Reducing the Sign Problem using Line Integrals

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Introduction

We wish to solve complex valued integrals of the form

$$\langle O\rangle = \int d^N x \exp(-E(x))O(x)/\langle 1\rangle, \quad E \in \mathbb{C} \; (1)$$

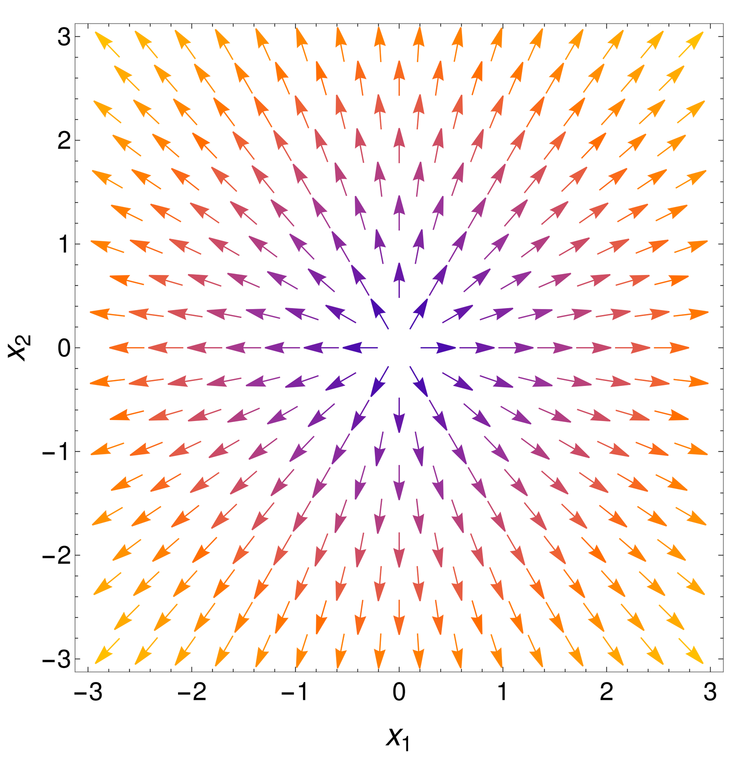
- Attempted solution: Sample lines instead of points, such that oscillations will cancel
- Line found by following the direction in which the imaginary part of the action $Im(E) \equiv E_{im}$ changes

Line is found by following the following differential equation

$$\frac{\partial E_{im}(x)}{\partial x_j} \equiv F_j(x)$$

$$F_j(x) = \frac{dx_j}{d\tau}$$

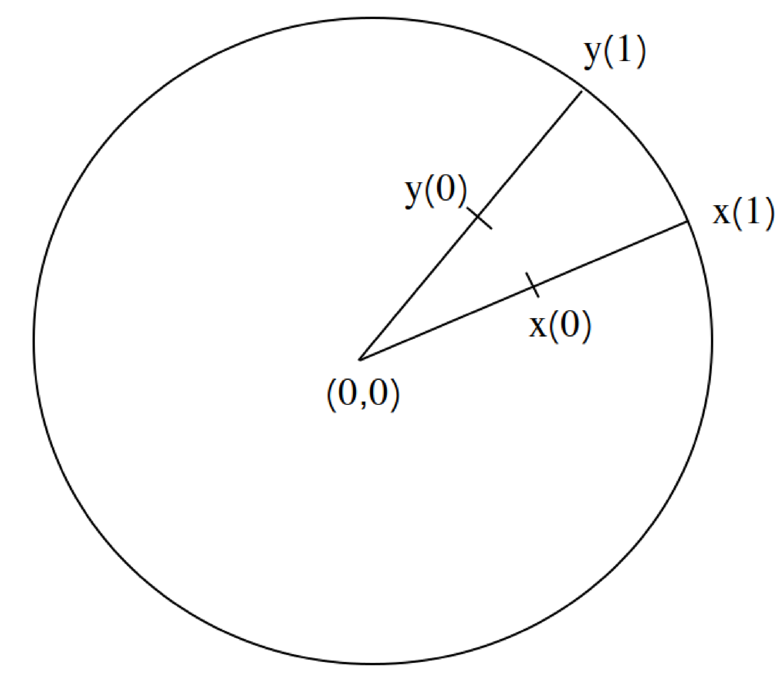
- Example: $E = i(x_1^2 + x_2^2)$



Plot of $F_j = 2x_j$

Relative Volume

- Example: $E = i(x_1^2 + x_2^2)$
  We start with 2 slightly different starting points $x(\tau = 0)$ and $y(\tau = 0)$. At later time $\tau = 1$ the points are further separated. Due to the change in the separation between 2 infinitesimally close points, we need to count contributions to the line integral as $V_{rel} = \left|\frac{x(1)}{x(0)}\right|$, in order to count every point in the integral equally.
- In general the relative change to the volume can be found as: $\frac{d\log(V_{rel}(\tau))}{d\tau} = \sum_j \frac{\partial^2 E_{im}(x(\tau))}{\partial^2 x_j}$



Sketch of example: The lines are created by starting at time $\tau = 0$. At time $\tau = 1$ the positions in the 2 lines are further away from each other.

Implementation

Line integral I_O implemented as a set of ordinary differential equations from initial position x_0

$$F_j(x) = \frac{\partial E_{im}}{\partial x_j} = \frac{dx_j}{d\tau}$$

$$\frac{ds}{d\tau} = \sqrt{\sum_j F_j(x)^2}$$

$$\frac{dJ}{d\tau} = \sum_j \frac{\partial^2 E_{im}}{\partial^2 x_j}$$

$$\frac{dI_O}{d\tau} = O(x(\tau))|F(x_0)| \times e^{-E(x(\tau))-g(s)+J}$$

- $J = \log(V_{rel}(\tau))$
- $|F(x_0)|$ can be absorbed into the initial conditions of J
- Arbitrary function $g(s)$ included. We will use $g(s) = (s/\sigma)^2$

Example: 1d real time anharmonic oscillator

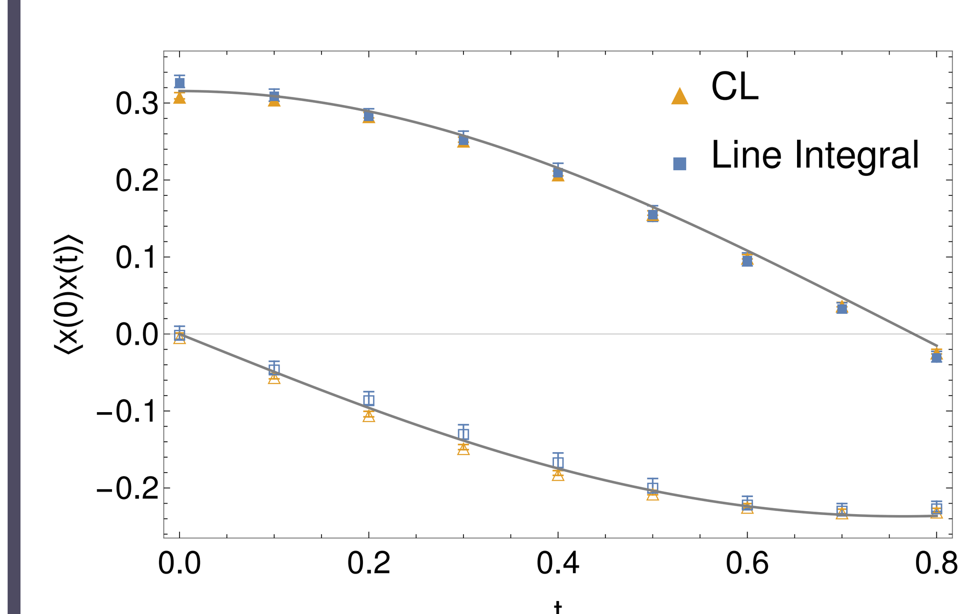
- As an example of the line integrals usage, we calculate the correlator $\langle x(0)x(t)\rangle$ for a real time 1d anharmonic oscillator, and compare with the solution from discretizing x

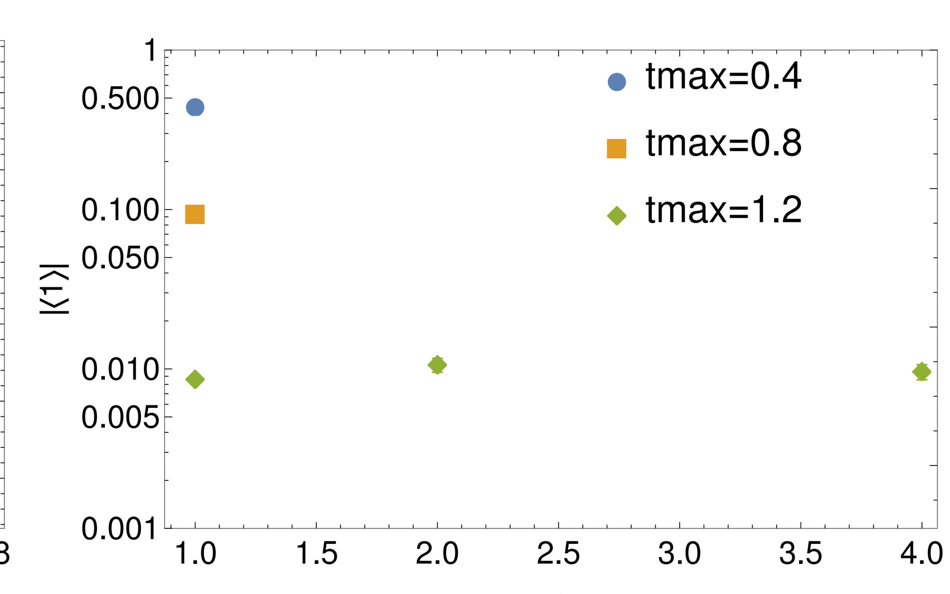
$$\langle O\rangle = \langle x(0)x(t)\rangle = Tr(e^{-\beta H} x e^{-itH} x e^{itH})/Tr(e^{-\beta H}) \quad (2)$$

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda x^4}{4!} \quad (3)$$

- We will look at the strongly coupled case $\lambda = 24$, $\beta = 1.0$, $\sigma = 1.0$
- We sample with the metropolis algorithm using $|I_1(x_j)|$:

$$\langle O\rangle = \frac{\sum_j I_O(x_j)/|I_1(x_j)|}{\sum_j I_1(x_j)/|I_1(x_j)|}, \quad I_O = I_O(\tau = \infty) - I_O(\tau = -\infty) \quad (4)$$





Conclusion

- Changed sampling from points to lines
- Written down the line integrals as a set of differential equations
- Reduced the sign problem for a 1d real time anharmonic oscillator
- Sign problem grows too much for large times, $|\langle 1\rangle|$ becomes too small