Position-space gluon propagator: positivity violation and gluon mass generation.

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Outline

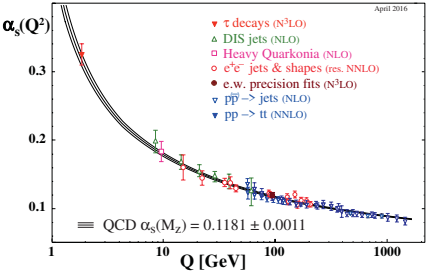
Quantum Chromo-Dynamics

QCD Lagrangian depends on a few parameters: one coupling, α_s, and quark masses (m_u, m_d, m_s, m_c, m_b and m_t).

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \sum_{f=u,\cdots,t} \bar{\psi}_f \left(i \not{D} - m_f \right) \psi_f$$

α_s acquires a renormalization scheme dependent running with the momentum.

The running of $\alpha_s(\mu^2) = \frac{g^2(\mu^2)}{4\pi}$ is controlled by its RGE, $\frac{d\alpha_s}{d\ln\mu^2} = \beta(\alpha_s)$



Quantum Chromo-Dynamics

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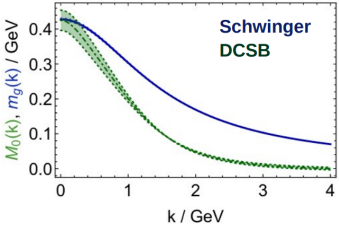
$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \sum_{f=u,\cdots,t} \bar{\psi}_f \left(i \not{D} - m_f \right) \psi_f$$

Emergent phenomena:

- Confinement (Hadron masses).
- Dynamically generated gluon-mass.
- Spontaneous chiral symmetry breaking.



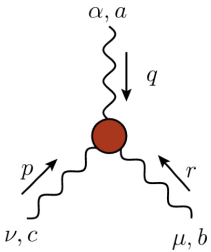
NP approaches:

- Lattice-QCD.
- Functional methods.
- QCD vacuum, sum rules,...

Gluon self-coupling

$$\mathcal{L}_{\rm YM} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} \; ; \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g \; f^{abc} A^b_\mu A^c_\nu$$

- Three-gluon coupling responsible for the main differences between gluon and photon dynamics.
- It is itself a non-perturbative object which can be computed from the lattice or DSE.
- Key ingredient in DSE of quark-gluon or ghost-gluon vertices, for example.

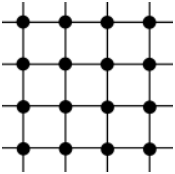


Poster by F. Pinto-Gómez & FS on Tuesday...

Lattice formulation

Path integral in imaginary time:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [dU d\psi d\bar{\psi}] \mathcal{O}(U, \psi, \bar{\psi}) e^{-S(U,\psi,\bar{\psi})} \to \frac{1}{N} \sum_{i=1}^{N} O_i$$



dimensionless; lattice spacing a fixed *a posteriori*.

Pros

- **Just QCD**.
- Regularized per se ($\Lambda \sim a^{-1}$).

Cons

- Finite volume and discretization errors.
- Broken rotational symmetry!
- Expensive chiral fermions.

Lattice setups

Exploited quenched gauge field configurations with:

β	L^4/a^4	a (fm)	confs
5.6	32^4	0.236	2000
5.6	48^4	0.236	500
5.7	32^4	0.182	500
5.8	32^4	0.144	2000
5.8	48^4	0.144	500
6.0	32^4	0.096	2000
6.2	32^4	0.070	2000

- Absolute calibration for $\beta = 5.8$ taken from [S. Necco and R. Sommer, NPB 622 (2002) 328].
- Relative calibrations based in gluon propagator scaling [Ph. Boucaud, FS, *et al.* PRD 98 (2018) 114515].

Computing gluon propagator in Landau gauge

Landau gauge

Landau gauge $\partial_\mu A^a_\mu = 0$ fixed numerically, allowing to compute gauge dependent quantities.

- In momentum-space:

$$\widetilde{\Delta}^{ab}_{\mu\nu}(q^2) = \langle \widetilde{A}^a_\mu(q)\widetilde{A}^b_\nu(-q)\rangle = \delta^{ab}\widetilde{\Delta}(q)P_{\mu\nu}(q)$$

- In position-space we considered the scalar part of the propagator and the Schwinger function:

$$\Delta(x) = \frac{1}{24}\langle A^a_\mu(x)A^a_\mu(0)\rangle\,; \qquad S(t) = \sum_{\vec{x}}\Delta(\vec{x},t)$$

In the lattice, $\Delta(x)$ and $\widetilde{\Delta}(q)$ are related by a discrete Fourier transform.

$$\Delta(x) = \sum_q \widetilde{\Delta}(q)e^{-iq\cdot x}$$

Gluon propagation and non-perturbative physics.

In the continuum, infinite-volume, position and momentum-space propagators are related through:

$$\Delta(x) = \frac{1}{4\pi^2 x}\int_0^\infty dq\; q^2 J_1(qx)\widetilde{\Delta}(q)$$

Massless gluon:

$$\frac{1}{q^2} \quad \rightarrow \quad \frac{1}{4\pi^2 x^2}$$

Spectral representation $\rho(m)$

$$\begin{aligned}
\widetilde{\Delta}(q) &= \int dm\, \frac{\rho(m)}{q^2+m^2} \\
\Delta(x) &= \frac{1}{4\pi^2 x}\int dm\, m\, \rho(m) K_1(xm) \\
S(t) &= \int dm\, \rho(m) e^{-mt}
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Massive gluon:

$$\frac{1}{q^2+m^2} \quad \rightarrow \quad \frac{mK_1(mx)}{4\pi^2 x} \sim \frac{e^{-mx}}{x^{3/2}}$$

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Yukawa-like:

$$\left(\frac{1}{q^2+m^2}\right)^{3/2} \rightarrow \frac{e^{-mx}}{12\pi^2 x}$$

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Positivity violation: a negative value of $\Delta(x)$ or $S(t)$ implies that $\rho(m)$ cannot be positively defined.

Momentum and position-space propagators

If we were able to fully eliminate discretization and finite-volume artifacts from the lattice propagators, we should recover propagators that are related by a Fourier Transform in the continuum.

Fourier transforms...

$$\begin{tikzcd}
\Delta_{Latt}(x) \arrow[r, leftrightarrow, "\mathtt{FFT}"] \arrow[d] & \widetilde{\Delta}_{Latt}(q) \arrow[d, "\mathtt{H4\ errors}"] \\
\Delta_{H4}(x) \arrow[d] & \widetilde{\Delta}_{H4}(q) \arrow[d, "\mathtt{O4\ errors}"] \\
\Delta(x) \arrow[r, leftrightarrow, "\mathcal{F}"] & \widetilde{\Delta}(q)
\end{tikzcd}$$

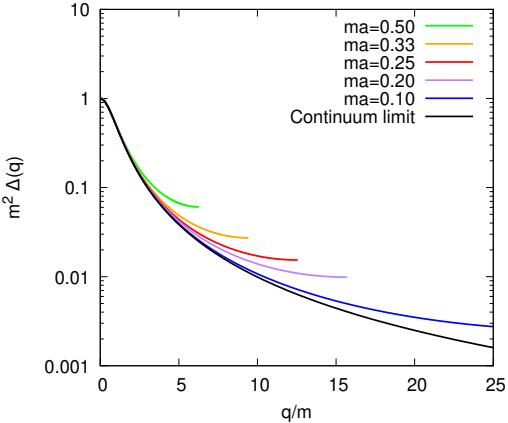
Lattice H(4) symmetry.

In 1D, we have limited information because:

- finite Brillouin zone
- finite number of points

In higher dimensions, we still have limited information and:

- rotational symmetry is broken
- we can use these effects to extrapolate to the continuum limit with a single β.



Boson propagator for an infinite discrete lattice

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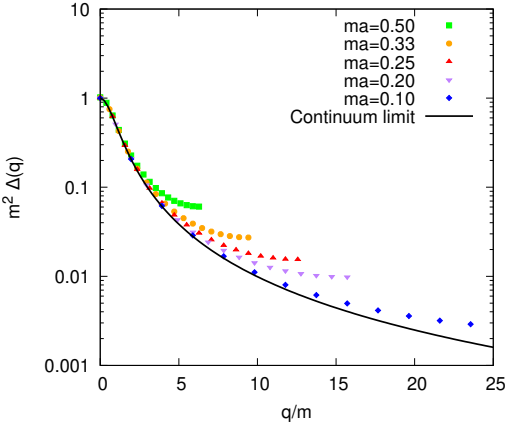
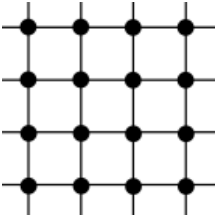


Boson propagator for a finite discrete lattice

Lattice H(4) symmetry.

In the lattice, O(4) rotational symmetry is broken into H(4), and as a result, the propagator does not only depend on q^2, but

$$\widetilde{\Delta}_{Latt}(q^2, q^{[4]}, q^{[6]}, q^{[8]}), \qquad with \qquad q^{[2n]} = \sum_i q_i^{2n}$$

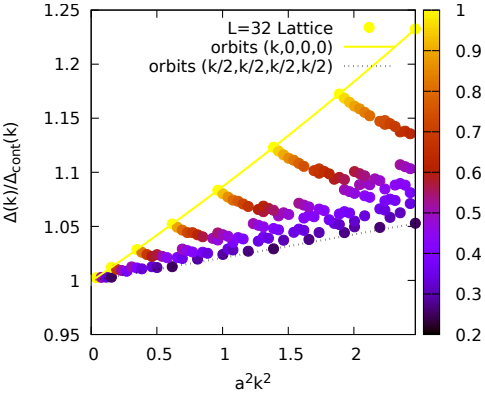


For a free massive boson, for example:

$$\widetilde{\Delta}_{Latt}(q^2, q^{[4]}, q^{[6]}, q^{[8]}) = \frac{1}{q^2+m^2} + \frac{a^2 q^{[4]}}{12}\left(\frac{1}{q^2+m^2}\right)^2 + \cdots$$

An extrapolation in $q^{[4]} \to 0$ eliminate the leading discretization errors [JHEP 09 (2007) 007].

H4 extrapolation method



For each k^2, there is a linear trend in

$$\frac{k^{[4]}}{(k^2)^2} \ ,$$

with limiting cases for orbits $(k,0,0,0)$ and $(\frac{k}{2},\frac{k}{2},\frac{k}{2},\frac{k}{2})$.

This dependence (almost) allows to take the continuum limit $a \to 0$ with a single β.

H4 extrapolation method

Assuming a *smooth* dependence on k^2 of the slopes we make the ansatz:

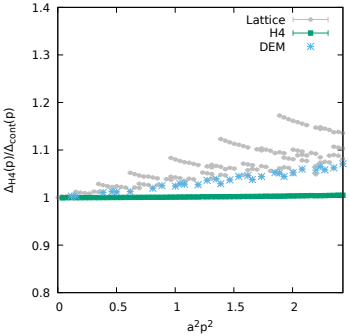
$$\widetilde{\Delta}_{Latt}(k^2,0) = \widetilde{\Delta}_{Latt}(k^2,k^{[4]}) \times \left[1 + a^2 \left(\frac{k^{[4]}}{(k^2)^2}\right) c(k^2) + \cdots\right]$$

where dimensional arguments suggest $c(k^2) \sim k^2$.

H4 extrapolation

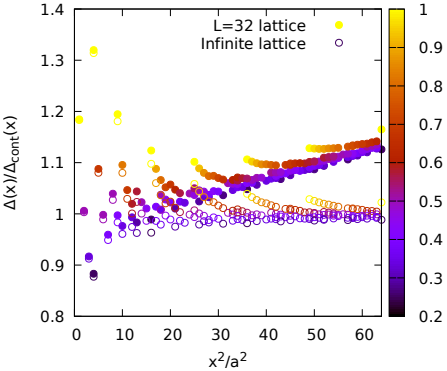
Linear least squares method used to determine $\widetilde{\Delta}_{Latt}(k^2,0)\ \forall k^2$ and c_i in $c(k^2) = \sum_{i=1}^{3} c_i (k^2)^i$.

[FS 2204.12189]



H4 extrapolation method *in position-space*

In position-space, discretization errors affect mainly short-distances:



The methods traditionally used in momentum-space do not work:

- Still there's a linear trend in

$$\frac{x^{[4]}}{(x^2)^2}$$

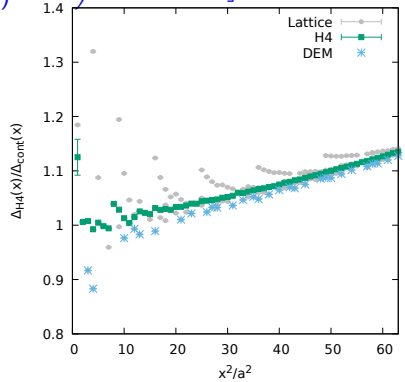
- *Democratic* orbits are not optimal!
- There is no equivalent to $\hat{k} = 2\sin(k/2)$.
- Finite-volume errors mix with discretization ones.

H4 extrapolation method *in position-space*

In analogy with momentum-space, we will assume:

$$\Delta_{Latt}(x^2,0) = \Delta_{Latt}(x^2,x^{[4]}) \times \left[1 + a^2\left(\frac{x^{[4]}}{(x^2)^2} - \alpha\right)c(x^2) + \cdots\right]$$

where dimensional arguments suggest $c(x^2) \sim 1/x^2$.



Best-results are obtained for $\alpha = 1/2$, and removes most of the discretization errors, although some noise still present for small x^2.

O4 symmetric errors.

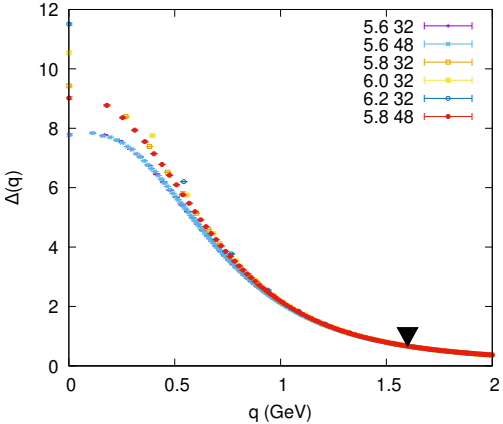
After H4 errors removal, smooth functions of k^2 (x^2) are obtained, but they can still be affected by discretization or finite-volume errors.

MOM renormalization:

$$\widetilde{\Delta}_{R,\mu^2}(q^2) = \lim_{a\to 0} \frac{\widetilde{\Delta}_0(q^2,a)}{\mu^2 \widetilde{\Delta}_0(\mu^2,a)}$$

Typical strategy: choose μ and compute the rhs ratio

$$\frac{\widetilde{\Delta}_0(q^2,a)}{\mu^2 \widetilde{\Delta}_0(\mu^2,a)} = \widetilde{\Delta}_{R,\mu^2}(q^2) + \mathcal{O}(a^2)$$



[Ph. Boucaud. FS. *et al* PRD 98 (2018) 114515]

Fitting strategy

Let's assume some parametrization for $\widetilde{\Delta}_{R,\mu}(q)$ and make a global non-linear fit for all lattice setups of the form:

$$\widetilde{\Delta}_{R,\mu}(q) = z_\beta \widetilde{\Delta}_0(q, a(\beta)) \times \left[1 + \frac{a^2}{a_0^2} a_0{}^2 C(q^2)\right] \times \left[1 + d_1 e^{-d_2 L q}/L\right]$$

with $C(q^2) = c_1 q^2$ keeping its leading term.

Remanent O4 lattice corrections fitted at once for all β's:

- Start with a reasonable guess of z_β's, ratios a^2/a_0^2 and $\widetilde{\Delta}_{R,\mu}$ from one β.
- Non-linear least squares problem (gradient descent).
- Jackknife error analysis.
- Check stability for different parametrizations of $\widetilde{\Delta}_{R,\mu}(q)$ and $C(q^2)$.

Continuum limit

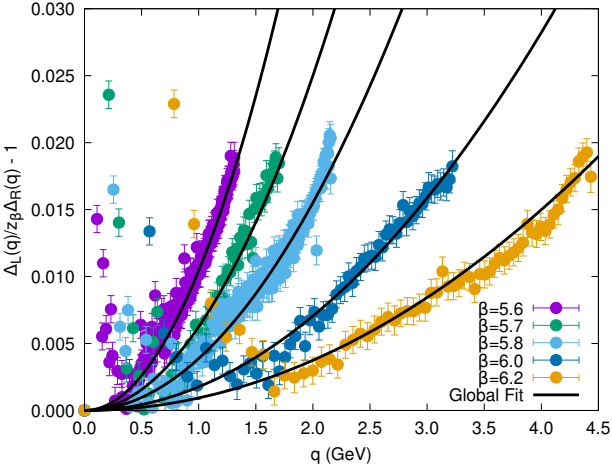
- Dominant term: $C(q^2) = c_1 q^2$ with

$$c_1 = 0.0077(6)(2)$$

(note that there is a single fit for all β's)

- Stable against different parametrizations of $C(q^2)$.
- Relative calibration for the ratios of lattice spacings $a(\beta)/a(\beta_0)$

[Ph. Boucaud, FS, *et al* PRD 98 (2018) 114515]



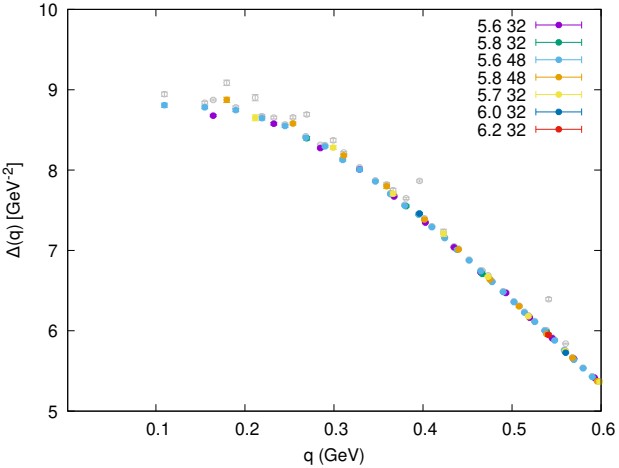
Finite-volume errors

In the IR, still some differences among setup's survive due to finite-volume errors.

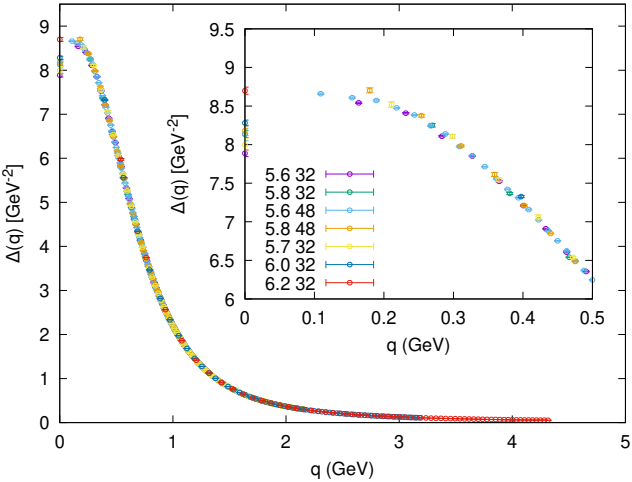
Excluding the $q = 0$, they've been fitted by:

$$\widetilde{\Delta}_R(q) = \widetilde{\Delta}_R(q, L) \times \left[1 + d_1 e^{-d_2 L q}/L\right]$$

with $d_1 \approx 3.7(2) GeV^{-1}$ and $d_2 \approx 0.26(3)$.



Gluon propagator in momentum-space



Gluon propagator in momentum-space

Three different ansätze for $\Delta_{R,\mu}(q)$:

- Fit 1 (log):

$$\widetilde{\Delta}^{-1}_{R,\mu}(q) = q^2\left[1 + \left(\kappa_1 - \frac{\kappa_2}{1+(q^2/\kappa_4^2)^2}\right)\log\frac{q^2}{\mu^2}\right] + R(q^2) - R(\mu^2)$$

  with $R(q^2)$ a Padé approximant.

[A.C. Aguilar, FS, et al. 2107.00768]

- Fit 2 (pow-n):

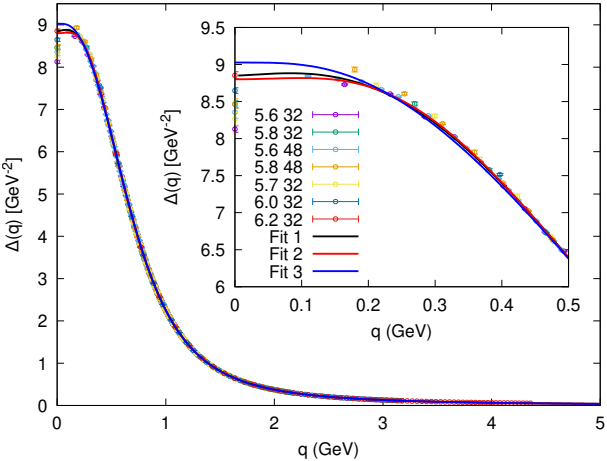
$$\widetilde{\Delta}_{R,\mu}(q) = \frac{1}{\mu^2}\left(\frac{\mu^2+m^2}{q^2+m^2}\right)^n \frac{R(q^2)}{R(\mu^2)}$$

  for $n = 3/2$.

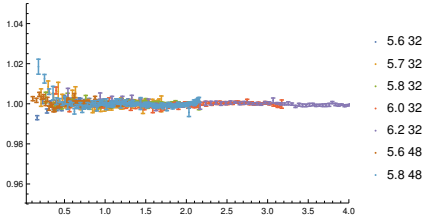
- Fit 3, Very Refined Gribov-Zwanziger D. Dudal et al. Annals Phys. 397 (2018) 351:

$$\widetilde{\Delta}_{R,\mu}(q) = Z\frac{q^4 + M_2^2 q^2 + M_1^4}{q^6 + M_5^2 q^4 + M_4^4 q^2 + M_3^6}$$

Gluon propagator in momentum-space



- After continuum-limit and infinite-volume extrapolation, very nice scaling for different β's and physical volumes.
- Effective gluon mass $\sim$ 337(2) MeV.



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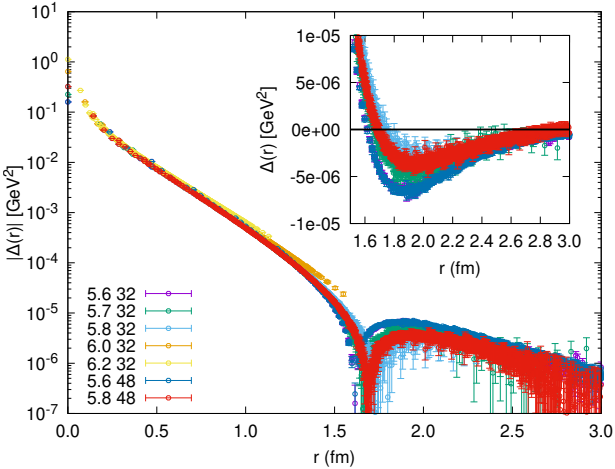
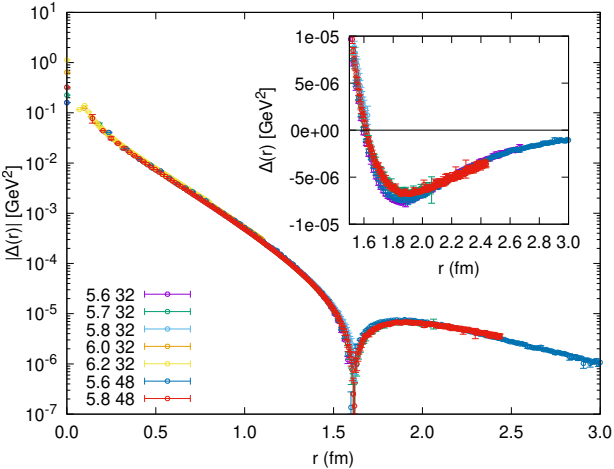


- H4 errors corrected.
- Renormalization constants (z_β) and lattice spacings (a) taken from $\widetilde{\Delta}_{R,\mu}$.
- O4 finite volume errors are fitted with the form:

$$\Delta(x, L=\infty) = \Delta(x, L) \times \left[1 + d\frac{x^2}{L^2}\right]$$

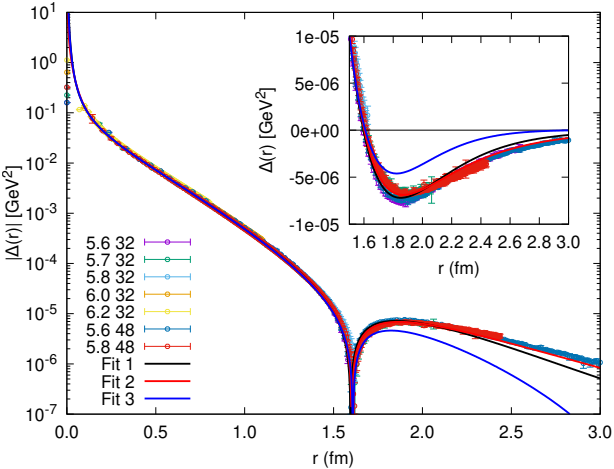
Gluon propagator in position-space



- For $r \lesssim 1$ fm, lattice data for the gluon propagator behave as e^{-mr}/r with $m \sim 340 MeV$.

  T. Iritani *et al.* PRD80 (2009) 114505.
- The Yukawa/exponential regime is broken for larger distances.
- Gluon propagator becomes negative for $r \gtrsim 1.6$ fm Positivity violation.
- Assuming the parametrizations used are valid $\forall q \in (0, \infty)$, we computed the continuous FT of the momentum-space fits.

Gluon propagator in position-space



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Schwinger function

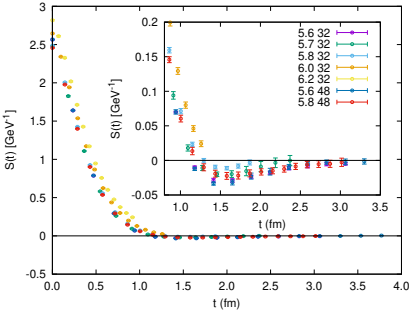
Lattice studies focus in the direct determination of the Schwinger function

$$S(t) = \sum_{\vec{x}} \Delta(\vec{x}, t)$$

which for a single massive pole would behave as $S(t) \sim e^{-mt}$

It can be computed from the continuum momentum or position-space propagators as:

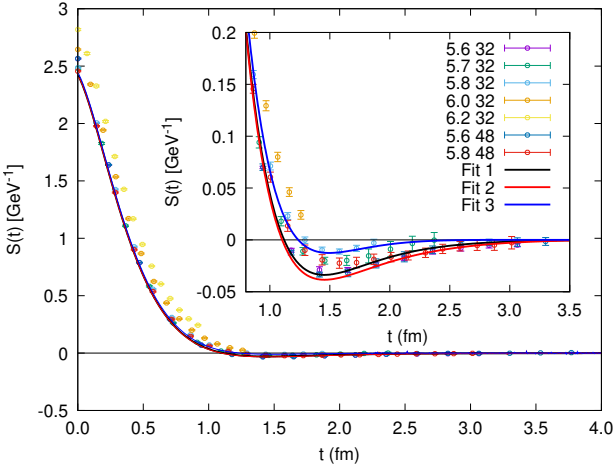
$$S(t) = \int d^3\vec{x}\ \Delta(\vec{x}, t) = \frac{1}{2\pi} \int dq_0\ \widetilde{\Delta}(\vec{0}, q_0) e^{iq_0 t}$$



Schwinger function

Using our fits in momentum space:

- we reproduce the zero-crossing position and minimum of $S(t)$. Positivity violation.
- sizable lattice artifacts for $S(t)$.
- the Fourier transform of $\widetilde{\Delta}(\vec{0}, p_0)$ seems a good approximation to the largest volumes $S(t)$.
- compatible with lattice [P. Bowman, U. Heller, *et al.* PRD 76 (2007) 094505] and DSE [C. Fischer, R. Alkofer, PRD 67 (2003) 094020].



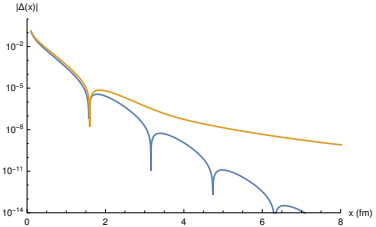
Analytic structure of the gluon propagator

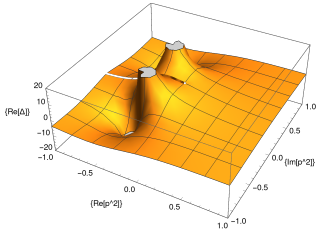
For a propagator with a pair of complex conjugated poles:

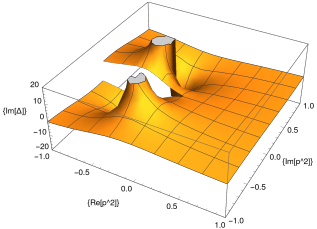
$$\widetilde{\Delta}_{R,\mu} \sim \frac{1}{p^2+\omega_+^2} + \frac{1}{p^2+\omega_-^2} \rightarrow S(t) \sim e^{-Re[\omega]t}\cos(Im[\omega]t)$$

which oscillates in sign. Similarly:

$$\Delta(x) \sim (K_1(\omega_+ x) + K_1(\omega_- x))/x$$







Proposal

$\Delta(x)$ may me a key ingredient for constraining the analytic structure of the gluon propagator.

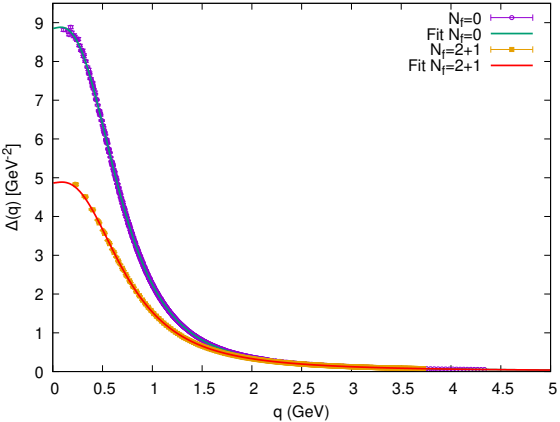
Gluon propagator in momentum-space

Let us confront our results with $N_f = 2+1$ unqueched data using Domain-Wall fermions from S. Zaferopulos, FS *et al.* PRL 122 (2019) 162002 with:

β	$a(\mathrm{fm})$	N^4	m_π/MeV	confs.
2.13	0.1139	$48^3 \times 96$	139.4	350
2.25	0.0835	$64^3 \times 128$	139.2	330

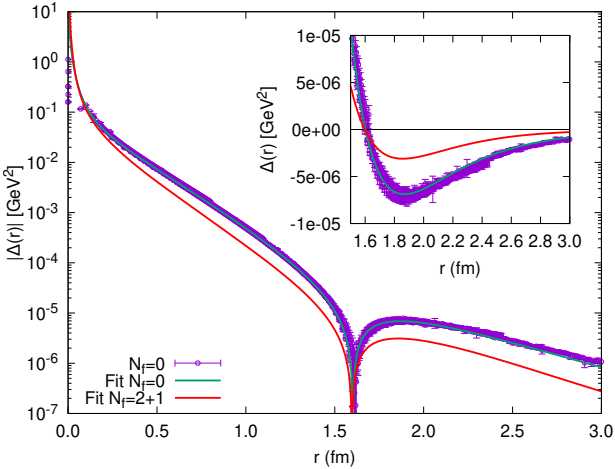
- Nice scaling also for unquenched data.
- Gluon propagator $\Delta_{R,4.3GeV}(q^2 \to 0) = 1/m^2$ with:

$$m_{N_f=0} \approx 337\mathrm{MeV}$$
$$m_{N_f=2+1} \approx 450\mathrm{MeV}$$



Gluon propagator in position-space

- Gluon propagator becomes negative at $r \approx 1.6\text{fm}$ for both $N_f = 0$ and $N_f = 2+1$.
- Larger effective mass for $N_f = 2+1$.

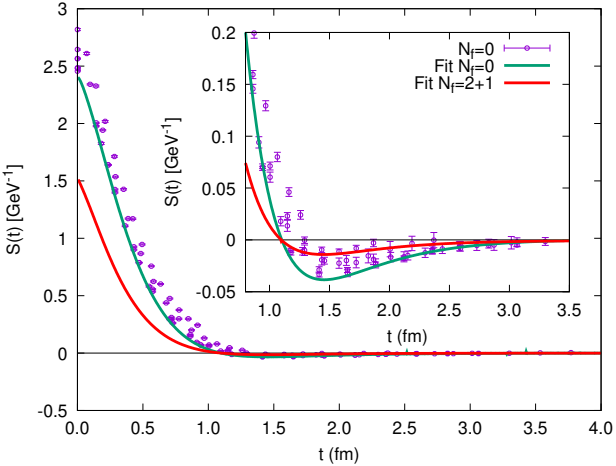


Schwinger function

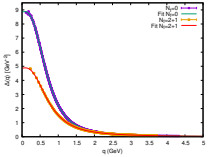
- Schwinger function negative for $r \gtrsim 1.10\mathrm{fm}$
- No apparent differences between $N_f = 0$ and $N_f = 2 + 1$.

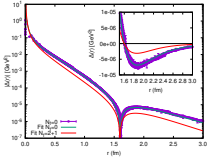
P. Bowman *et al.* PRD 76, 094505 (2007)

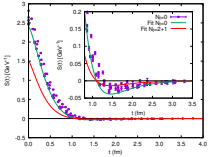
R. Alkofer *et al.* PRD 70, 014014 (2004)



Conclusions







- ✓ Very accurate data for gluon propagator in both position and mometum-space.
- ✓ Continuous FT as a guarantee of the continuum and infinite-volume limits.
- ✓ Positivity violation manifest both from $\Delta(r)$ and $S(t)$ not modified by quarks.
- ✓ Sea-mass effects do not alter the zeros of $\Delta(r)$ and $S(t)$.
- ... Quark propagator, renormalization constants,...
- ... Spectral function analysis.
- ... H(4) breaking for $L \neq T$ (Matsubara frequencies).
- ... Non periodic BC.

Thanks!