarks, $A_0 = 0$ gauge

Parametrize gauge field as function of arbitrary parameter ξ,

$$A_i(\bar{x}, \xi) = (1-\xi) A_i(\bar{x}) + \xi A_i^{\Omega}(\bar{x})$$

gauge transf. of A_i

If $\Omega \to 1$ @ $\bar{x} \to \infty$,

$$Q_{top} = \frac{1}{8\pi^2} \int \mathrm{tr}\, G_{\mu\nu} \tilde{G}^{\mu\nu} = \frac{1}{24\pi^2} \int d^3x \; \mathrm{tr} \left(\Omega^{\dagger} \partial_i \Omega \right)^3$$

$$= \text{integer}$$

Above isn't soln. $\bar{c}$ minimal action, but gets Q_{top} right

$Z(N)$ vacua

But alternately, we can choose $\Omega_\infty = \omega_j = e^{2\pi i j/N}\mathbb{1}$

$$\Omega_\infty(\xi) = e^{i\chi^j \xi} \qquad \chi^j = \frac{2\pi j}{N} t_N, \quad t_N = \begin{pmatrix} \mathbb{1}_{N-1} & \\ & -(N-1) \end{pmatrix}$$

For finite $r = \sqrt{x^2+y^2+z^2}$, need more involved ansatz

$$Y_{ij} = \frac{\sigma\cdot\hat{x}}{2} + \frac{1}{N} - \frac{1}{2} \quad i,j=1,2 \qquad Y_{ij} = \frac{\delta^{ij}}{N} \quad i,j=3\ldots N$$

$$\Omega(\bar{x},\xi) = e^{iY\theta(r,\xi)}$$

$\theta = 0$ @ $r=0$, $\xi=0$

$= 2\pi\xi$ @ $r=\infty$

Above illustrative, <u>not</u> minimal action

Topological Chg.

$$G_{\mu\nu} \to \Omega^{\dagger} G \Omega + \frac{\partial}{\partial \xi} A^{\Omega} = \Omega^{\dagger} (G - Da) \Omega$$

Variation in ξ like "time" $\quad a = \frac{d\Omega}{d\xi} \Omega^{-1}$

"a" analogous to A_0

$$Q_{top} = \frac{1}{8\pi^2} \int tr(G - Da)^2 \, d^4x$$

$$= \frac{1}{4\pi^2} \int_{x=\infty} d^2 S^i \int dx \, tr(a B^i)$$

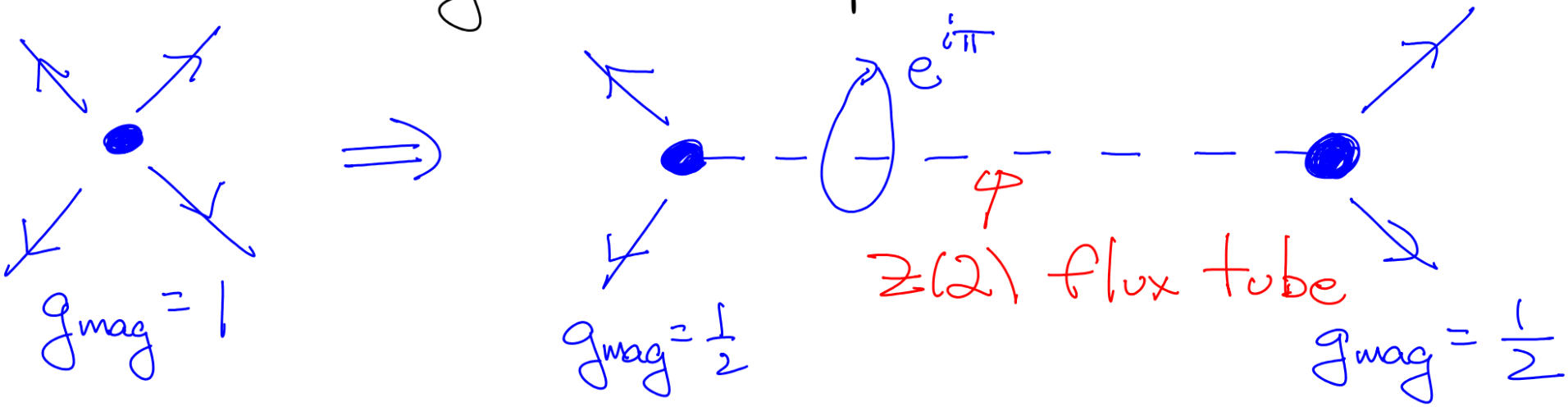
<u>Need</u> magnetic chg. For SU(2):

$$G_{ij} \sim -\frac{i}{2} \vec{\sigma} \cdot \hat{x} \, \varepsilon_{ijk} \frac{\hat{x}^k}{r^2} \Rightarrow Q_{top} = 1$$

So?

"Split" Z_2 monopole

For a SU(2) magnetic monopole



Without quarks, Z(2) flux tube invisible

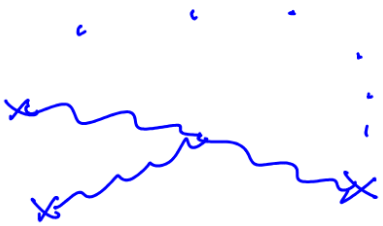
Each end has $Q_{top} = +\frac{1}{2}$.

To observe each end seperately, do need Z(2) twisted boundary conditions

Also: von Smekal

Vacuum of frac. inst.'s

For $SU(N)$, instanton $\bar{c}$ $Q_{top} = 1 = N$ frac.'s



Natural length scale of $Z(N)$ flux tubes $\sim \Lambda_{QCD}^{-1}$

Manifestly <u>non</u>-pert.

Unsal, 2007.03880 - on femto-slab,

θ-dependence $= f(\theta/N)$, not $f(\theta)$.

Presumably carries over

$Z(N)$ dyon

A explicit construction, @ $T > T_{deconf}$. Now $A_0 \neq 0$

$$A_0^\infty = \frac{2\pi T}{g N} k, \qquad k = t_N = \begin{pmatrix} \mathbb{1}_{N-1} & \\ & -(N-1) \end{pmatrix}$$

$$\mathbb{L} = e^{ig\int_0^{1/T} A_0 dx} = e^{\frac{2\pi i}{N} k} \qquad \text{or} \qquad t_N' = \begin{pmatrix} -(N-1) & \\ & \mathbb{1}_{N-1} \end{pmatrix}$$

k = nontrivial holonomy. Above value is a minimum of the holonomous potential

$$A_0 = \frac{2\pi T}{g N} g k \Rightarrow V_{holonomous} \sim T^4 g^2 (1-g)^2$$

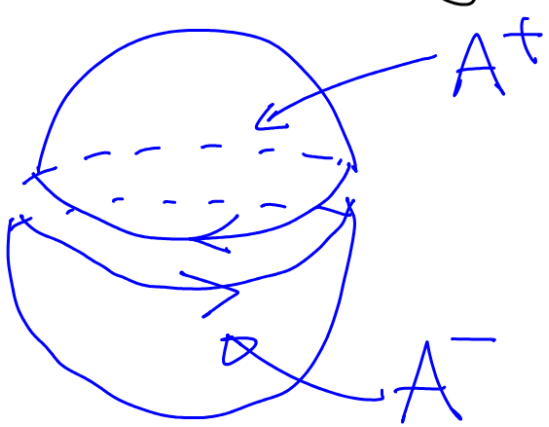
$|g|$ mod 1

$Z(N)$ monopole

At spatial ∞,

$$A_\varphi^{\pm} = \frac{1}{N\,2r}\, m\, \frac{(\pm 1 - \cos\theta)}{\sin\theta} = \frac{1}{N} * \text{Dirac monopole}$$

m = magnetic chg, $\sim k_1$ or k_2



$$e^{i\oint A^+}\, e^{-i\oint A^-} = e^{\frac{2\pi i}{N} m}$$

Above A_φ at spatial ∞. For A_0

$$A_0(r) \underset{r\to\infty}{\sim} \frac{2\pi T}{N} k - \frac{1}{2Nr} m + \cdots,$$

$Z(N)$ dyon

Assume there is a regular solution for all r, esp. $r=0$. For simplicity, take it to be static

$$Q_{top} = \frac{1}{4\pi^2}\int \partial_i \, \mathrm{tr}\, A_0 B_i = \frac{1}{N^2}\,\mathrm{tr}(mk)$$

$m=k$: $Q_{top} = \frac{N-1}{N}$ $\qquad$ $m \neq k$: $Q_{top} = \frac{1}{N}$

Identical to 't Hooft, but $Z(N)$ twist not in a box, but in radial direction

Dyons vs calorons

Lee & Lu: th/9802108
Kraan & van Baal: th/9805168 } KvBLL

Also:
Sharma
Faber

Show instanton @ $T \neq 0$
$= N$ constituents $\bar{c}$ $Q_{top} = 1/N$

	KvBLL	$Z(N)$ dyon
magnetic chg	integer	$1/N$
$V_{hol}(q)$	maximum $\sim 1/N$	minimum $\Rightarrow$ integer

But max. of $V_{hol.}$ $\Rightarrow$ @ 1-loop, free energy
$\sim$ + volume of space!

$Z(N)$ dyons

$T > T_d$: elec. chg. <u>un</u>confined, but mag. chg. <u>confined</u>

$\Rightarrow$ $Z(N)$ dyons only relevant for $T: T_d \to T_{inst}$

Instantons dominate $T > T_{inst}$.

$T < T_d$: $Z(N)$ dyons do <u>not</u> propagate straight in time

Equivalent to vortices? (Also: Leinweber)

Tangled over size $\sim \Lambda_{QCD}$

Entropy of config.'s $\bar{c}$ $Q_{top} = \pm 1/N \ldots > Q_{top} = \pm 1 \ldots$

Lattice-pure glue

Edwards, Heller, Narayanan lat/9806011

To measure $Q_{top} \sim 1/N$, use χ-symmetric gf. prop. in adjoint, <u>not</u> fund., representation

Fund. rep.: 2 zero modes for $Q_{top} = 1$

Adj. rep.: $2N$ " " " "

2 zero modes for $Q_{top} = 1/N$

From eigenvector, estimate size

If $Q_{top} \sim 1/N$, are they dilute or densely packed?

probably

Lattice simulations

Fodor + ... 0905.3586 - SU(3) $\bar{c}$ sextet rep.

No evidence for frac. Q_{top}

But sextet only sensitive to $Q_{top} = \frac{1}{5}$, not $\frac{1}{3}$

S. Sharma: $N_c = 2$, near T_{deconf}

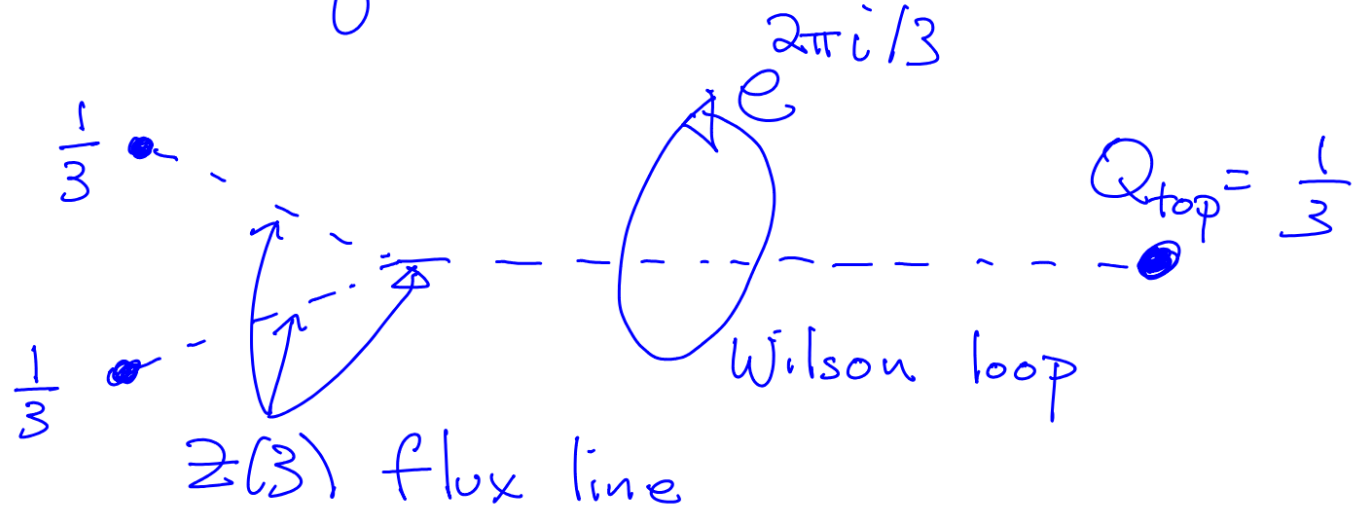
N. Karthik & R. Narayanan: large N_c

} in progress

No need for cooling.

+ Dynamical quarks

"Split" monopoles in $SU(3)$:



$Z(3)$ flux tube:
invisible to gluons, <u>not</u> to fund. qks

$\Rightarrow$ Dynamical quarks <u>confine</u> objects $\bar{c}$ $Q_{top} = \pm \frac{1}{N}$

Complicated interactions between qks & dyons

$SU(3)$: $T \neq 0$, $\mu = 0$

Lattice : c/o qks, $T_{deconf} \sim 270$ MeV

2+1 flavors, $T_\chi \sim 155$ "

Three regimes ?

$T > 300$: instanton dominated
($Z(3)$ dyons confined)

$300 > T > T_\chi$: $Z(3)$ dyons + $\approx$ massless qks

$T_\chi > T$: $Z(3)$ dyons + massive qks

Quanitative tests of <u>dynamical</u> qks + dyons ?

Finite μ vs finite T

Instantons have color $\vec{E}$ & $\vec{B}$.

Suppressed by Debye mass:

$$m_{Debye}^2 = g^2\left(\underbrace{\frac{1}{3}\left(N_c + \frac{N_f}{2}\right)T^2}_{\text{big}} + \underbrace{2N_f\left(\frac{\mu_{qk}}{2\pi}\right)^2}_{\text{small}}\right)$$

Because of # d.o.f., Debye suppression is <u>much</u> larger @ $T \neq 0$ than $T = 0$, $\mu \neq 0$

Three regimes @ $\mu \neq 0$, $T = 0$

RDP & Rennecke: 1910.14052 –

instantons @ $T \sim 150$ MeV $\approx$ $\mu_{qk} \sim 2$ GeV!

<u>deep</u> in perturbative regime

(Also: Kurkela)

$T = 0$:

$\mu > \mu_{qk} > 2$ GeV – instantons

$\mu_{qk} > \mu > \mu_\chi$ – $Z(N)$ dyons + massless qks

$\mu_\chi > \mu > 313$ MeV – " + massive qks

χ transition

Summary

For pure gauge, two sources of fluc's in topological charge

instantons in weak coupling – <u>all</u> sizes

$Z(N)$ dyons " strong " – <u>one</u> size

confine?

With quarks: <u>much</u> more complicated

$\mu = 0$, $T > 300$ MeV – dyons confined into inst's

$T <$ " " – dyons & quarks int'g

$T = 0$, $\mu \neq 0$: dyons & qks relevant for <u>all</u> densities in neutron stars

instantons never matter

Polyakov loop in real time

Go from $\mathcal{L} \to \mathcal{H}$amiltonian RDP 2202, 11122

$$\left\langle \mathrm{tr}\, e^{ig\int A_0(x)\,dx} \right\rangle \Longrightarrow$$

$$\sum \langle A_i | \int \mathcal{D}\chi \; e^{-\frac{1}{T}\int \mathcal{H} + ig\,\mathrm{tr}\,\chi D\cdot E} \; \mathrm{tr}\, e^{+ig\chi(x)} | A_i \rangle$$

A_0 in $\mathcal{L} \Rightarrow$ constraint field χ in $\mathcal{H}$ to impose Gauss' law

At non-zero holonomy $\Rightarrow$ $\exp(ig A_0^{cl}/T) = \Omega_\infty$

$$Z(\Omega_\infty) = \sum \langle A_i | \underbrace{e^{-\mathcal{H}/T}\, \delta(D\cdot E)}_{} | A_i^{\Omega_\infty} \rangle$$

$$A_i^{\Omega_\infty} \to \Omega_\infty^\dagger A_i \Omega_\infty$$

= "twisted" partition function