Density of states for gravitational waves

Felix Springer
David Schaich

University of Liverpool
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Outline

- Underlying physics and motivation: Composite Dark Matter
- Numerical tool: LLR
  - LLR explained
  - Review U(1) case
  - Bulk phase transitions SU(N)
  - Confinement phase Transition SU(4)

Composite Dark Matter

- Broad class of composite Dark Matter models
  - Dark mesons
  - Dark glueballs
  - Dark baryons

Composite Dark Matter

- Broad class of composite Dark Matter models
  - "Stealth dark matter confinement transition and gravitational waves", LSD Collaboration, arXiv:2006.16429
  - "Testing the Dark Confined Landscape:From Lattice to Gravitational Waves", Huang et al., arXiv:2012.11614
  - "Dark Confinement-Deconfinement Phase Transition:A Roadmap from Polyakov Loop Models to Gravitational Waves", Kang et al., arXiv:2101.03795
  - "Review of strongly-coupled composite dark matter models and lattice simulations", Kribs and Neil, arXiv:1604.04627

Baryon Dark Matter

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- New strongly coupled $SU(N)$ gauge sector coupled to new massive fermions

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- Dark Matter candidate: composite 'dark baryon' DM
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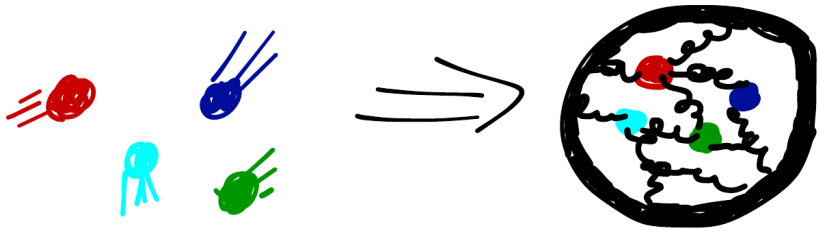
  If confinement transition is first order $\rightarrow$ stochastic background of gravitational waves



Stealth Dark Matter

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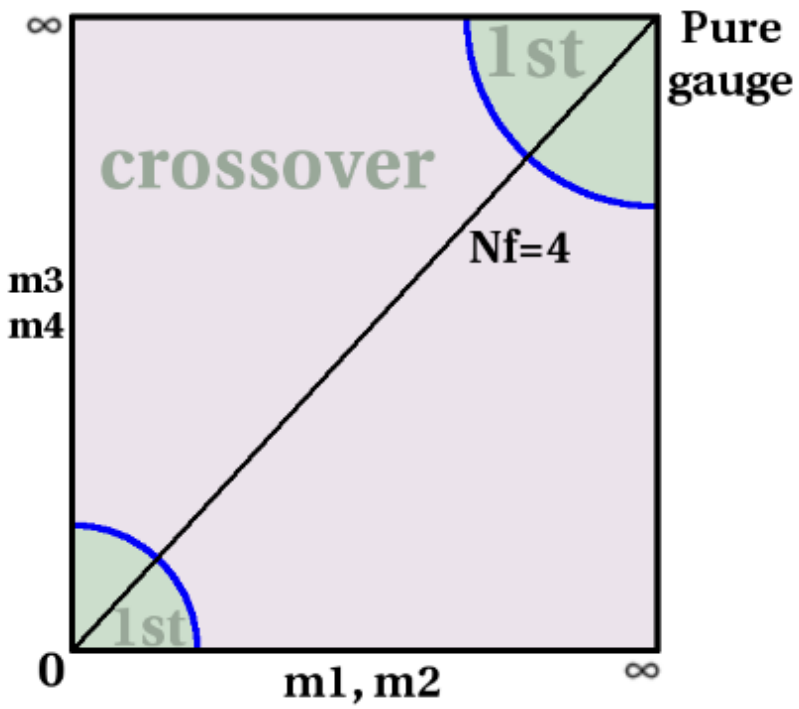
- $SU(4)$ gauge theory coupled to four flavors of moderately heavy fermions in the fundamental representation
- Pure gauge $SU(4)$ theory recovered in the 'quenched' case, fermions infinitely massive



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Solution:
Wang-Landau type algorithms

Standard Technique:

$$\langle \mathcal{O} \rangle = \frac{1}{Z(\beta)} \int \mathrm{d}\phi \mathcal{O}[\phi] e^{\beta S[\phi]}$$

approximate by

$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{\mathcal{C}} \mathcal{O}[\mathcal{C}]$$

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High dimensional integral:

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Density of states:

$$\rho(E) = \int \mathcal{D}\phi \delta(S[\phi] - E)$$

1 dimensional integral:

$$Z(\beta) = \int \mathrm{d}E \rho(E) e^{\beta E}$$

Calculate observable:

$$\langle \mathcal{O} \rangle = \frac{1}{Z(\beta)} \int \mathrm{d}E \mathcal{O} \rho(E) e^{\beta E}$$

Density of States

Task:
Calculate $\rho(E)$ with near constant relative error over a wide range of E

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Solution:
Linear Logarithmic Relaxation

Re-weighted Expectation Value

Re-weighted expectation value ΔE

$$\begin{aligned}\langle\langle E - E_0\rangle\rangle_\delta(a) &= \frac{1}{N}\int \mathcal{D}\phi\theta_{E_0,\delta}(E - E_0)e^{-aS} \\ &= \frac{1}{N}\int_{E_0-\frac{\delta}{2}}^{E_0+\frac{\delta}{2}} \mathrm{d}E(E - E_0)\rho(E)e^{-aE} \\ &= \Delta E\end{aligned}$$

Normalization factor

$$N = \int \mathcal{D}\phi\theta_{E_0,\delta}e^{-aS[\phi]}$$

Set re-weighted expectation value to zero

$$\Delta E(a) = \frac{1}{N} \int_{E_0 - \frac{\delta}{2}}^{E_0 + \frac{\delta}{2}} \mathrm{d}E (E - E_0) \rho(E) e^{-aE} = 0$$

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Approximate integral with trapezium rule

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Let interval go to zero $\delta \to 0$

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Approximate integral with trapezium rule

Let interval go to zero $\delta \to 0$

Solve for a

$$a_0 = \frac{1}{\rho(E_0)} \frac{\mathrm{d}\rho(E)}{\mathrm{d}E}\Big|_{E=E_0} = \frac{\mathrm{d}ln(\rho(E))}{\mathrm{d}E}\Big|_{E=E_0}$$

numerically integrate a and we get ρ with constant relative error!

Integrate the exponent using the Trapezium rule

$$\begin{aligned}\rho(E_N) &= e^{\int_{-\infty}^{E_N} \mathrm{d}E' a(E')} \\ &= \rho_c e^{a_0 \frac{\delta_E}{2} + \sum_{k=1}^{N-1} a_k \delta + a_N \delta + \mathcal{O}(\delta^2)}\end{aligned}$$

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Get $\rho(E)$ up to a constant with exponential error suppression

$$\rho(E_N) = \rho_c \tilde{\rho}(E_N) e^{\mathcal{O}(\delta^2)}$$

Solve $\Delta E(a) = 0$ via Robbins-Monro iteration

$$a^{(n+1)} = a^{(n)} + \frac{12}{\delta^2(n+1)} \Delta E(a^{(n)})$$

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Evaluate $\Delta E(a^{(n)})$ with standard Monte Carlo

$$\begin{aligned} \Delta E(a^{(n)}) &= \langle\langle E - E_0 \rangle\rangle_\delta (a^{(n)}) \\ &= \frac{1}{N} \int \mathcal{D}\phi \theta_{E_0,\delta}(E - E_0) e^{-a^{(n)} S} \end{aligned}$$

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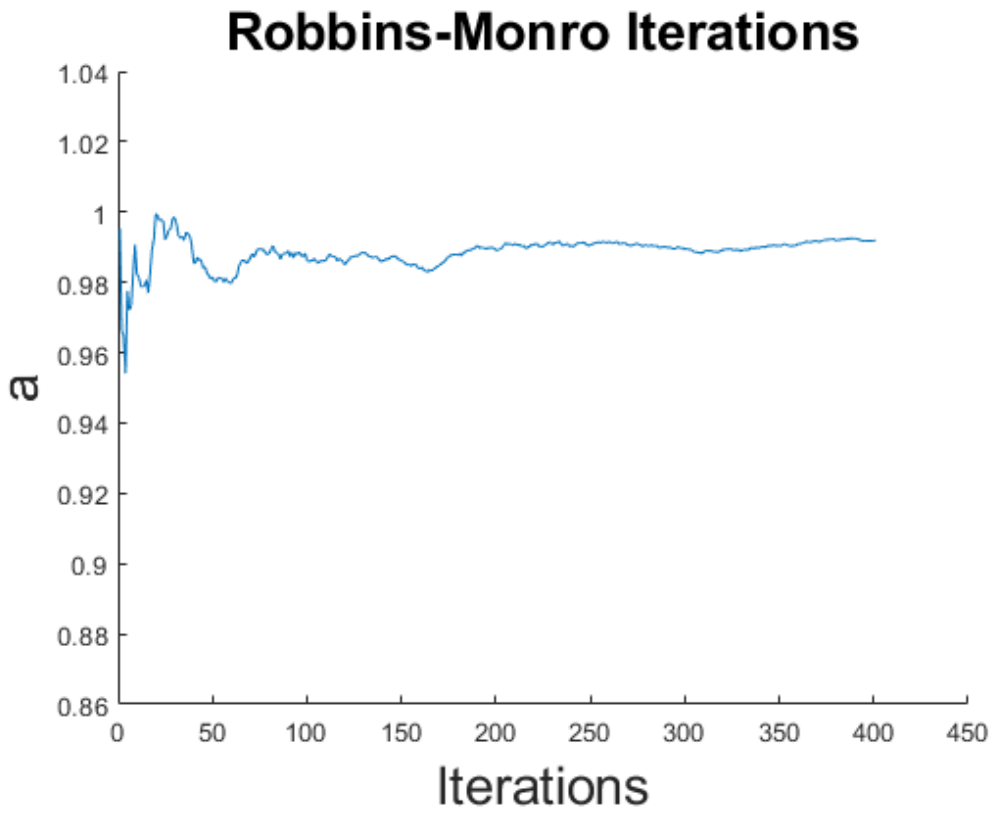
New Boltzmann weight!

Alternatively: $\Delta E(a^{(n)}) = \frac{1}{N} \int \mathcal{D}\phi (E - E_0) e^{-a^{(n)} S - \frac{(S - E_0)}{2\delta^2}}$


Robbins-Monro Iterations
1.04
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0.96
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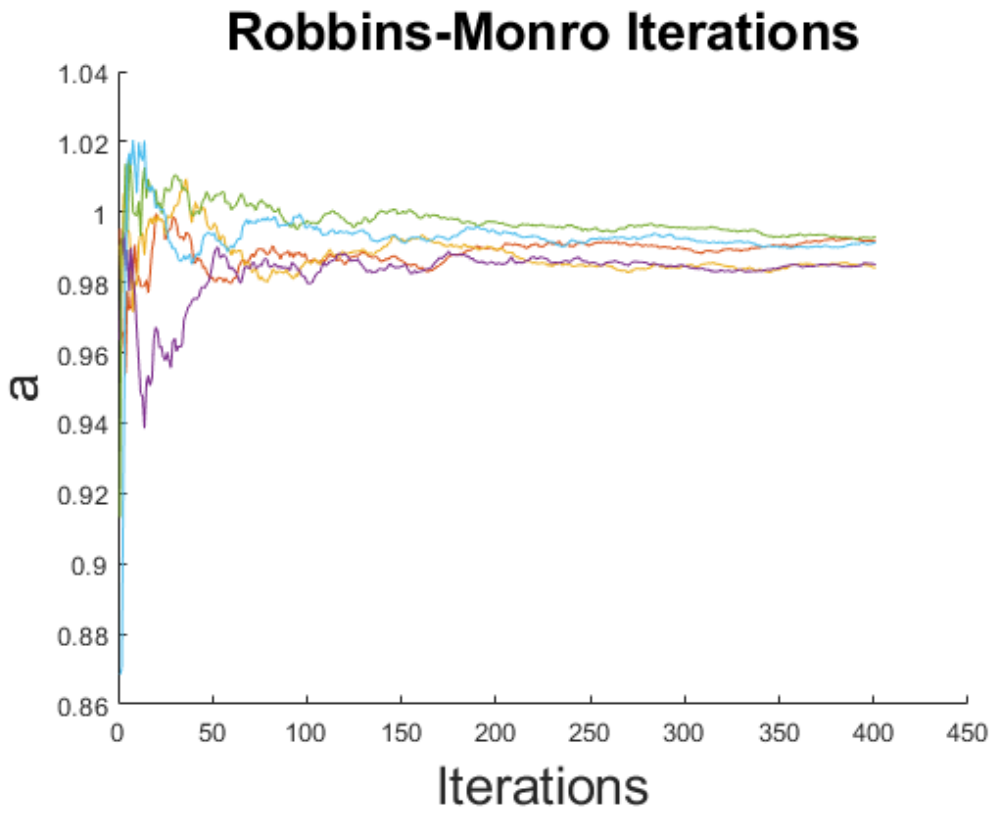
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Iterations

Pseudocode

loop over energy interval $[E_i \pm \frac{\delta}{2}]$

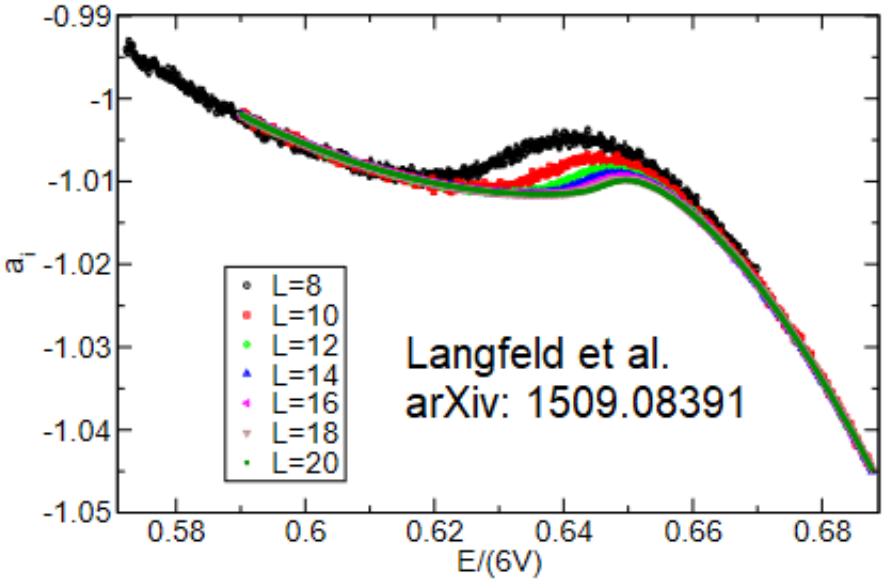
loop over Robbins-Monro iteration

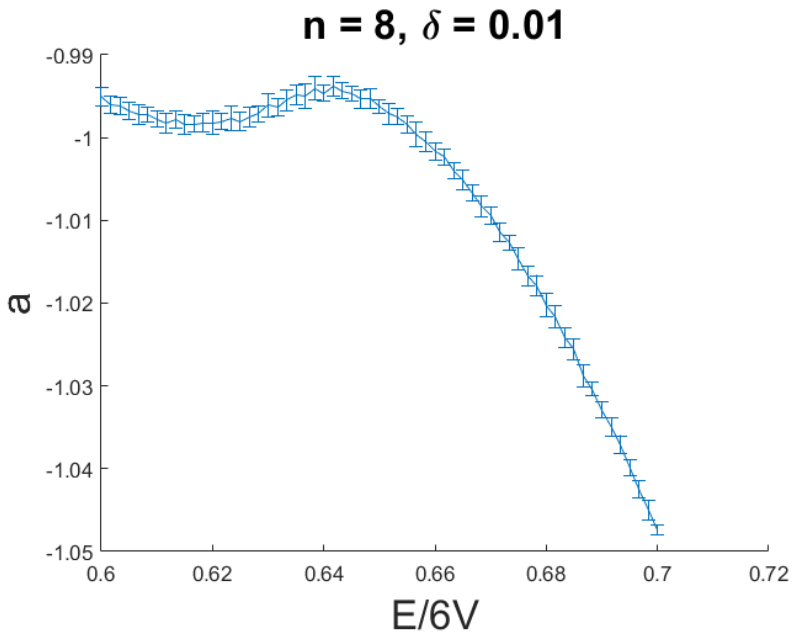
loop over Monte Carlo sweeps through the lattice
Evaluate $\Delta E_i(a_i^{(n)})$ using standard Monte Carlo
New Boltzmann weight and restricted energy interval!
End loop

Calculate new $a_i^{(n+1)}$
$a_i^{(n+1)} = a_i^{(n)} + \frac{12}{\delta^2(n+1)} \Delta E_i(a_i^{(n)})$
End loop

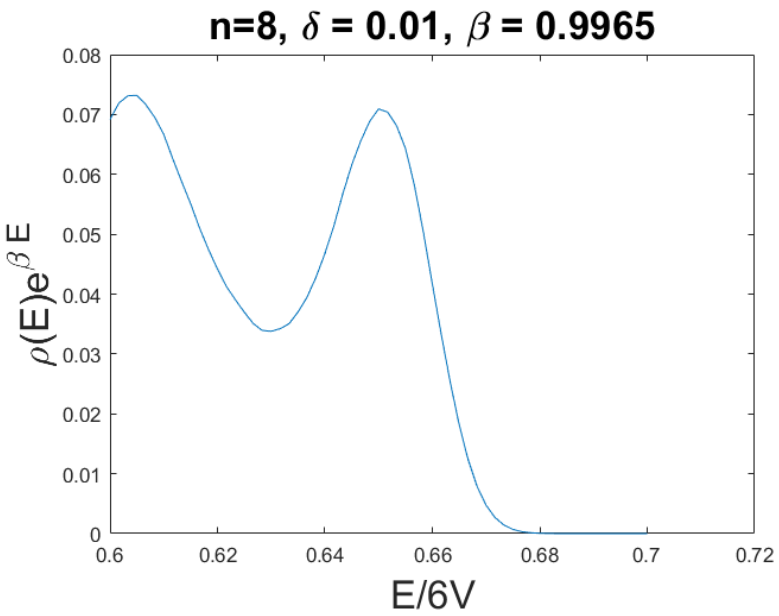
End loop

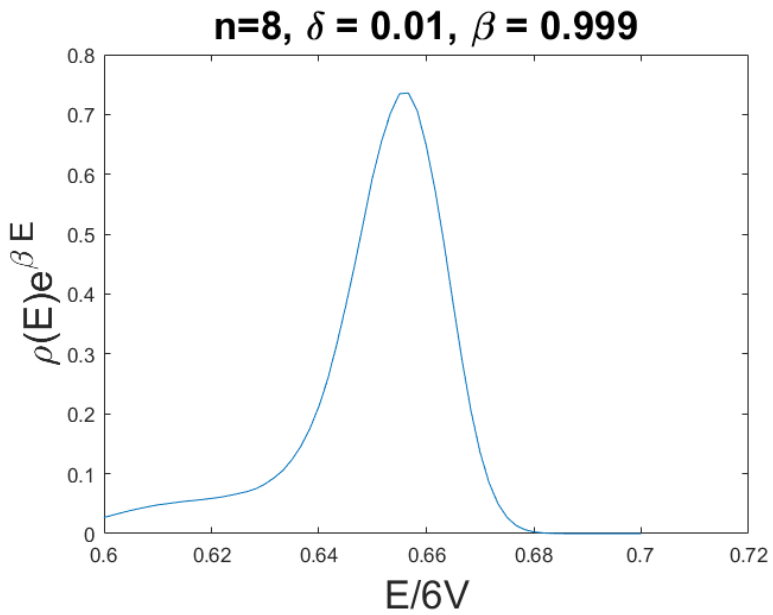
Compact U(1) Lattice Gauge Theory





U(1) Density of States





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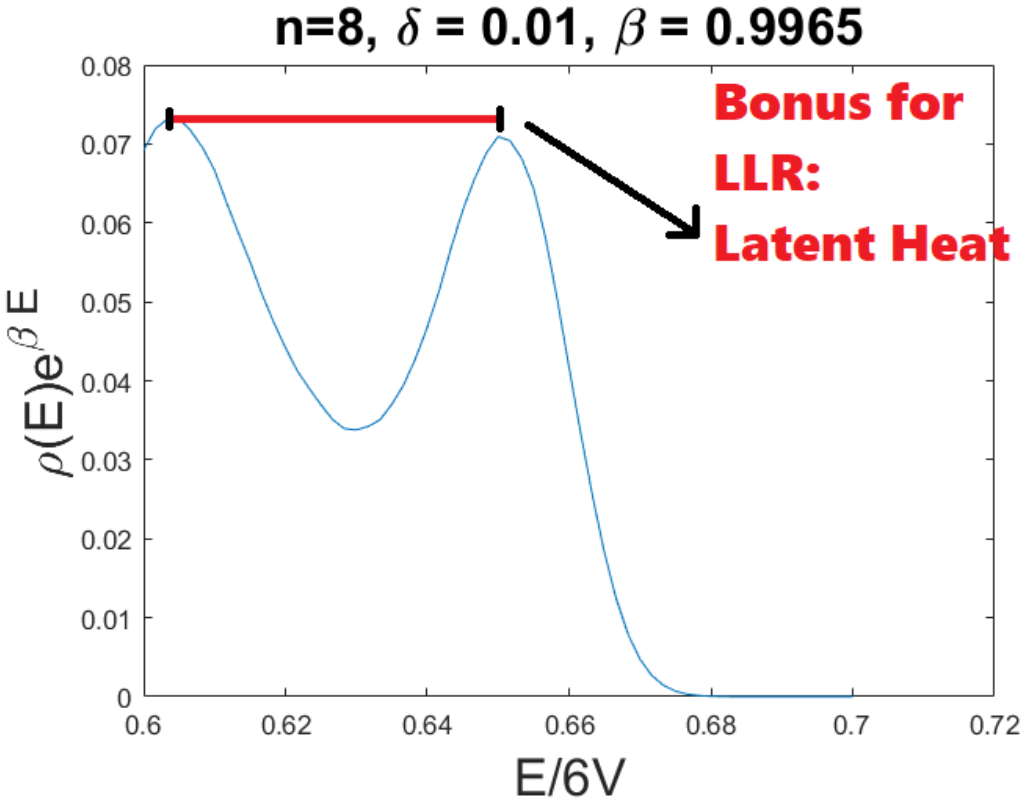

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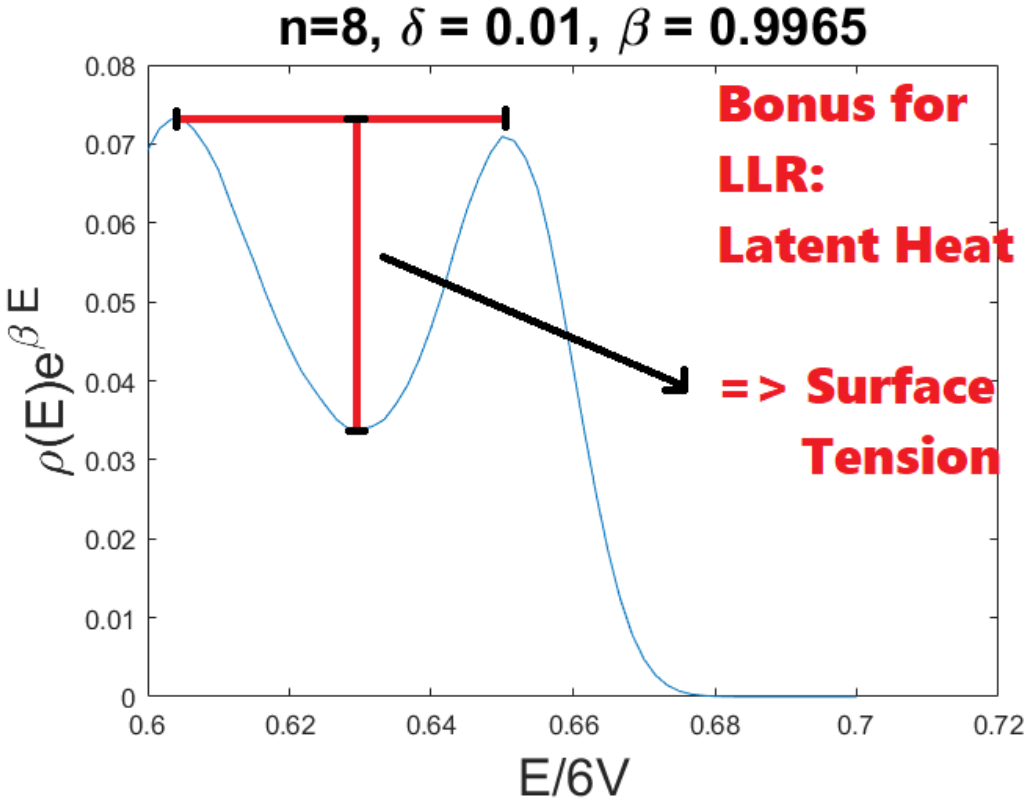
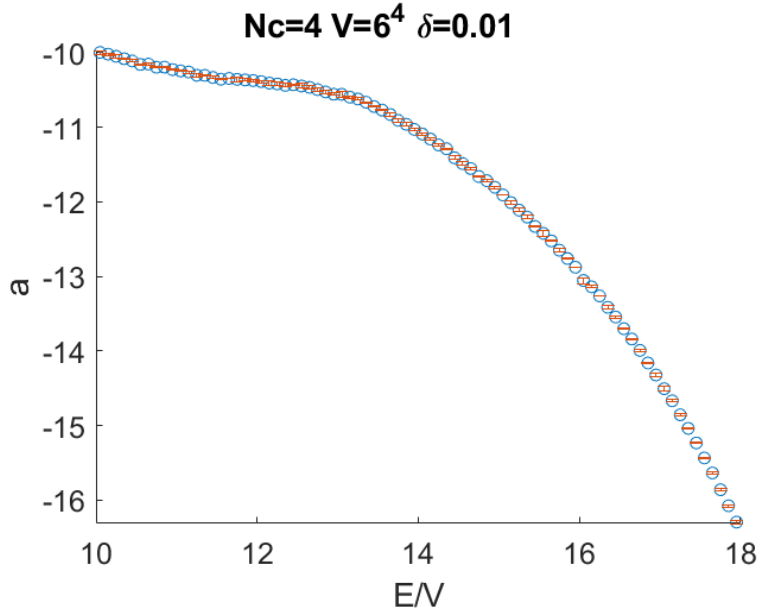
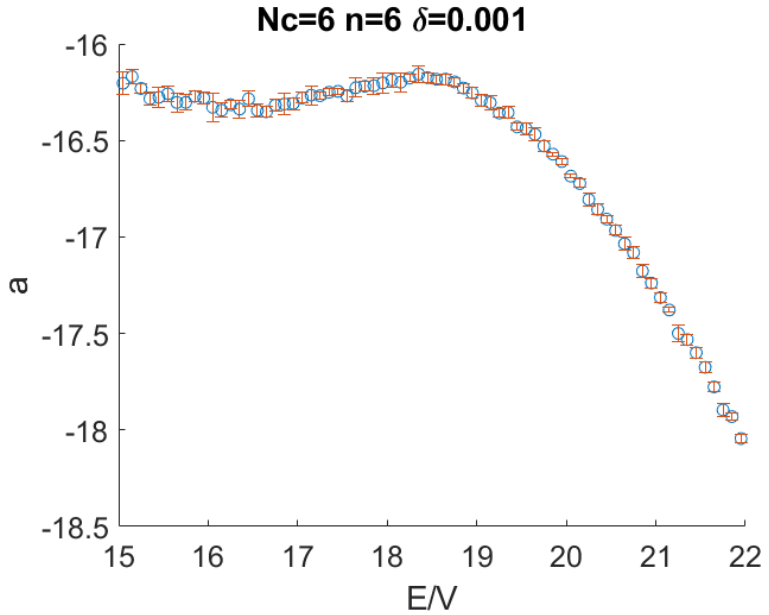


SU(N) Bulk Phase Transition





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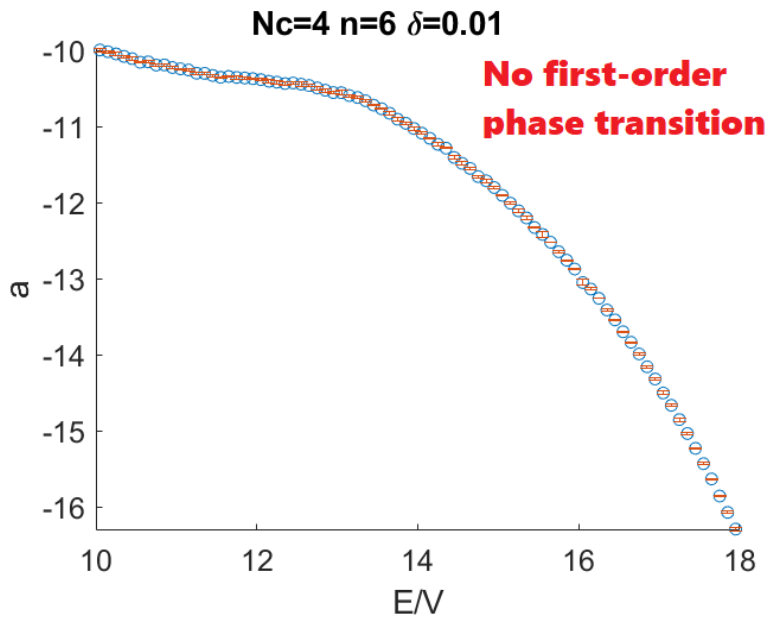
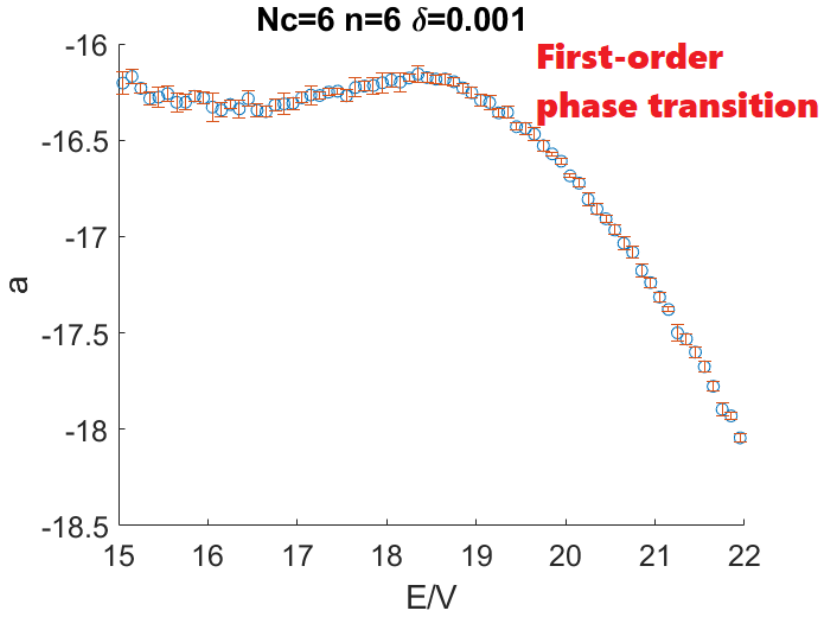
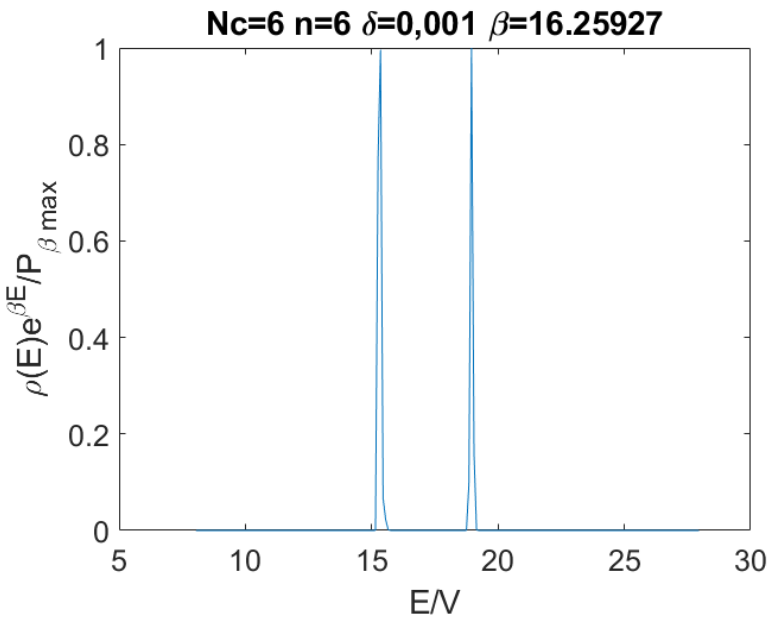
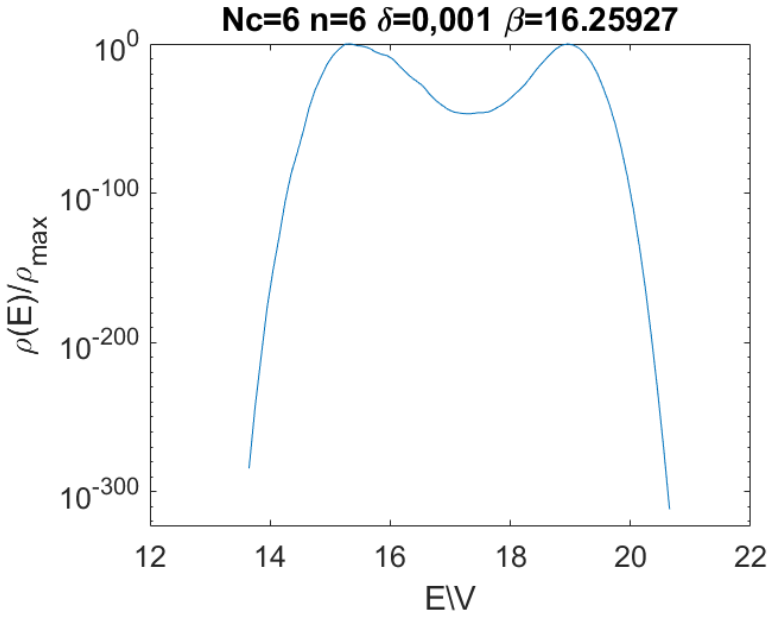


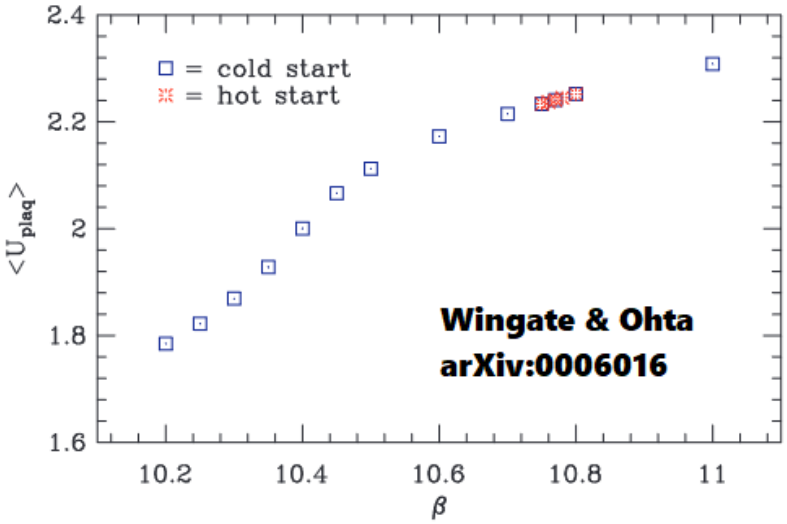


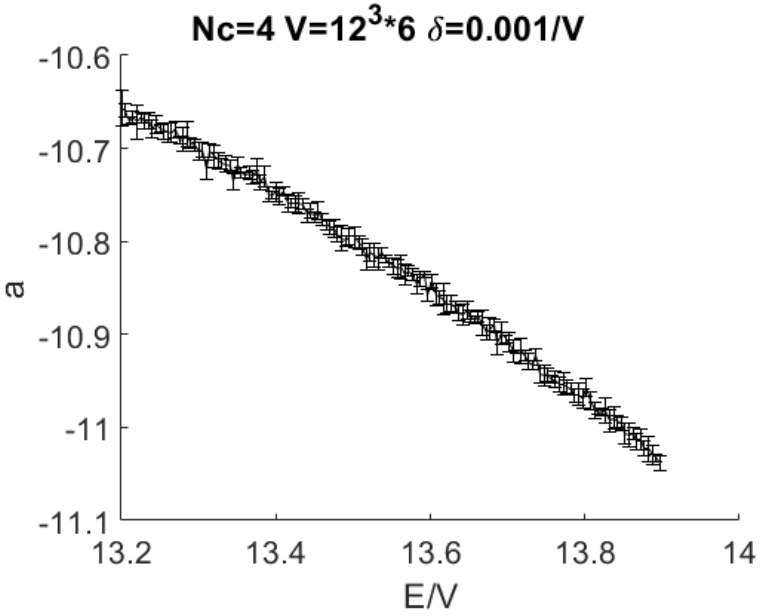
SU(6) Density of States





SU(4) Thermal Phase Transition





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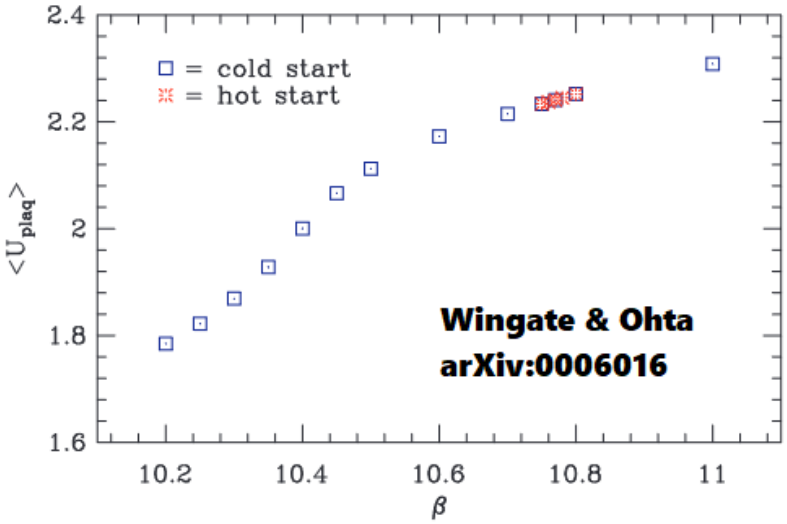
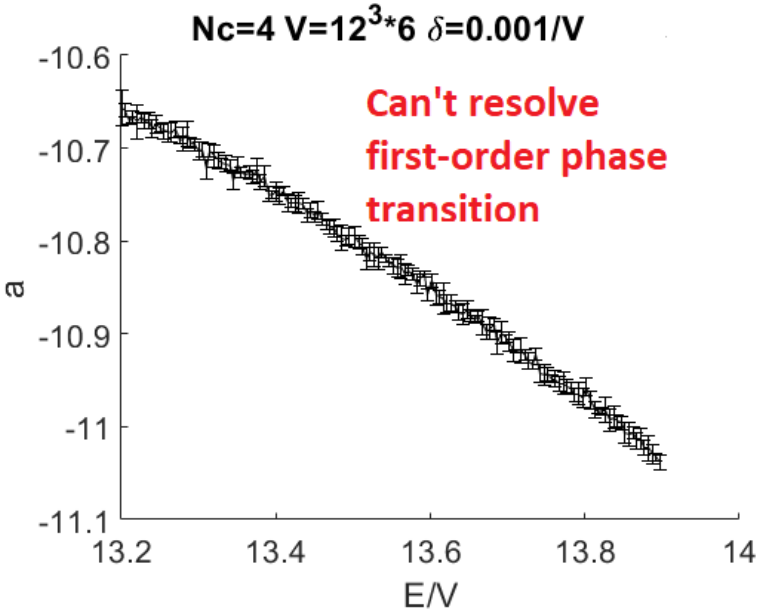
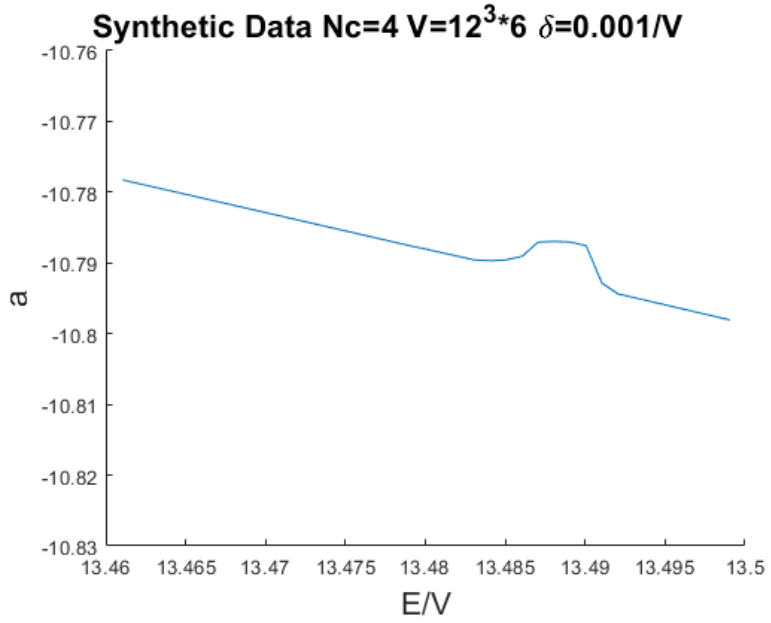
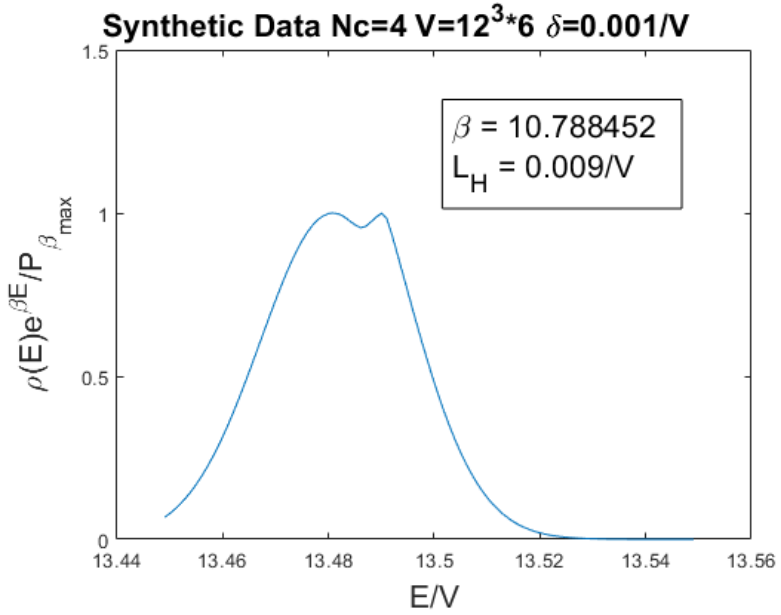




Synthetic Data





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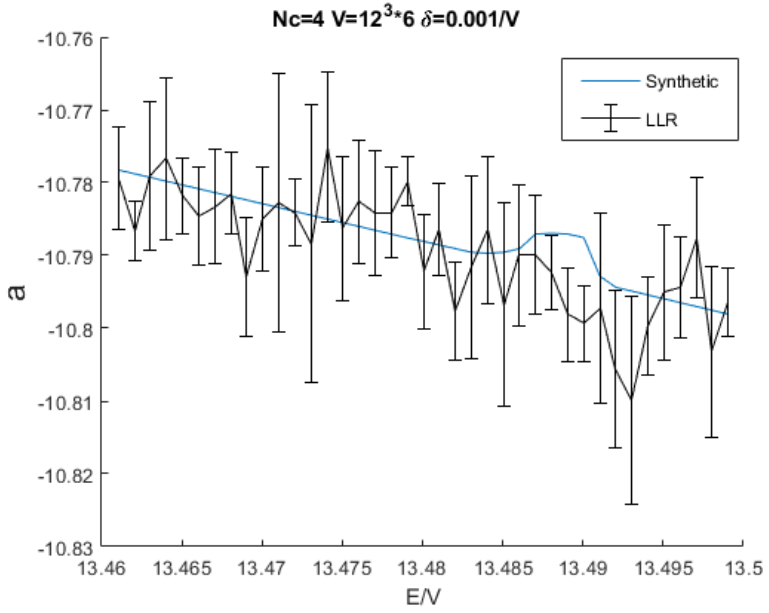

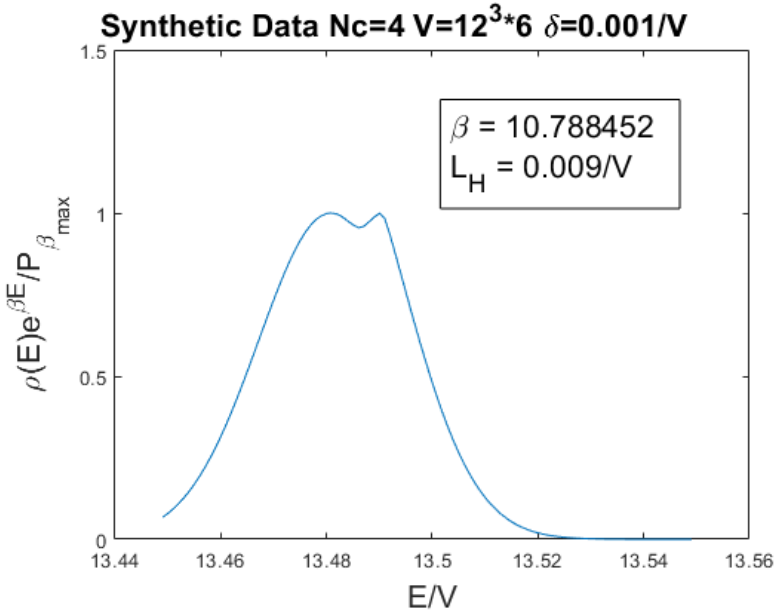

Recap

- Broad class of Composite Dark Matter models
- Gravitational waves offer new way to probe these models
- First order phase transition difficult with MCMC$\rightarrow$ LLR
- LLR works for strong bulk phase transitions (U(1), SU(N))
- Weaker confinement Transition not yet resolved
  - Improve statistics
  - Go to bigger lattice sizes and increase aspect ratios
  - Decrease $\delta = 0.001/V \rightarrow \delta = 0.0001/V$
  - Go to SU(N) with N $\geq$ 6