Effective quarks and gluons in QCD

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Juan José Gálvez Viruet (U. Granada)

XVth Quark Confinement and the Hadron Spectrum

Motivation

1. Hadrons in terms of quarks and gluons and their dynamics

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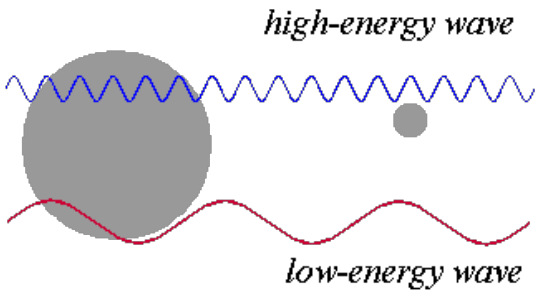
2. QCD at different scales: from asymptotic freedom to confinement



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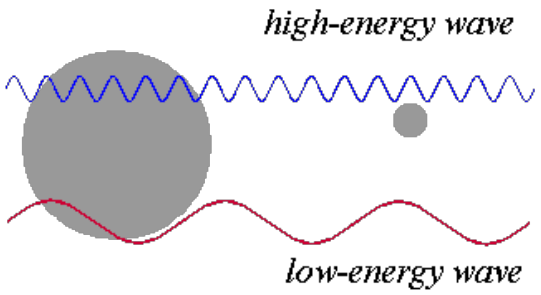

3. Hamiltonian of QCD is complicated
   → It needs regularization and has counterterms
   → How to deal with an infinite number of particles??
   → Starting from QCD, derive an effective Hamiltonian

Hadronic states in QCD

Every state in QCD is a superposition of infinitely many Fock components

$$|\Psi_{\text{meson}}\rangle = c_1|q\bar{q}\rangle + c_2|q\bar{q}g\rangle + c_3|q\bar{q}gg\rangle + c_4|q\bar{q}q\bar{q}gg\rangle + \dots$$

$$P_{|q\bar{q}\rangle} = |\langle q\bar{q}|\Psi_{\text{meson}}\rangle|^2 = c_1^2$$

Masses are given by the eigenvalues of Hamiltonian operator acting on the Fock space

$$H_{QCD}|\Psi_{\text{meson}}\rangle = E_n|\Psi_{\text{meson}}\rangle$$

E_n are the energy levels of the system, i.e. **masses of hadrons**

Problem: In practice it is not feasible

The method of calculation

Renormalization Group Procedure for Effective Particles

The method of calculation

Originally Similarity Renormalization Group
[S.D. Głazek, K.G. Wilson, PRD49, PRD57]

Lagrangian density of QCD $\mathcal{L}_{\rm QCD} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2}{\rm tr}F^{\mu\nu}F_{\mu\nu}$

1. **Canonical Hamiltonian** Use front-form dynamics:
   - $\mathcal{L}_{\rm QCD} \to \mathcal{T}^{\mu\nu}_{\rm QCD} \to H_{\rm QCD} = \int_{{\rm x}^+=0} \mathcal{H}_{\rm QCD}({\rm x})d{\rm x}, \quad A^+ = 0$
     $k^+ = k^0 + k^3, \quad k^- = k^0 - k^3, \quad \vec{k}^\perp = (k^1, k^2); \quad x = k^+/P^+,$
2. **Regularization** Interaction vertices:
   - UV and small-x cutoff $\int dx d\kappa^\perp \to \int dk^+ d\kappa^\perp r_\delta(x) r_\Delta(\kappa^\perp)$
     $\lim_{\delta\to 0} r_\delta(x) = 1, \ \lim_{\Delta\to\infty} r_\Delta(x) = 1$

     See poster by J.J. Gálvez Viruet for reg. with gluon mass
3. **Renormalization**
   Scale parameter λ introduced by RGPEP equation

$$H_0 \quad \to \quad H_\lambda$$

The method of calculation: RGPEP

Renormalization group procedure for effective particles

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Change of basis through a similarity transformation

$$H_\lambda \;=\; U_\lambda H_0 U_\lambda^\dagger \,, \qquad U_\lambda|\psi_0\rangle \;=\; |\psi_\lambda\rangle \,, \quad H_0|\psi_0\rangle = H_\lambda|\psi_\lambda\rangle = E|\psi_0\rangle$$

U_λ depends on an *energy-scale* parameter λ preserves the eigenvalues
H_λ satisfies

$$\frac{dH_\lambda}{d\lambda^{-4}} = [G_\lambda, H_\lambda], \text{ where } G_\lambda \text{ is a generator}$$

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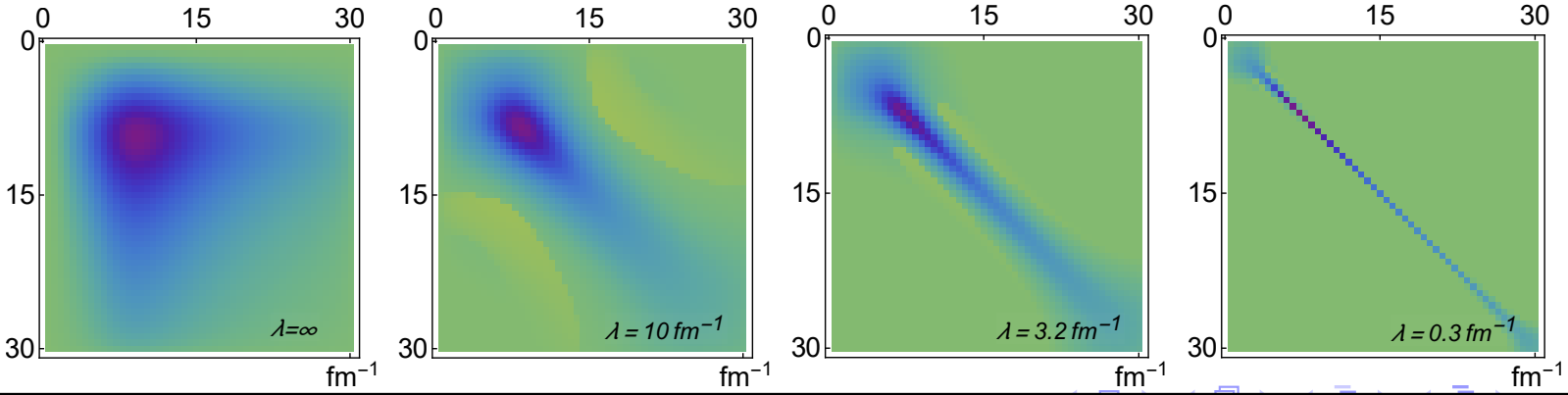
Example: [Gomez-Rocha, Arriola, PLB 800 (2020), APP Sup.14 (2021)]

Evolution of a model potential $V(p,p')$



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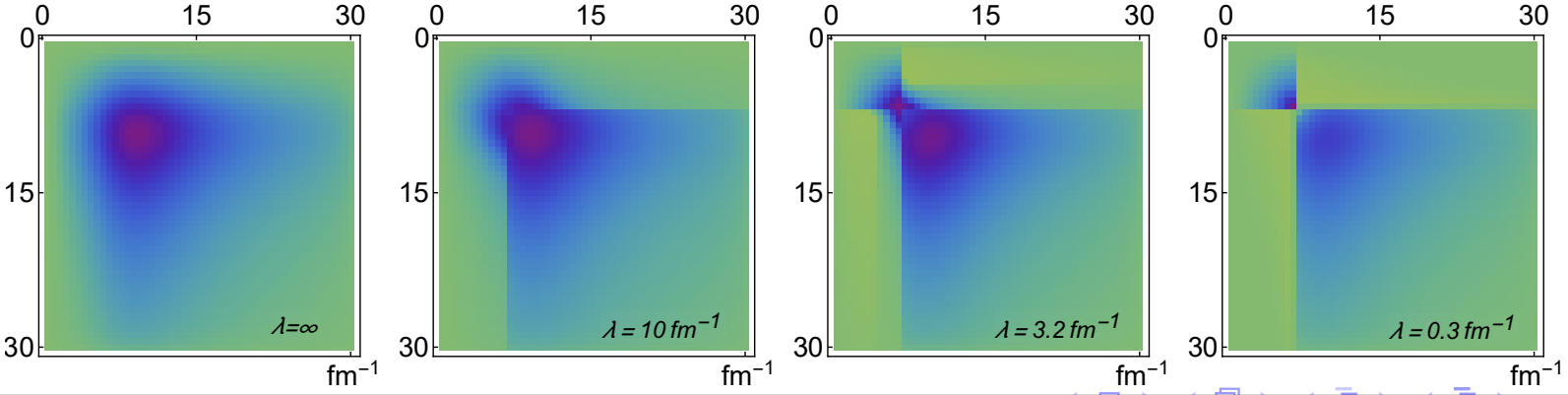


The method of calculation

Effective quanta

Creation and annihilation operators become *effective*

$$a^\dagger|0\rangle = |g\rangle \quad \rightarrow \quad a^\dagger_\lambda|0\rangle = |g_\lambda\rangle \qquad a^\dagger_\lambda = \mathcal{U}_\lambda\, a^\dagger\, \mathcal{U}^\dagger_\lambda$$

They create/annihilate particles of type λ or of *size* $s = 1/\lambda$
$[\lambda] =$ energy $\sim$ scale
$[s = 1/\lambda] =$ length $\sim$ size

Hadrons in terms effective particles

Hadrons can be described in terms of effective particles of type λ

$$|\Psi_{\text{meson}\,\lambda}\rangle = c_{1,\lambda}|q\bar{q}\rangle_\lambda + c_{2,\lambda}|q\bar{q}g\rangle_\lambda + c_{3,\lambda}|q\bar{q}gg\rangle_\lambda + c_{4,\lambda}|q\bar{q}q\bar{q}gg\rangle_\lambda + \ldots$$

and describe from asymptotic freedom to bound states

RGPEP

Solve the RGPEP equation perturbatively

$$\frac{dH_\lambda}{d\lambda^{-4}} = [[H_f, H_{P\lambda}], H_\lambda] \qquad a_\lambda = \mathcal{U}_\lambda \, a \, \mathcal{U}_\lambda^\dagger$$

perturbatively, order by order

$$H_\lambda = H_f + gH_{1,\lambda} + g^2 H_{2,\lambda} + g^3 H_{3,\lambda} + ...$$

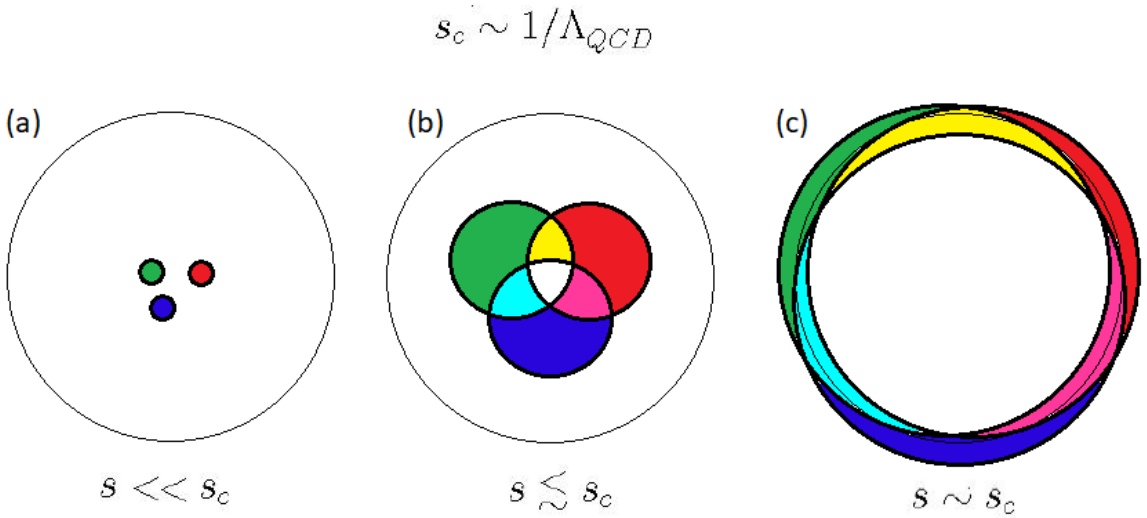
$$\begin{aligned}
\mathcal{H}'_f &= 0 \,, \\
g\mathcal{H}'_{\lambda\,1} &= [[\mathcal{H}_f, g\mathcal{H}_{1P\lambda}], \mathcal{H}_f] \,, \\
g^2\mathcal{H}'_{\lambda\,2} &= \left[\left[\mathcal{H}_f, g^2\mathcal{H}_{2P\lambda}\right], \mathcal{H}_f\right] + [[\mathcal{H}_f, g\mathcal{H}_{1P\lambda}], g\mathcal{H}_{1\lambda}] \,, \\
g^3\mathcal{H}'_{\lambda\,3} &= \left[\left[\mathcal{H}_f, g^3\mathcal{H}_{3P\lambda}\right], \mathcal{H}_f\right] + \left[\left[\mathcal{H}_f, g^2\mathcal{H}_{2P\lambda}\right], g\mathcal{H}_{1\lambda}\right] + \left[[\mathcal{H}_f, g\mathcal{H}_{1Pt\lambda}], g^2\mathcal{H}_{2\lambda}\right] \,, \\
&\vdots
\end{aligned}$$

→ Integration produces functions with *form factors*

$$e^{-(\mathcal{M}_a^2 - \mathcal{M}_b^2)^2/\lambda^4}$$

The concept of effective particles

Effective particles of type λ can change their relative motion kinetic energy through a single effective interaction by no more than about λ



$s = 1/\lambda$
$f_t = e^{-(\mathcal{M}_1^2 - \mathcal{M}_2^2)^2/\lambda^4}$

Figure adapted from Patryk Kubiczek

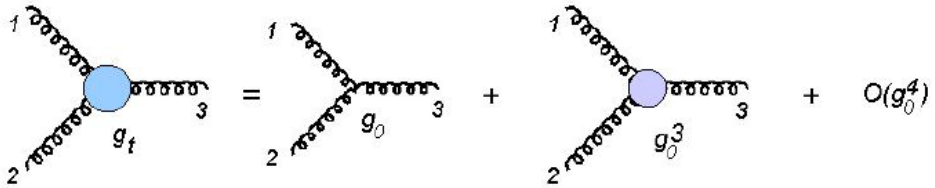
Asymptotic freedom

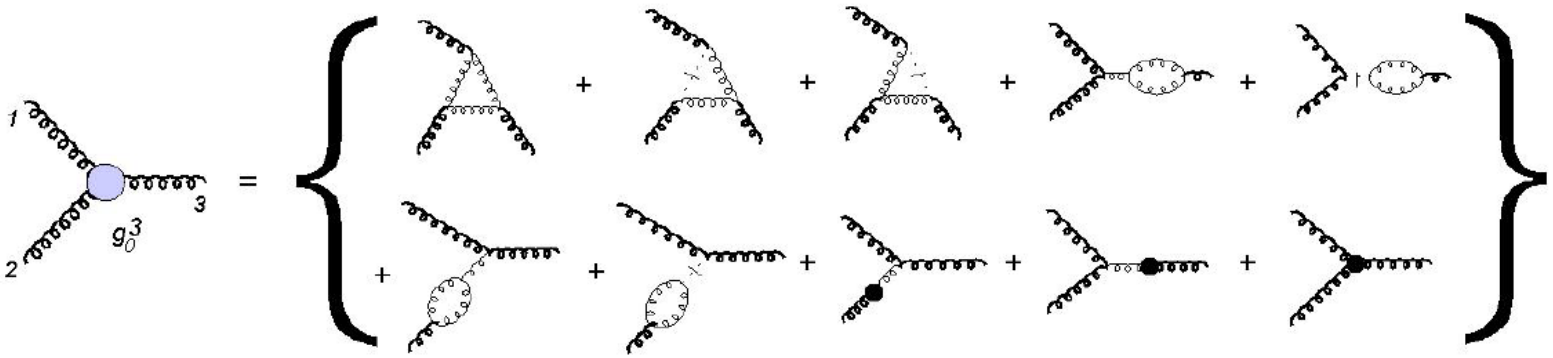
Example of third-order calculation

Example of 3rd-order calculation:

→ **The three-gluon vertex:**

$$Y_\lambda \quad = \quad gH_{1\lambda} + g^3 H_{3\lambda}$$





$$Y_\lambda \quad = \quad \sum_{123} \int [123]\, \tilde{Y}_\lambda(k_1, k_2, k_3, \sigma)\, a^\dagger_{1,\lambda} a^\dagger_{2,\lambda} a_{3,\lambda} + H.c.$$

→ **We obtain the running coupling with the correct AF behavior:**

$$\Rightarrow g_\lambda \quad = \quad g_0 - \frac{g_0^3}{48\pi^2} N_c\, 11\, \ln \frac{\lambda}{\lambda_0} \ , \quad \text{[MGR, Głazek, PRD 92]}$$

→ BUT: finite dependence on regularization

Example of 3rd-order calculation:

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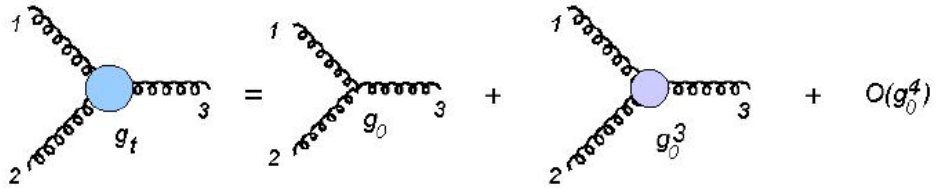

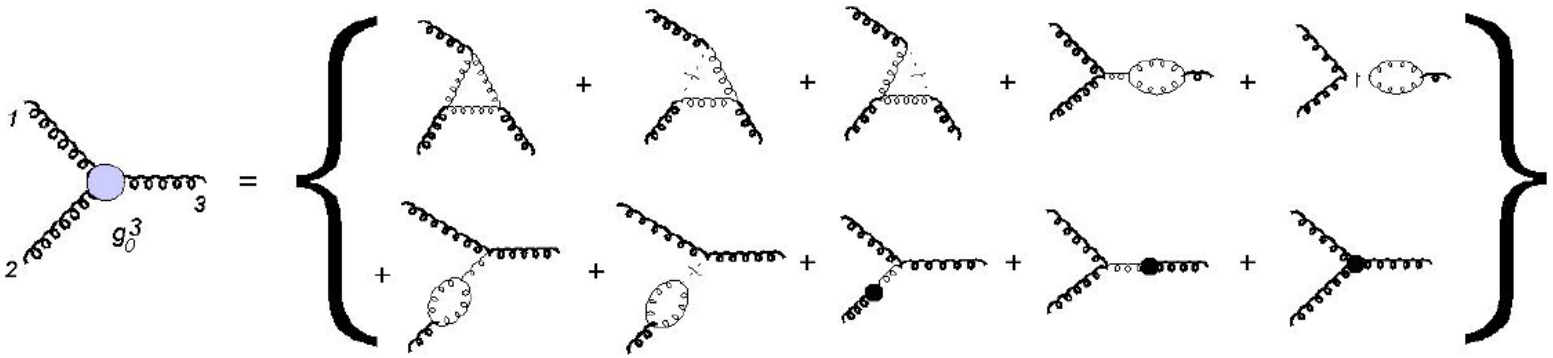

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→ BUT: finite dependence on regularization

See poster by J.J. Gálvez Viruet

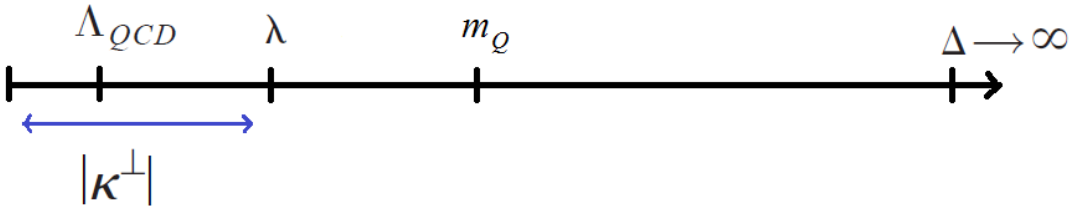
Bound states

Example of second-order calculation

Effective theory for heavy quarks

Heavy quarks make calculations much simpler
Heavy quarkonium is the simplest bound state

Assumptions



- ⋆ QCD with only quarks of heavy mass m_b (4.18 GeV/c^2), m_c (1.5 GeV/c^2)
- ⋆ No $Q\bar{Q}$ pair production (too heavy)
- ⋆ 2nd-order perturbative RGPEP:

$$H_{QCD\,\lambda} = H_f + gH_{1,\lambda} + g^2 H_{2,\lambda} \qquad |\Psi_\lambda\rangle = |Q_\lambda \bar{Q}_\lambda\rangle + |Q_\lambda \bar{Q}_\lambda\, g_\lambda\rangle$$

- ⋆ Non relativistic limit $k/m_Q \to 0$ simplifies the equations

Structure of the eigenvalue problem

Gluon-mass ansatz

$$\begin{bmatrix} \dots & \dots & \dots \\ \dots & H_f + g^2 H_{2,\lambda} & gH_{1,\lambda} \\ \dots & gH_{1,\lambda} & H_f + g^2 H_{2,\lambda} \end{bmatrix} \begin{bmatrix} \dots \\ |Q_\lambda \bar{Q}_\lambda G_\lambda\rangle \\ |Q_\lambda \bar{Q}_\lambda\rangle \end{bmatrix} = E \begin{bmatrix} \dots \\ |Q_\lambda \bar{Q}_\lambda G_\lambda\rangle \\ |Q_\lambda \bar{Q}_\lambda\rangle \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} H_f + g^2 H_2 + \mu^2 & gH_1 \\ gH_1 & H_f + g^2 H_2 \end{bmatrix} \begin{bmatrix} |Q\bar{Q}G\rangle \\ |Q\bar{Q}\rangle \end{bmatrix} = E \begin{bmatrix} |Q\bar{Q}G\rangle \\ |Q\bar{Q}\rangle \end{bmatrix}$$

Reduction to the $|Q_\lambda \bar{Q}_\lambda\rangle$ component

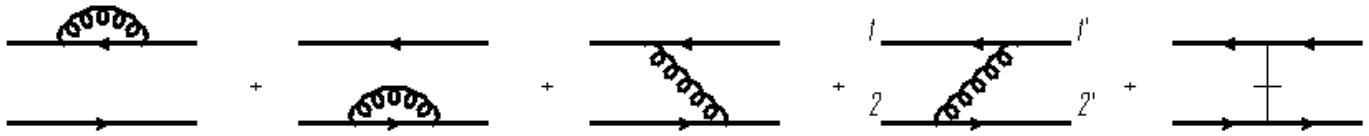
We follow [Wilson PRD 2 (1970) 1438]

$$H_{Q\bar{Q}\,\mathrm{eff},\lambda} |Q_\lambda \bar{Q}_\lambda\rangle = E |Q_\lambda \bar{Q}_\lambda\rangle$$

The effective eigenvalue equation

$$H_{Q\bar{Q}\,\mathrm{eff}\,\lambda}|Q_\lambda \bar{Q}_\lambda\rangle = E|Q_\lambda \bar{Q}_\lambda\rangle$$

$H_{\mathrm{eff}\,Q\bar{Q}}$ =



mass terms gluon exch. terms inst. int.

$$|Q_t \bar{Q}_t\rangle = \int [1'2']\, P^+\, \tilde{\delta}(P - k_1' - k_2')\, \psi_{1'2'}(\kappa^\perp_{1'2'}, x_{1'})\, \frac{\delta_{c_{1'}c_{2'}}}{\sqrt{3}}\, b^\dagger_{1',\lambda} d^\dagger_{2',\lambda}|0\rangle$$

The effective eigenvalue equation

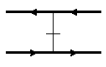
The eigenvalue equation in the NR limit is

$$\left[\frac{|\vec{k}_{12}|^2}{m} - B + \frac{\delta m_t^2}{2m} + \frac{\delta m_t^2}{2m}\right] \psi_{12}(\kappa_{12}^{\perp}, x_1) + \int \frac{d^3\vec{k}_{1'2'}}{(2\pi)^3} V_{Q\bar{Q}}(\vec{k}_{12} - \vec{k}_{1'2'})\, \psi_{1'2'}(\kappa_{1'2'}^{\perp}, x_{2'}) = 0$$

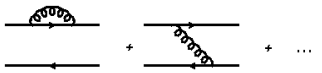
where

$$V_{Q\bar{Q}}(\vec{q}) \quad = \quad V_{C,BF}(\vec{q}) + W(\vec{q})$$

with



$$\Rightarrow \; V_{C,BF}(\vec{q}) \quad = \quad -\frac{4}{3}\frac{4\pi\alpha}{|\vec{q}|^2} \; + \; BF$$



$$\Rightarrow \; W(\vec{q}) \quad = \quad \frac{4}{3}\, 4\pi\alpha \left[\frac{1}{\vec{q}^2} - \frac{1}{q_z^2}\right] \frac{\mu^2}{\mu^2 + \vec{q}^2}\, e^{-2m^2 \frac{|\vec{q}^2|^2}{q_z^2\, \lambda^4}}$$

Remark: If $\mu^2 = 0$, $W = 0 \Rightarrow$ QED

The effective eigenvalue equation

Coulomb + Harmonic Oscillator

$$\left[\frac{\vec{k}^2}{m} - B\right]\psi(\vec{k}) + \int \frac{d^3q}{(2\pi)^3} V_{C,\,BF}(\vec{q})\,\psi(\vec{k}-\vec{q}) - \frac{4}{3}\frac{\alpha}{2\pi} b^{-3} \sum_i \tau_i \frac{\partial^2}{dk_i^2}\psi(\vec{k}) = 0$$

$$b = \frac{\sqrt{2m}}{\lambda_0^2}$$

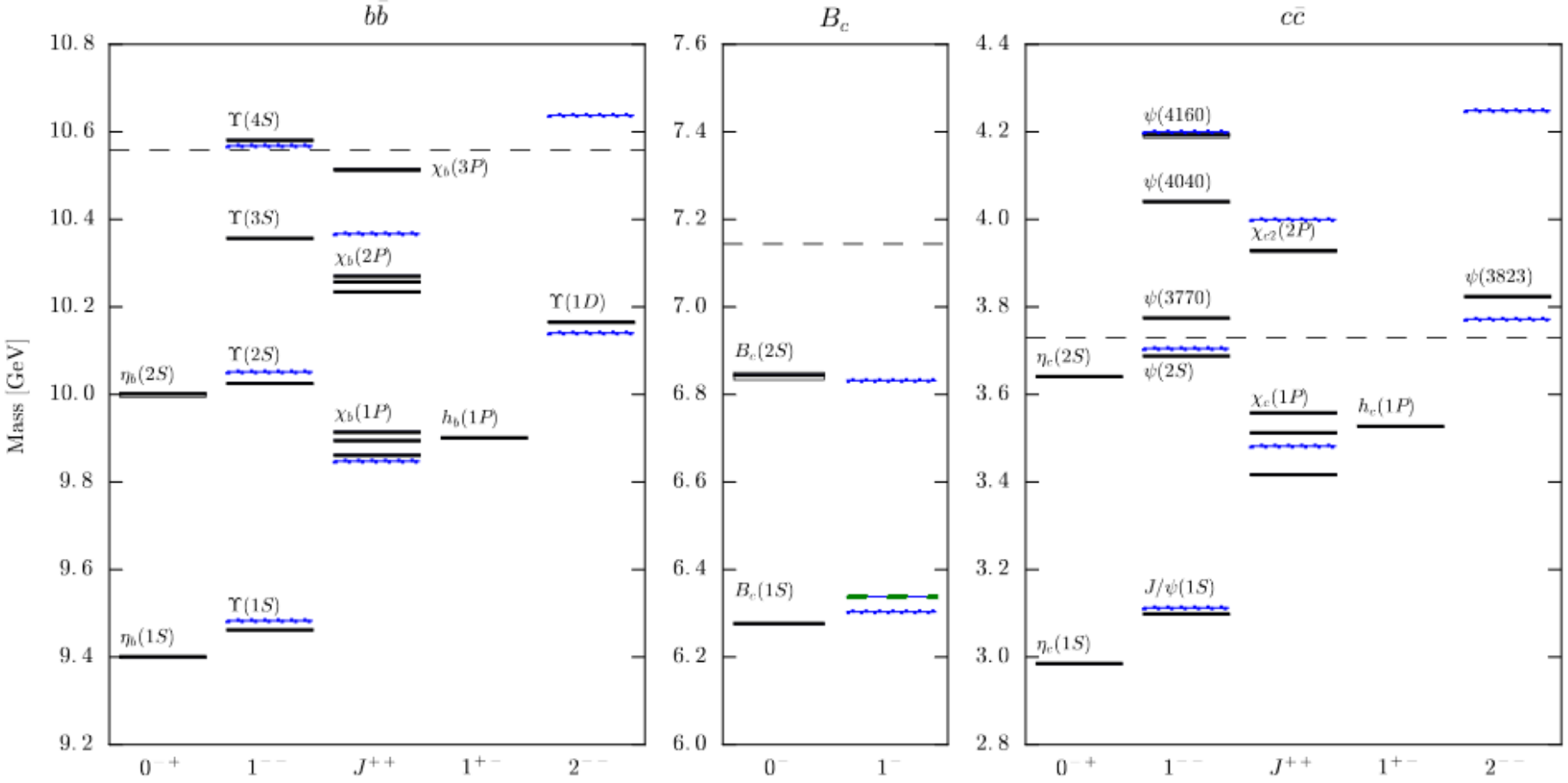
The term coming from the gluon-mass Ansatz yields an additional harmonic oscillator interaction

Position space

$$\left[2m - \frac{\Delta_{\vec{r}}}{m} - \frac{4}{3}\alpha\left(\frac{1}{r} + BF\right) + \frac{1}{2}\,\tilde{\kappa}\,r^2\right]\psi(\vec{r}) \quad = \quad (2m+B)\,\psi(\vec{r}) \;=\; M\,\psi(\vec{r})\;.$$

Analogous calculation for baryons presented in
[Serafin, Gomez-Rocha, More, Głazek, EPJ C78 (2018)]

Some numerical results: heavy mesons



[Serafin, Gomez-Rocha, More, Głazek, EPJ C78 (2018)]

Black: PDG masses, **Blue:** Our calculation

Green: average of many different approaches [MGR, Hilger, Krassnigg, PRD 93 (2016)]

Remarks

- ⋆ Eigenvalue equation for a single particle

$$H_{\mathrm{eff}Q}|Q\rangle = \infty|Q\rangle$$

$$H_{\mathrm{eff}\bar{Q}}|\bar{Q}\rangle = \infty|\bar{Q}\rangle$$

- ⋆ The divergence is canceled when the quarks are bound
  $\rightarrow$ result compatible with *confinement*

Remarks

The front-form (FF) eigenvalue equation

$$H_\lambda|\Psi_\lambda\rangle = P_\lambda^-|\Psi_\lambda\rangle = E|\Psi_\lambda\rangle$$

$$P_\lambda^-|\Psi_\lambda\rangle = \frac{M^2 + P^{\perp 2}}{P^+}|\Psi_\lambda\rangle \qquad \Rightarrow \quad (P_\lambda^- P^+ - P^{\perp 2})|\Psi_\lambda\rangle = M^2|\Psi_\lambda\rangle$$

Remark:

- ⋆ The eigenvalue is M^2 in FF instead of M in instant form (IF);
- ⋆ At large distances: $U_{\text{eff FF}} \approx V^2_{\text{eff IF}}$
  Linear potential in IF $\Rightarrow$ quadratic potential in FF

$$V_{IF}(r) \sim \sigma\, r \;\Rightarrow\; V_{FF}(r) \sim \sigma^2\, r^2$$

[Trawiński *et al.* PRD**90** (2014) 074017]

Summary and Conclusions

1. RGPEP is a Hamiltonian approach to QCD that connects phenomena at different energy regimes
2. $\mathcal{L}_{\mathrm{QCD}} \to \mathcal{H}_{\mathrm{QCD}} \to H_{\mathrm{QCD}} \to H_{\lambda} \to H_{Q\bar{Q}\mathrm{e}ff}$
3. Asymptotic freedom in 3-gluon vertex ✓
4. Effective potential for quarkonium <u>including effective gluon explicitly</u> $\Rightarrow H_{Q\bar{Q}\mathrm{eff}} = \text{Coulomb} + \text{harmonic oscillator}$
5. Even in this crude approximation $\to$ reasonable spectra

Ongoing applications of the RGPEP method

Spectrum of exotic states

- **Tetraquarks:** K. Serafin (Lanzhou Inst., China) *et al.*
  [Phys.Rev.D 105 (2022) 094028]
- **Hybrid mesons:** M. Gomez-Rocha (Granada U.) and collaborators.

Hadron Structure

- **Proton** Structure in High-Energy High-Multiplicity p-p Collisions
  S.D. Glazek (Warsaw U.) and P. Kubiczek (Jagiellonian U.)
  [Few Body Syst. 57 (2016) 7, 509-513]
- Some on **structure functions** for heavy hadrons:
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Thank you for your attention!

End

Appendix

RGPEP

Effective quanta

Hamiltonian can be re-written in terms of *effective quanta*

$$H(q) \quad = \quad H_\lambda(q_\lambda)$$

$$q^\dagger|0\rangle = |q\rangle \quad \rightarrow \quad q_\lambda^\dagger|0\rangle = |q_\lambda\rangle \qquad q_\lambda = \mathcal{U}_\lambda\, q\, \mathcal{U}_\lambda^\dagger$$

$s = size$
$\lambda = 1/s$ momentum scale

$$\mathcal{U}_t \;=\; T e^{-\int_0^t d\tau\, \mathcal{G}_\tau} \;, \quad \mathcal{G}_t = [H_f, H_{Pt}]$$

RGPEP equation

$$H'_\lambda \;=\; [[H_f, H_{P\lambda}], H_\lambda]$$

Initial condition

$$H_{\lambda=\infty}(= H_{s=0}) = H_{QCD}^{\text{canonical}} + \text{CT}_{\Delta\delta}$$

Counterterms $\text{CT}^{\Delta\,\delta}$ remove UV-cutoff $\Delta\delta$.

RGPEP

Solve the RGPEP equation perturbatively

$$H'_\lambda \quad = \quad [[H_f, H_{P\lambda}], H_\lambda] \qquad q_\lambda = \mathcal{U}_\lambda \, q \, \mathcal{U}_\lambda^\dagger$$

perturbatively, order by order

$$H_\lambda \quad = \quad H_f + gH_{1,\lambda} + g^2 H_{2,\lambda} + g^3 H_{3,\lambda} + g^4 H_{4,\lambda} + \ldots$$

$$\begin{aligned}
\mathcal{H}'_f &= 0 \,, \\
g\mathcal{H}'_{\lambda\,1} &= [[\mathcal{H}_f, g\mathcal{H}_{1P\lambda}], \mathcal{H}_f] \;, \\
g^2\mathcal{H}'_{\lambda\,2} &= \left[\left[\mathcal{H}_f, g^2\mathcal{H}_{2P\lambda}\right], \mathcal{H}_f\right] + [[\mathcal{H}_f, g\mathcal{H}_{1P\lambda}], g\mathcal{H}_{1\lambda}] \;, \\
g^3\mathcal{H}'_{\lambda\,3} &= \left[\left[\mathcal{H}_f, g^3\mathcal{H}_{3P\lambda}\right], \mathcal{H}_f\right] + \left[\left[\mathcal{H}_f, g^2\mathcal{H}_{2P\lambda}\right], g\mathcal{H}_{1\lambda}\right] + \left[\left[\mathcal{H}_f, g\mathcal{H}_{1Pt\lambda}\right], g^2\mathcal{H}_{2\lambda}\right] \;, \\
&\vdots
\end{aligned}$$

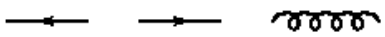
$\rightarrow$ Integration yields functions with *form factors*

$$e^{-(\mathcal{M}_a^2 - \mathcal{M}_b^2)^2/\lambda^4}$$

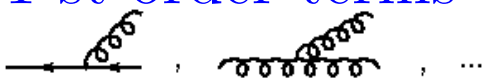
Renormalized Hamiltonian

Examples of terms in $H_{\lambda\,QCD} = H_f + gH_{1,\lambda} + g^2_{2,\lambda} + g^3 H_{3,\lambda} + g^4 H_{4,\lambda} + \ldots$

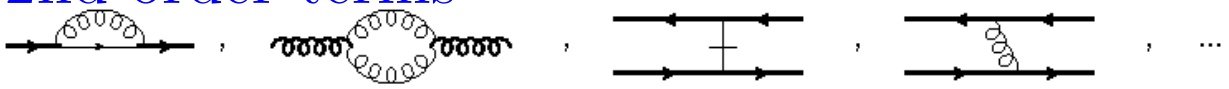
0-th order terms



1-st order terms

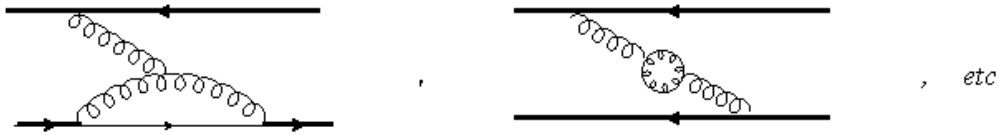


2nd-order terms



3rd-order terms

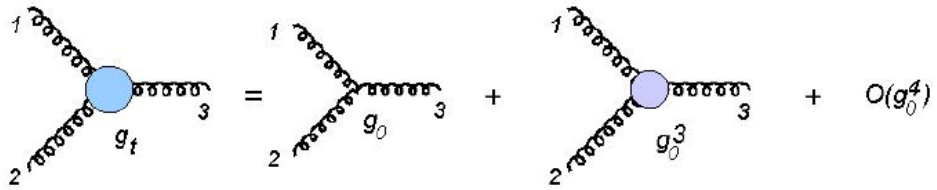
4th-order terms

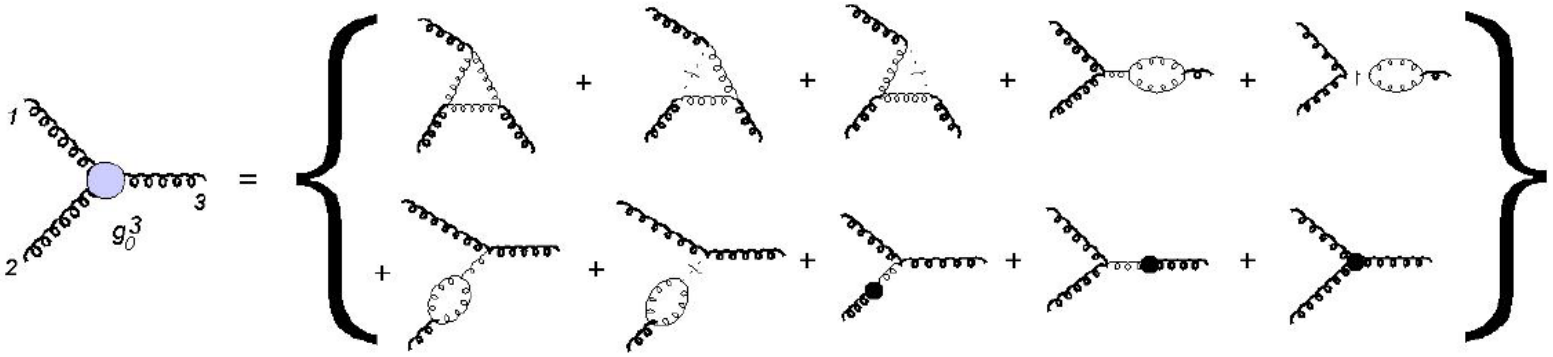


Example of 3rd-order calculation:

→ **The three-gluon vertex:**

$$Y_\lambda \quad = \quad gH_{1\lambda} + g^3 H_{3\lambda}$$





$$Y_\lambda \quad = \quad \sum_{123} \int [123]\, \tilde{Y}_\lambda(k_1, k_2, k_3, \sigma)\, a^\dagger_{1,\lambda} a^\dagger_{2,\lambda} a_{3,\lambda} + H.c.$$

→ **We obtain the running coupling with the correct AF behavior:**

$$\Rightarrow g_\lambda \quad = \quad g_0 - \frac{g_0^3}{48\pi^2} N_c\, 11\, \ln \frac{\lambda}{\lambda_0} \ ,$$

[MGR, Głazek, PRD 92]

Structure of the eigenvalue problem

The front-form (FF) eigenvalue equation

$$H_\lambda|\Psi_\lambda\rangle = P_\lambda^-|\Psi_\lambda\rangle = E|\Psi_\lambda\rangle$$

$$P_\lambda^-|\Psi_\lambda\rangle = \frac{M^2 + P^{\perp 2}}{P^+}|\Psi_\lambda\rangle \qquad \Rightarrow \quad (P_\lambda^- P^+ - P^{\perp 2})|\Psi_\lambda\rangle = M^2|\Psi_\lambda\rangle$$

Remark:

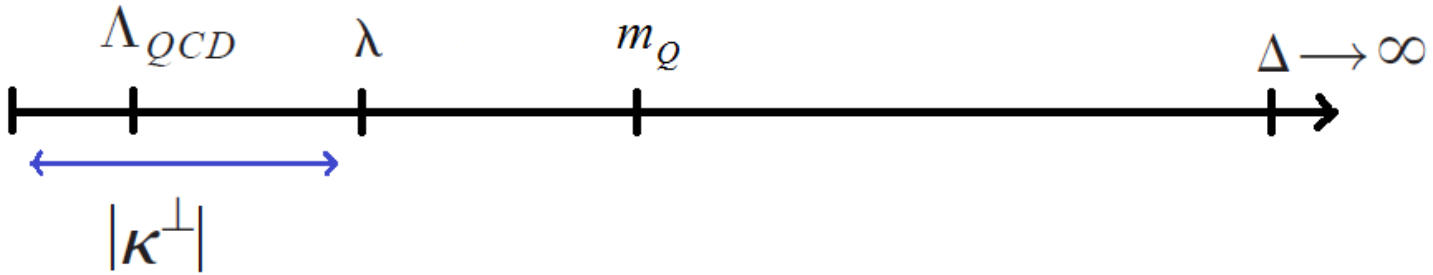
- ⋆ The eigenvalue is M^2 in FF instead of M in instant form (IF);
- ⋆ At large distances: $U_{\text{eff FF}} \approx V^2_{\text{eff IF}}$
  Linear potential in IF $\Rightarrow$ quadratic potential in FF

$$V_{IF}(r) \sim \sigma\, r \;\Rightarrow\; V_{FF}(r) \sim \sigma^2\, r^2$$

[Trawiński *et al.* PRD**90** (2014) 074017]

Effective theory for heavy quarks

Hierarchy of scales:



Assumptions

- ⋆ QCD with only one flavor
- ⋆ No $Q\bar{Q}$ pairs (too heavy)
- ⋆ 2nd-order perturbative RGPEP:

$$H_{QCD\,\lambda} = H_f + gH_{1,\lambda} + g^2 H_{2,\lambda} \qquad |\Psi_\lambda\rangle = |Q_\lambda \bar{Q}_\lambda\rangle + |Q_\lambda \bar{Q}_\lambda\, g_\lambda\rangle$$

- ⋆ A gluon-mass ansatz will account for non-Abelian terms

Structure of the eigenvalue problem

Gluon-mass ansatz

$$\begin{bmatrix} \dots & \dots & \dots \\ \dots & H_f + g^2 H_{2,\lambda} & gH_{1,\lambda} \\ \dots & gH_{1,\lambda} & H_f + g^2 H_{2,\lambda} \end{bmatrix} \begin{bmatrix} \dots \\ |Q_\lambda \bar{Q}_\lambda G_\lambda\rangle \\ |Q_\lambda \bar{Q}_\lambda\rangle \end{bmatrix} = E \begin{bmatrix} \dots \\ |Q_\lambda \bar{Q}_\lambda G_\lambda\rangle \\ |Q_\lambda \bar{Q}_\lambda\rangle \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} H_f + g^2 H_2 + \mu^2 & gH_1 \\ gH_1 & H_f + g^2 H_2 \end{bmatrix} \begin{bmatrix} |Q\bar{Q}G\rangle \\ |Q\bar{Q}\rangle \end{bmatrix} = E \begin{bmatrix} |Q\bar{Q}G\rangle \\ |Q\bar{Q}\rangle \end{bmatrix}$$

Reduction to the $|Q_\lambda \bar{Q}_\lambda\rangle$ component

We follow [Wilson PRD 2 (1970) 1438]

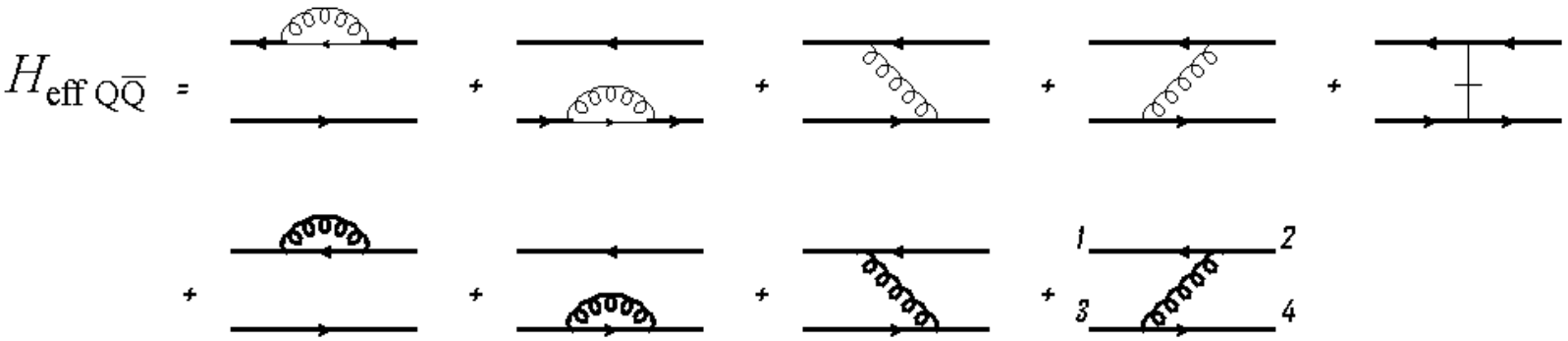
$$H_{Q\bar{Q}\,\mathrm{eff}\,\lambda} = H_f + g^2 H_{2,\lambda} + \frac{g^2}{2} H_{1,\lambda} \left(\frac{1}{E_l - H_f - \mu^2} + \frac{1}{E_{l'} - H_f - \mu^2} \right) H_{1,\lambda}$$

$$H_{Q\bar{Q}\,\mathrm{eff},\lambda} |Q_\lambda \bar{Q}_\lambda\rangle = E |Q_\lambda \bar{Q}_\lambda\rangle$$

The effective eigenvalue equation

$$H_{Q\bar{Q}\,\mathrm{eff}\,\lambda}|Q_\lambda \bar{Q}_\lambda\rangle = E|Q_\lambda \bar{Q}_\lambda\rangle$$

$$H_{Q\bar{Q}\,\mathrm{eff}\,\lambda} = H_f + g^2 H_{2,\lambda} + \frac{g^2}{2} H_{1,\lambda}\left(\frac{1}{E_l - H_f - \mu^2} + \frac{1}{E_{l'} - H_f - \mu^2}\right) H_{1,\lambda}$$



mass terms gluon exch. terms inst. int.

$$|Q_t \bar{Q}_t\rangle = \int [24]\, P^+\, \tilde{\delta}(P - k_2 - k_4)\, \psi_{24}(\kappa_{24}^{\perp}, x_2)\, \frac{\delta_{c_2 c_4}}{\sqrt{3}}\, b_{2,\lambda}^{\dagger} d_{4,\lambda}^{\dagger}|0\rangle$$

The effective eigenvalue equation

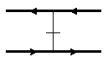
The eigenvalue equation in the NR limit is

$$\left[\frac{|\vec{k}_{13}|^2}{m} - B + \frac{\delta m_t^2}{2m} + \frac{\delta m_t^2}{2m}\right] \psi_{13}(\kappa_{13}^{\perp}, x_1) + \int \frac{d^3\vec{k}_{24}}{(2\pi)^3} V_{Q\bar{Q}}(\vec{k}_{13} - \vec{k}_{24})\, \psi_{24}(\kappa_{24}^{\perp}, x_2) = 0$$

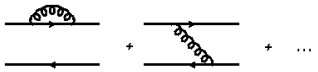
where

$$V_{Q\bar{Q}}(\vec{q}) \quad = \quad V_{C,BF}(\vec{q}) + W(\vec{q})$$

with



$$\Rightarrow \; V_{C,BF}(\vec{q}) \quad = \quad -\frac{4}{3}\frac{4\pi\alpha}{|\vec{q}|^2} \; + \; BF$$



$$\Rightarrow \; W(\vec{q}) \quad = \quad \frac{4}{3}\, 4\pi\alpha \left[\frac{1}{\vec{q}^2} - \frac{1}{q_z^2}\right] \frac{\mu^2}{\mu^2 + \vec{q}^2}\, e^{-2m^2 \frac{|\vec{q}^2|^2}{q_z^2\, \lambda^4}}$$

Remark: If $\mu^2 = 0$, $W = 0 \Rightarrow$ QED

The effective eigenvalue equation

Coulomb + Harmonic Oscillator

$$\left[\frac{\vec{k}^2}{m} - B\right]\psi(\vec{k}) + \int \frac{d^3q}{(2\pi)^3} V_{C,\,BF}(\vec{q})\,\psi(\vec{k}-\vec{q}) - \frac{4}{3}\frac{\alpha}{2\pi} b^{-3} \sum_i \tau_i \frac{\partial^2}{dk_i^2}\psi(\vec{k}) = 0$$

$$b = \frac{\sqrt{2m}}{\lambda_0^2}$$

Position space

$$\left[2m - \frac{\Delta_{\vec{r}}}{m} - \frac{4}{3}\alpha\left(\frac{1}{r} + BF\right) + \frac{1}{2}\,\tilde{\kappa}\,r^2\right]\psi(\vec{r}) \quad = \quad (2m+B)\,\psi(\vec{r}) \;=\; M\,\psi(\vec{r}) \qquad (1)$$

Appendix

RGPEP equation:

$$\mathcal{H}_t' \;=\; [[\mathcal{H}_f, \mathcal{H}_{Pt}], \mathcal{H}_t]$$

The Hamiltonian can be expressed in the following way

$$\mathcal{H}_t \;=\; \mathcal{H}_0 + g\mathcal{H}_{t1} + g^2\mathcal{H}_{t2} + g^3\mathcal{H}_{t3} + g^4\mathcal{H}_{t4}\ .$$

And thus then the RGPEP equation reads,

$$\begin{aligned} &\mathcal{H}_0' + g\mathcal{H}_{t1} + g^2\mathcal{H}_{t2}' + g^3\mathcal{H}_{t3}' + g^4\mathcal{H}_{t4}' \\ =\; &\left[\left[\mathcal{H}_0, \mathcal{H}_0 + g\mathcal{H}_{1Pt} + g^2\mathcal{H}_{2Pt} + g^3\mathcal{H}_{3Pt} + g^4\mathcal{H}_{4Pt}\right], \mathcal{H}_0 + g\mathcal{H}_{t1} + g^2\mathcal{H}_{t2} + g^3\mathcal{H}_{t3} + g^4\mathcal{H}_{t4}\right]\ . \end{aligned}$$

with

$$\mathcal{H}_t(a_0) = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \ldots, i_n} c_t(i_1, \ldots, i_n)\; a_{0i_1}^{\dagger} \cdots a_{0i_n}$$

$$\mathcal{H}_{Pt}(a_0) \;=\; \sum_{n=2}^{\infty} \sum_{i_1, i_2, \ldots, i_n} c_t(i_1, \ldots, i_n) \left(\frac{1}{2}\sum_{k=1}^{n} p_{i_k}^{+}\right)^2 a_{0i_1}^{\dagger} \cdots a_{0i_n}$$

Appendix

Equations order by order:

$$
\begin{aligned}
\mathcal{H}'_0 &= 0 \,, \\
g\mathcal{H}'_{t\,1} &= \left[\left[\mathcal{H}_0, g\mathcal{H}_{1Pt}\right], \mathcal{H}_0\right] \,, \\
g^2\mathcal{H}'_{t\,2} &= \left[\left[\mathcal{H}_0, g^2\mathcal{H}_{2Pt}\right], \mathcal{H}_0\right] + \left[\left[\mathcal{H}_0, g\mathcal{H}_{1Pt}\right], g\mathcal{H}_{1t}\right] \,, \\
g^3\mathcal{H}'_{t\,3} &= \left[\left[\mathcal{H}_0, g^3\mathcal{H}_{3Pt}\right], \mathcal{H}_0\right] + \left[\left[\mathcal{H}_0, g^2\mathcal{H}_{2Pt}\right], g\mathcal{H}_{1t}\right] + \left[\left[\mathcal{H}_0, g\mathcal{H}_{1Pt}\right], g^2\mathcal{H}_{2t}\right] \,, \\
g^4\mathcal{H}'_{t\,4} &= \left[\left[\mathcal{H}_0, g^4\mathcal{H}_{4Pt}\right], \mathcal{H}_0\right] + \left[\left[\mathcal{H}_0, g^3\mathcal{H}_{3Pt}\right], g\mathcal{H}_{1t}\right] + \left[\left[\mathcal{H}_0, g^2\mathcal{H}_{2Pt}\right], g^2\mathcal{H}_{2t}\right] + \left[\left[\mathcal{H}_0, g\mathcal{H}_{1Pt}\right], g^3\mathcal{H}_{3t}\right]
\end{aligned}
$$

Solutions order by order:
We present solutions to these equations in terms of matrix elements $\mathcal{H}_{t\,ab} = \langle a|\mathcal{H}|b\rangle$.

$$
\begin{aligned}
\mathcal{H}_{t1\,ab} &= f_{t\,ab}\,\mathcal{H}_{01\,ab} \,, \\
\mathcal{H}_{t2\,ab} &= f_{t\,ab}\sum_x \mathcal{H}_{01\,ax}\mathcal{H}_{01\,xb}\,\mathcal{B}_{t\,axb} \;+\; f_{t\,ab}\,\mathcal{G}_{02\,ab} \,, \\
\mathcal{H}_{t3\,ab} &= f_{t\,ab}\,\mathcal{G}_{03\,ab} + f_{t\,ab}\sum_{xy}\mathcal{H}_{01\,ax}\,\mathcal{H}_{01\,xy}\,\mathcal{H}_{01\,yb}\,\mathcal{C}_{t\,axyb} \;+\; f_{t\,ab}\sum_x\left(\mathcal{H}_{01\,ax}\,\mathcal{G}_{02\,xb} + \mathcal{G}_{02\,ax}\,\mathcal{H}_{01\,xb}\right)\mathcal{B}_{t\,axb} \,, \\
\mathcal{H}_{t4\,ab} &= f_{t\,ab}\sum_{xyz}\mathcal{H}_{01\,ax}\,\mathcal{H}_{01\,xy}\,\mathcal{H}_{01\,yz}\,\mathcal{H}_{01\,zb}\,\mathcal{D}_{t\,axyzb} \\
&\quad + f_{\tau\,ab}\sum_{xy}\left(\mathcal{H}_{01\,ax}\,\mathcal{H}_{01\,xy}\,\mathcal{G}_{02\,yb} \;+\; \mathcal{H}_{01\,ax}\,\mathcal{G}_{02\,xy}\,\mathcal{H}_{01\,yb} \;+\; \mathcal{G}_{02\,ax}\,\mathcal{H}_{01\,xy}\mathcal{H}_{01\,yb}\right)\mathcal{C}_{t\,axyb} \\
&\quad + f_{\tau\,ab}\sum_x\left(\mathcal{G}_{02\,ax}\,\mathcal{G}_{02\,xb} + \mathcal{H}_{01\,ax}\,\mathcal{G}_{03\,xb} + \mathcal{G}_{03\,ax}\,\mathcal{H}_{01\,xb}\right)\mathcal{B}_{t\,axb} \;+\; f_{\tau\,ab}\,\mathcal{G}_{04\,ab} \,.
\end{aligned}
$$

Appendix

RGPEP factors:

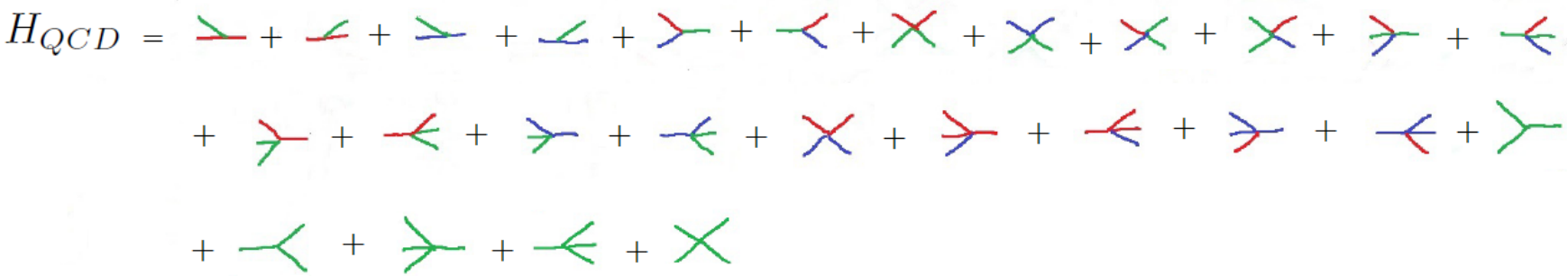
$$
\begin{aligned}
\mathcal{A}_{t\,axb} &= [ax\,p_{ax} + bx\,p_{bx}]\, f^{-1}_{t\,ab} f_{t\,ax} f_{t\,bx} \ , \\
\mathcal{B}_{t\,axb} &= \int_0^t \mathcal{A}_{\tau\,axb}\, d\tau \ , \\
\mathcal{C}_{t\,axyb} &= \int_0^t [\mathcal{A}_{\tau\,axb}\, \mathcal{B}_{\tau\,xyb} + \mathcal{A}_{\tau\,ayb}\, \mathcal{B}_{\tau\,axy}]\, d\tau \ , \\
\mathcal{D}_{t\,axyzb} \ , &= \int_0^t [\mathcal{A}_{\tau\,axb}\, \mathcal{C}_{\tau\,xyzb} + \mathcal{A}_{\tau\,azb}\, \mathcal{C}_{\tau\,axyz} + A_{\tau\,ayb}\, \mathcal{B}_{\tau\,axy}\, \mathcal{B}_{\tau\,yzb}] \ ,
\end{aligned}
$$

with $f_{t\,ab} := \exp\left[-ab^2\, t\right]$ and $ab := \mathcal{M}^2_{ab} - \mathcal{M}^2_{ba}$.

Appendix: Toward 4th order

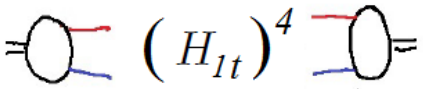
$$H_{QCD} = H_{\psi^2} + H_{A^2} + H_{A^3} + H_{\psi A \psi} + H_{\psi A A \psi} + H_{[\partial A A](\psi\psi)} + H_{(\psi\psi)^2}$$

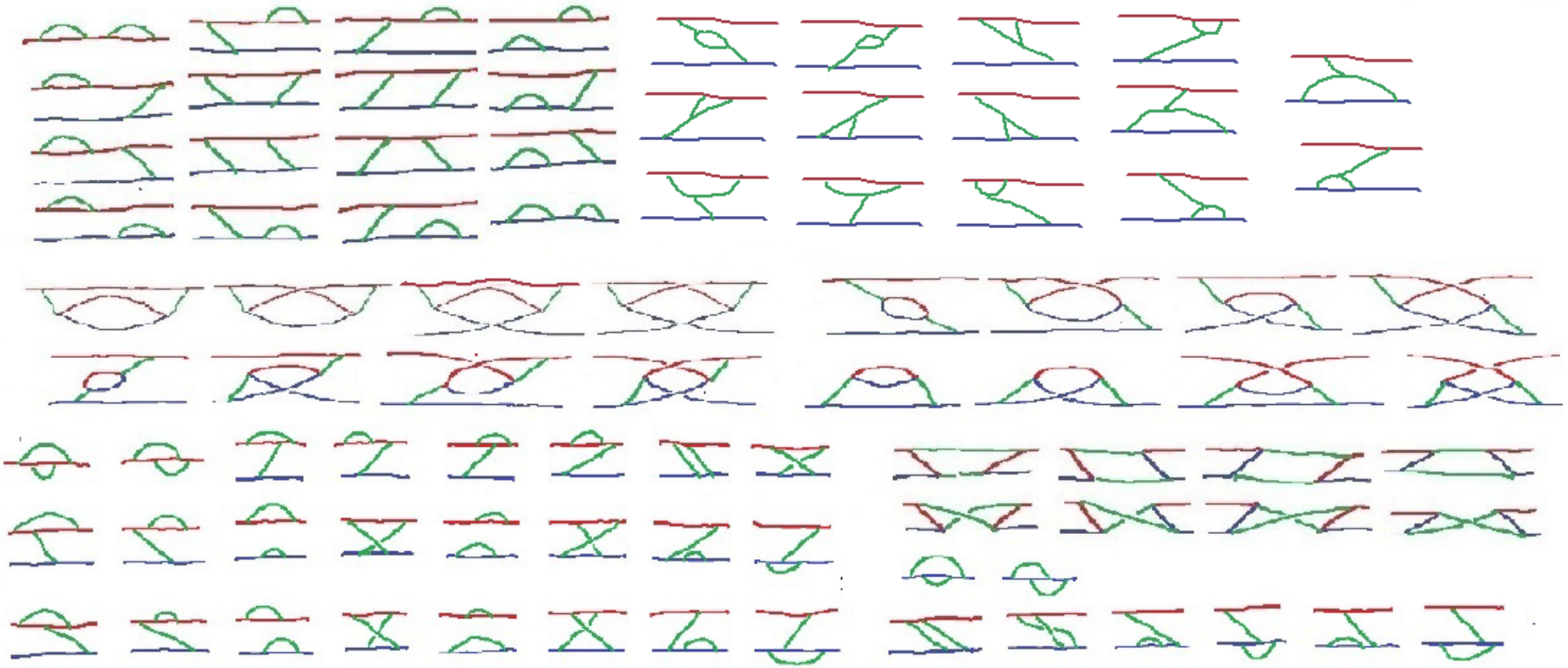
Canonical Hamiltonian expressed with diagrams:



Appendix: Example of 4th-order diagrams

Forth power of the 1st-order Hamiltonian





Appendix. H. O. parameters

Defining

$$\tau(T) \quad = \quad \int du \left(\frac{\mu_{253}^2 \, u^2 \, \frac{b^2}{T^2}}{\mu_{253}^2 \, \frac{b^2}{T^2} + u^2} + \frac{\mu_{154}^2 \, u^2 \, \frac{b^2}{T^2}}{\mu_{154}^2 \, \frac{b^2}{T^2} + u^2} \right) e^{-u^2} \ , \tag{2}$$

and

$$\vec{\tau} \quad = \quad \int dT \, T(1-T^2) \, \vec{w}(T) \, \tau(T) \ , \tag{3}$$

one can write:

$$W_{Q\bar{Q}} \quad = \quad \int \frac{d^3q}{(2\pi)^3} W(\vec{q}) \frac{1}{2} q^2 w_i(t) \frac{\partial^2}{\partial k_i^2} \psi(\vec{k}) \ = \ -\frac{4}{3} \frac{\alpha}{4\pi} b^{-3} \sum_i \tau_i \frac{\partial^2}{dk_i^2} \tag{4}$$