Determination of α_s value from tau decays with a renormalon-motivated approach

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Talk presented by G.C.

[work in progress, continuation of our work 2112.01992 and EPJC **81** (2021) 930 (arxiv: 2105.00356) and EPJC **82** (2022) 362 (arxiv: 2112.01992)]

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Introduction

- We use ALEPH data for the strangeless semihadronic tau decay in the sum rules.
- The weight functions in the sum rules are Borel-Laplace, as an additional filter we use the momenta $a^{(2,0)}$ and $a^{(2,1)}$.
- For the theoretical expression of the sum rules we will use an improved version of the truncated OPE.
  1. For $D = 0$ part we use Adler function $d(Q^2)_{D=0}$ whose higher order expansion coefficients are predicted by the structure of the leading IR and UV renormalons.
  2. The $D = 4$ term has the known structure $\sim 1/(Q^2)^2$, while $D = 6$ terms $\sim 1/(Q^2)^3$ are *supplemented* by the required terms $\sim \alpha_s (Q^2)^{k_j}/(Q^2)^3$ reflected by the renormalon structure of the constructed Adler function extension; we truncate at $D = 6$.

Introduction

- In the evaluation of the $D = 0$ part of the sum rules we apply the (truncated) FOPT, and the Borel resummation with PV (with truncated correction polynomial).
- For each truncation index, we extract the value of α_s from the fit of the sum rules with doubly-pinched Borel-Laplace weight function.
- The best truncation index is inferred from consideration of the (doubly-pinched) FESRs momenta $a^{(2,0)}$ and $a^{(2,1)}$.
- The averaged extracted value of α_s will be presented.

Sum rules

The Adler function $\mathcal{D}(Q^2)$ is logarithmic derivative of the quark current polarisation function $\Pi(Q^2)$

$$\mathcal{D}(Q^2) \equiv -2\pi^2 \frac{d\Pi(Q^2)}{d\ln Q^2}, \tag{1}$$

where $Q^2 \equiv -q^2$ $(= -(q^0)^2 + \vec{q}^2)$. We will consider the total (V+A)-channel, i.e., $\Pi(Q^2)$ will be the total (V+A)-channel polarisation function

$$\Pi(Q^2) = \Pi_{\rm V}^{(1)}(Q^2) + \Pi_{\rm A}^{(1)}(Q^2) + \Pi_{\rm A}^{(0)}(Q^2). \tag{2}$$

We neglect $\Pi_{\rm V}^{(0)}$ because $\mathrm{Im}\Pi_{\rm V}^{(0)}(-\sigma + i\epsilon) \propto (m_d - m_u)^2$ is negligible. According to the general principles of Quantum Field Theory, $\Pi(Q^2;\mu^2)$ and $\mathcal{D}(Q^2)$ are holomorphic (i.e., analytic) functions of Q^2 in the complex Q^2-plane with the excepcion of the real negative axis $(-\infty, -m_\pi^2)$. If $g(Q^2)$ is a (arbitrary) holomorphic function of Q^2, and we apply the Cauchy theorem to the integral $\oint dQ^2 g(Q^2)\Pi(Q^2;\mu^2)$ along a closed path in the complex Q^2-plane in Fig. 1, we obtain

Sum rules

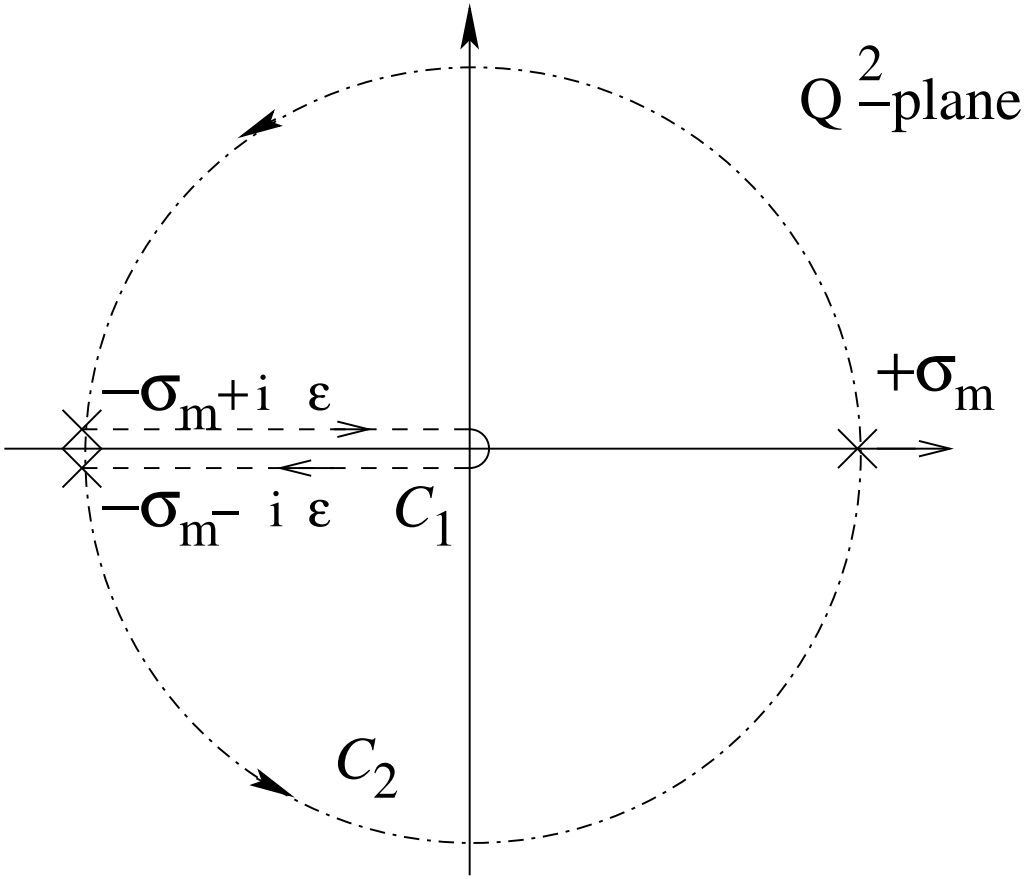

Figure: The closed integration path $C_1 + C_2$ for $\oint dQ^2 g(Q^2)\Pi(Q^2)$. The radius of the circle C_2 is $|Q^2| = \sigma_{\rm m}$ $(= 2.8\ {\rm GeV}^2)$. On the path C_1 we have $\varepsilon \to +0$.

Sum rules

$$\oint_{C_1+C_2} dQ^2 g(Q^2)\Pi(Q^2) = 0 \tag{3a}$$

$$\Rightarrow \int_0^{\sigma_{\rm m}} d\sigma g(-\sigma)\omega_{\rm exp}(\sigma) = -i\pi \oint_{|Q^2|=\sigma_{\rm m}} dQ^2 g(Q^2)\Pi_{\rm th}(Q^2), \tag{3b}$$

where $\omega(\sigma)$ is proportional to the discontinuity (spectral) function of the (V+A)-channel polarisation function

$$\omega(\sigma) \equiv 2\pi \ {\rm Im}\ \Pi(Q^2 = -\sigma - i\epsilon)\ . \tag{4}$$

Sum rules

Integration by parts replaces the theoretical polarisation function in the sum rule (3b) by the Adler function (1)

$$\int_0^{\sigma_{\rm m}} d\sigma g(-\sigma)\omega_{\rm exp}(\sigma) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, \mathcal{D}_{\rm th}(\sigma_{\rm m} e^{i\phi}) G(\sigma_{\rm m} e^{i\phi}), \tag{5}$$

where $\mathcal{D}_{\rm th}(Q^2)$ is given by the theoretical OPE expansion of the Adler function, and the (holomorphic) function G is an integral of g:

$$G(Q^2) = \int_{-\sigma_{\rm m}}^{Q^2} dQ'^2 g(Q'^2). \tag{6}$$

Sum rules

The quantity $\omega(\sigma)$ was measured to a high precision by ALEPH Collaboration, in semihadronic strangeless τ-decays, Fig. 2.

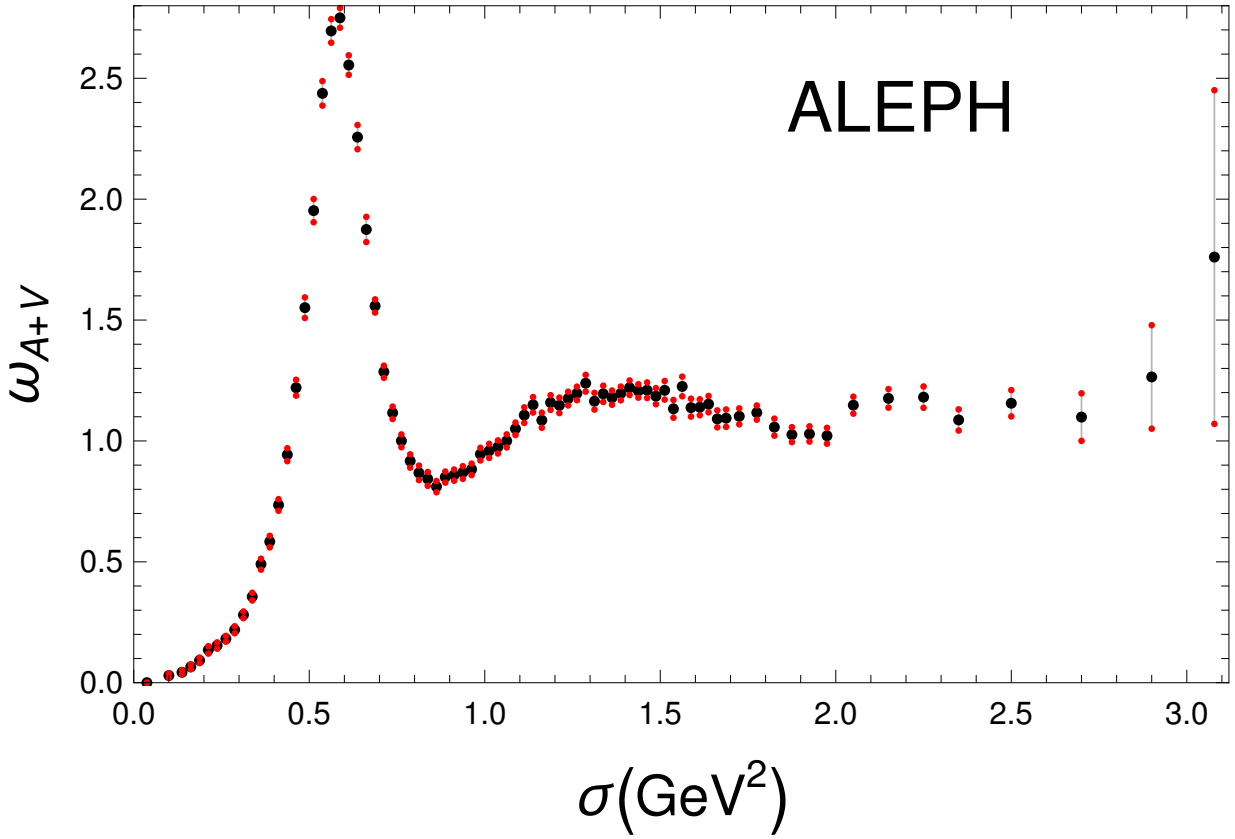

Figure: The spectral function $\omega(\sigma)$ for the (V+A)-channel, as measured by ALEPH Collaboration. The extremely narrow pion peak contribution $2\pi^2 f_\pi^2 \delta(\sigma - m_\pi^2)$ ($f_\pi = 0.1305$ GeV) has to be added to this. The last two bins have large uncertainties, so we exclude them, and this means that $\sigma_{\mathrm{m}} = 2.80\ \mathrm{GeV}^2$ in the sum rules.

Sum rules

The theoretical OPE expression of the polarisation function that is usually used in the literature has the following form:

$$\Pi_{\rm th}(Q^2;\mu^2) = -\frac{1}{2\pi^2}\ln\left(\frac{Q^2}{\mu^2}\right) + \Pi(Q^2)_{D=0} + \sum_{k\geq 2}\frac{\langle O_{2k}\rangle}{(Q^2)^k}\left(1+\mathcal{O}(a)\right). \quad (7)$$

The corresponding Adler function (1) is then

$$\mathcal{D}_{\rm th}(Q^2) \equiv -2\pi^2\frac{d\Pi_{\rm th}(Q^2)}{d\ln Q^2} = 1 + d(Q^2)_{D=0} + 2\pi^2\sum_{k\geq 2}\frac{k\langle O_{2k}\rangle}{(Q^2)^k}. \quad (8)$$

Sum rules

However, the theoretically better motivated OPE expansion of the Adler function has the following form, based on a complicated RGE-dependence of the higher D operators:

$$\mathcal{D}_{\rm th}(Q^2) \equiv -2\pi^2 \frac{d\Pi_{\rm th}(Q^2)}{d\ln Q^2} = d(Q^2)_{D=0} + 1 + 4\pi^2 \frac{\langle O_4 \rangle}{(Q^2)^2} + \frac{6\pi^2}{(Q^2)^3}\left[\langle O_6^{(2)}\rangle a(Q^2)^{k^{(2)}} + \langle O_6^{(1)}\rangle a(Q^2)^{k^{(1)}}\right] + \ldots, \qquad (9)$$

which has two different condensates at $D = 6$, with the approximate values where $k^{(2)} = \gamma^{(1)}(O_6^{(2)})/\beta_0 \approx 0.222$ $(= 1 - \kappa_2)$ and $k^{(1)} = \gamma^{(1)}(O_6^{(1)})/\beta_0 \approx 0.625$ $(= 1 - \kappa_1)$ (Boito, Hornung, Jamin (BHJ), 2015), where $\gamma^{(1)}(O_6^{(j)})$ is the effective leading-order anomalous dimension of the operator $O_6^{(j)}$ (BHJ, 2015).

Renormalon-motivated Adler function $d(Q^2)_{D=0}$

The perturbation expansion of $d(Q^2)_{D=0}$ in powers of $a(\mu^2) \equiv \alpha_s(\mu^2)/\pi$ is

$$d(Q^2)_{D=0,\mathrm{pt}} = d_0 a(\kappa Q^2) + d_1(\kappa)\, a(\kappa Q^2)^2 + \ldots + d_n(\kappa)\, a(\kappa Q^2)^{n+1} + \ldots, \tag{10}$$

(where $d_0 = 1$), and the expansion of the Borel transform $\mathcal{B}[d](u;\kappa)$ is

$$\mathcal{B}[d](u;\kappa) \equiv d_0 + \frac{d_1(\kappa)}{1!\beta_0} u + \ldots + \frac{d_n(\kappa)}{n!\beta_0^n} u^n + \ldots \ . \tag{11}$$

Renormalon-motivated Adler function $d(Q^2)_{D=0}$

The behaviour of $\mathcal{B}[d](u;\kappa)$ close to the renormalon IR singularities $u = 2, 3, \ldots$ is

$$\mathcal{B}[d](u;\kappa) \sim 1/(2-u)^{1+2\beta_1/\beta_0^2},\ 1/(2-u)^{2\beta_1/\beta_0^2}, \ldots\ (p=2), \tag{12a}$$

$$\mathcal{B}[d](u;\kappa) \sim 1/(p-u)^{\kappa_j^{(p)}+p\beta_1/\beta_0^2}, 1/(p-u)^{\kappa_j^{p)}-1+p\beta_1/\beta_0^2}, \ldots (p=3,4,\ldots) \tag{12b}$$

Here, $\kappa_j^{(p)} = 1 - \gamma^{(1)}(O_{2p}^{(j)})/\beta_0$, where $\gamma^{(1)}(O_{2p}^{(j)})$ is the effective leading-order anomalous dimension of the $D = 2p$ dimensional OPE operator $O_{2p}^{(j)}$ appearing in the OPE of the Adler function; $\beta_0 = (11 - 2N_f/3)/4$ and $\beta_1 = (1/16)(102 - 38N_f/3)$ are the first two β-function coefficients appearing in the RGE

$$\frac{d\,a(Q^2)}{d\ln Q^2} = -\beta_0 a(Q^2)^2 - \beta_1 a(Q^2)^3 - \bar{\beta}_2 a(Q^2)^4 - \ldots \tag{13}$$

Renormalon-motivated Adler function $d(Q^2)_{D=0}$

This, and the requirement that the Q^2-dependence of the dimension $D = 2p$ terms of the corresponding OPE for Adler $\mathcal{D}(Q^2)$ is the same as that of the renormalon ambiguity of the inverse Borel transformation (Borel integral)

$$\delta\mathcal{D}(Q^2)_{p,\kappa_j^{(p)}} \sim \frac{1}{\beta_0}\mathrm{Im}\int_{+i\varepsilon}^{+\infty+i\varepsilon} du \exp\left(-\frac{u}{\beta_0 a(Q^2)}\right) 1/(p-u)^{\kappa_j^{(p)}+p\beta_1/\beta_0^2}, \tag{14}$$

implies that the corresponding OPE terms are

$$\begin{aligned}\mathcal{D}(Q^2)_{D=2p,\kappa_j^{(p)}} &\sim \frac{1}{(Q^2)^p} a(Q^2)^{1-\kappa_j^{(p)}} \left[1+\mathcal{O}(a)\right] \\ &= \frac{1}{(Q^2)^p} a(Q^2)^{\gamma^{(1)}(O_{2p}^{(j)})/\beta_0} \left[1+\mathcal{O}(a)\right]. \end{aligned} \tag{15}$$

Renormalon-motivated Adler function $d(Q^2)_{D=0}$

In addition, there are also UV renormalon poles at $u = -p = -1, -2, \ldots$

$$\mathcal{B}[d](u;\kappa) \quad \sim \quad 1/(p+u)^{2-p\beta_1/\beta_0^2}, 1/(p+u)^{1-p\beta_1/\beta_0^2}, \ldots \ (p = 1, 2, \ldots), \tag{16}$$

where we assumed for the effective leading-order anomalous dimensions the values of the large-β_0 approximation.
If we, on the other hand, reorganise the expansion (10) of $d(Q^2)_{D=0}$ in powers of logarithmic derivatives

$$\widetilde{a}_{n+1}(Q^2) \equiv \frac{(-1)^n}{n!\beta_0^n}\left(\frac{d}{d\ln Q^2}\right)^n a(Q^2) \qquad (n = 0, 1, 2, \ldots), \tag{17}$$

[note: $\widetilde{a}_{n+1}(Q^2) = a(Q^2)^{n+1} + \mathcal{O}(a^{n+2})$] we obtain

$$d(Q^2)_{D=0,\mathrm{lpt}} = \widetilde{d}_0 a(\kappa Q^2) + \widetilde{d}_1(\kappa)\, \widetilde{a}_2(\kappa Q^2) + \ldots + \widetilde{d}_n(\kappa)\, \widetilde{a}_{n+1}(\kappa Q^2) + \ldots. \tag{18}$$

By use of the $(\overline{\mathrm{MS}})$ RGE, we can relate the new coefficients $\widetilde{d}_n$ with the original ones $d_n, d_{n-1}, \ldots$.

Renormalon-motivated Adler function $d(Q^2)_{D=0}$

We now construct the Borel transform $\mathcal{B}[\widetilde{d}](u)$ [related to the original Borel transform $\mathcal{B}[d](u)$ of the Adler function (11) by $d_n \mapsto \widetilde{d}_n$]

$$\mathcal{B}[\widetilde{d}](u;\kappa) \equiv \widetilde{d}_0 + \frac{\widetilde{d}_1(\kappa)}{1!\beta_0} u + \ldots + \frac{\widetilde{d}_n(\kappa)}{n!\beta_0^n} u^n + \ldots . \tag{19}$$

It turns out that this transform has the simple one-loop renormalisation scale dependence (in contrast to the original $\mathcal{B}[d](u)$)

$$\frac{d}{d\ln\kappa}\widetilde{d}_n(\kappa) = n\beta_0 \widetilde{d}_{n-1}(\kappa) \quad \Rightarrow \quad \mathcal{B}[\widetilde{d}](u;\kappa) = \kappa^u \mathcal{B}[\widetilde{d}](u). \tag{20}$$

Therefore, we will take the structure of renormalon singularities with $\beta_1 \mapsto 0$), i.e.,

$$\mathcal{B}[\widetilde{d}](u;\kappa) \sim 1/(2-u)^1, 1/(3-u)^{\kappa_2}, 1/(3-u)^{\kappa_1}, \ldots ; 1/(1+u)^2, 1/(1+u)^1, \ldots \tag{21}$$

where the powers $\kappa_2 = 0.778$ $[= 1 - k_2 = 1 - \gamma^{(1)}(O_6^{(2)})/\beta_0]$ and $\kappa_2 = 0.375$ $[= 1 - k_1 = 1 - \gamma^{(1)}(O_6^{(1)})/\beta_0]$ are obtained from BHJ (2015) [cf. Eq. (9)].
In addition, terms of "0" powers $\ln(1 - u/2), \ln(1 - u/3)$, etc. can appear.

Renormalon-motivated Adler function $d(Q^2)_{D=0}$

It can be shown then that this will imply for $\mathcal{B}[d](u;\kappa)$ the corresponding structures

$$
\begin{aligned}
\mathcal{B}[d](u;\kappa) &\sim \frac{1}{(2-u)^{1+2\beta_1/\beta_0^2}}\left[1+\mathcal{O}(2-u)\right], \\
\mathcal{B}[d](u;\kappa) &\sim \frac{1}{(3-u)^{\kappa_2+3\beta_1/\beta_0^2}}\left[1+\mathcal{O}(3-u)\right], \\
\mathcal{B}[d](u;\kappa) &\sim \frac{1}{(3-u)^{\kappa_1+3\beta_1/\beta_0^2}}\left[1+\mathcal{O}(3-u)\right], \\
\mathcal{B}[d](u;\kappa) &\sim \frac{1}{(1+u)^{2-1\beta_1/\beta_0^2}}\left[1+\mathcal{O}(1+u)\right], \\
\mathcal{B}[d](u;\kappa) &\sim \frac{1}{(1+u)^{1-1\beta_1/\beta_0^2}}\left[1+\mathcal{O}(1+u)\right]. \qquad (22)
\end{aligned}
$$

This is in agreement with the teoretical expectations Eqs. (12).

Renormalon-motivated Adler function $d(Q^2)_{D=0}$

The ansatz made for $\mathcal{B}[\widetilde{d}](u)$ (with $\kappa = 1$) in the $\overline{\mathrm{MS}}$ scheme includes the singularities at the locations $u = 2, 3$ and $u = -1$

$$\mathcal{B}[\widetilde{d}](u) = \exp\left(\widetilde{K} u\right) \pi \Big\{ \widetilde{d}_{2,1}^{\mathrm{IR}} \left[\frac{1}{(2-u)} + \widetilde{\alpha}(-1) \ln\left(1 - \frac{u}{2}\right) \right] + \frac{\widetilde{d}_{3,2}^{\mathrm{IR}}}{(3-u)^{\kappa_2}} + \frac{\widetilde{d}_{3,1}^{\mathrm{IR}}}{(3-u)^{\kappa_1}} + \frac{\widetilde{d}_{1,2}^{\mathrm{UV}}}{(1+u)^2} \Big\}, \tag{23}$$

where the value of the parameter $\widetilde{\alpha}$ in the $\overline{\mathrm{MS}}$ scheme is fixed, $\widetilde{\alpha} = -0.255$ (G.C.(2019, PRD)). The other five parameters ($\widetilde{K}$ and the residues $\widetilde{d}_{2,1}^{\mathrm{IR}}, \widetilde{d}_{3,2}^{\mathrm{IR}}, \widetilde{d}_{3,1}^{\mathrm{IR}}, \widetilde{d}_{1,2}^{\mathrm{UV}}$) are determined by the knowledge of the first five coefficients d_n (and thus $\widetilde{d}_n$), $n = 0, 1, 2, 3, 4$. We take as the central value of d_4 the value $d_4 = 275.$, obtained by ECH (Kataev,Starshenko(1995)). Other estimates are $d_4 = 277 \pm 51$ (Boito et al.(2018)); $d_4 = 283$ (Beneke, Jamin(2008)); $d_4 = 338.19$ from this type of model in the miniMOM scheme (G.C.(2019)). The variation from $d_4 = 275.$ to $d_4 = 338.19$ we include by

$$d_4 = 275. \pm 63. \tag{24}$$

Renormalon-motivated Adler function $d(Q^2)_{D=0}$

Table: The values of $\widetilde{K}$ and of the renormalon residues $\widetilde{d}^{\rm X}_{i,j}$ (X=IR,UV) for the five-parameter ansatz (23) in the $\overline{\rm MS}$ scheme, when d_4 is taken according to Eq. (24).

d_4	$\widetilde{K}$	$\widetilde{d}^{\rm IR}_{2,1}$	$\widetilde{d}^{\rm IR}_{3,2}$	$\widetilde{d}^{\rm IR}_{3,1}$	$\widetilde{d}^{\rm UV}_{1,2}$
275.	0.16010	0.661852	2.04546	-0.68316	-0.0121699
275. - 63.	-0.33879	0.986155	6.75278	-2.74029	-0.011647
275. + 63.	0.5190	1.10826	-0.481538	-0.511642	-0.0117704

Renormalon-motivated Adler function $d(Q^2)_{D=0}$

Table: The $\overline{\mathrm{MS}}$ coefficients $\widetilde{d}_n$ and d_n (with $\kappa = 1$) ($n \leq 10$) for the case 275. [cf. Eq. (24)].

n	$d_4 = 275.$: $\widetilde{d}_n$	d_n
0	1	1
1	1.63982	1.63982
2	3.45578	6.37101
3	26.3849	49.0757
4	-25.4181	275.
5	1859.36	3206.48
6	-19035.2	16901.6
7	421210.	358634.
8	-7.80444×10^6	621177.
9	1.82502×10^8	7.52194×10^7
10	-4.43137×10^9	-5.21168×10^8

Specific sum rules used

Weight functions $g(Q^2)$ used in the sum rules (5):
The double-pinched Borel-Laplace transforms $B(M^2)$ where M^2 is a complex squared energy parameter:

$$g_{M^2}(Q^2) \;=\; \left(1+\frac{Q^2}{\sigma_{\rm m}}\right)^2 \frac{1}{M^2}\exp\left(\frac{Q^2}{M^2}\right) \qquad \Rightarrow \tag{25}$$

On the other hand, one can use FESRs with (double-pinched) momenta $a^{(2,n)}$ which are associated with the following weight functions $g^{(2,n)}$ $(n=0,1,\ldots)$:

$$g^{(2,n)}(Q^2) \;=\; \left(\frac{n+3}{n+1}\right)\frac{1}{\sigma_m}\left(1+\frac{Q^2}{\sigma_m}\right)^2 \sum_{k=0}^{n}(k+1)(-1)^k\left(\frac{Q^2}{\sigma_m}\right)^k, \tag{26}$$

Methods of evaluation of the $D = 0$ contribution

1.) Fixed Order Perturbation Theory using powers (FO): The truncated power expansion $d(\sigma_{\rm m} e^{i\phi})^{[N_t]}_{D=0,{\rm pt}}$ [cf. Eq. (10)]

$$d(\sigma_{\rm m} e^{i\phi})^{[N_t]}_{D=0,{\rm pt}} = a(\sigma_{\rm m} e^{i\phi}) + \sum_{n=1}^{N_t-1} d_n(\kappa) a(\sigma_{\rm m} e^{i\phi})^{n+1}, \qquad (27)$$

which appears in the contour integrals in the sum rules, Eqs. (45) and (48b), is written as truncated Taylor expansion in powers of $a(\sigma_{\rm m})$ up to (and including) $a(\sigma_{\rm m})^{N_t}$.

Methods of evaluation of the $D = 0$ contribution

3.) Inverse Borel Transformation with Principal Value (PV): The expression for the $D = 0$ part of the Adler function in the contour integrals is written as

$$\left(d(\sigma_{\rm m} e^{i\phi})_{D=0}\right)^{({\rm PV},[N_t])} = \frac{1}{\beta_0}\frac{1}{2}\left(\int_{\mathcal{C}_+} + \int_{\mathcal{C}_-}\right) du \exp\left[-\frac{u}{\beta_0 a(\kappa\sigma_{\rm m} e^{i\phi})}\right] \mathcal{B}[d](u;\kappa)_{\rm sing} + \delta d(\sigma_{\rm m} e^{i\phi};\kappa)^{[N_t]}_{D=0}, \quad (28)$$

where $\mathcal{B}[d](u;\kappa)_{\rm sing}$ is the singular part of the Borel transform of $d(Q^2)_{D=0}$, the arithmetic average over the integration paths $\mathcal{C}_\pm$ gives the Principal Value, and $\delta d(\sigma_{\rm m} e^{i\phi};\kappa)^{[N_t]}_{D=0}$ is the truncated series in powers of $a(\sigma_{\rm m} e^{i\phi})$ which completes the power terms corresponding to the Inverse Borel Transform of the singular part; we refer for a more detailed explanation to (C.A., G.C., D.T. 2021, EPJC [Sec. IV.B there]).

Results of fitting

We use the V+A channel of ALEPH, with $\sigma \leq \sigma_{\rm m} = 2.8\ {\rm GeV}^2$. The Borel-Laplace sum rules are applied in practice to the Real parts

$$\mathrm{Re}B_{\rm exp}(M^2;\sigma_{\rm m}) = \mathrm{Re}B_{\rm th}(M^2;\sigma_{\rm m}), \tag{29}$$

where for the Borel-Laplace scale parameters M^2 we take $M^2 = |M^2|\exp(i\Psi)$, where $0 \leq \Psi < \pi/2$. Specifically, we take $0.9\ {\rm GeV}^2 \leq |M^2| \leq 1.5\ {\rm GeV}^2$, and $\Psi = 0, \pi/6, \pi/4$.
We minimised the difference between the two quantities (29) by minimising, with respect to 4 parameters (α_s, $\langle O_4\rangle$, $\langle {O_4}^{(1)}\rangle$ and $\langle {O_4}^{(2)}\rangle$) the sum of squares

$$\chi^2 = \sum_{\alpha=0}^{n}\left(\frac{\mathrm{Re}B_{\rm th}({M^2}_\alpha;\sigma_{\rm m}) - \mathrm{Re}B_{\rm exp}({M^2}_\alpha;\sigma_{\rm m})}{\delta_B({M^2}_\alpha)}\right)^2, \tag{30}$$

where M^2_α is a set of 9 points along the chosen rays with $\Psi = 0, \pi/6, \pi/4$ and $0.9\ {\rm GeV}^2 \leq |M|^2 \leq 1.5\ {\rm GeV}^2$. Further, $\delta_B({M^2}_\alpha)$ are the experimental standard deviations of $\mathrm{Re}B_{\rm exp}({M^2}_\alpha;\sigma_{\rm m})$. We usually get very small $\chi^2 \lesssim 10^{-3}$.

Results of fitting

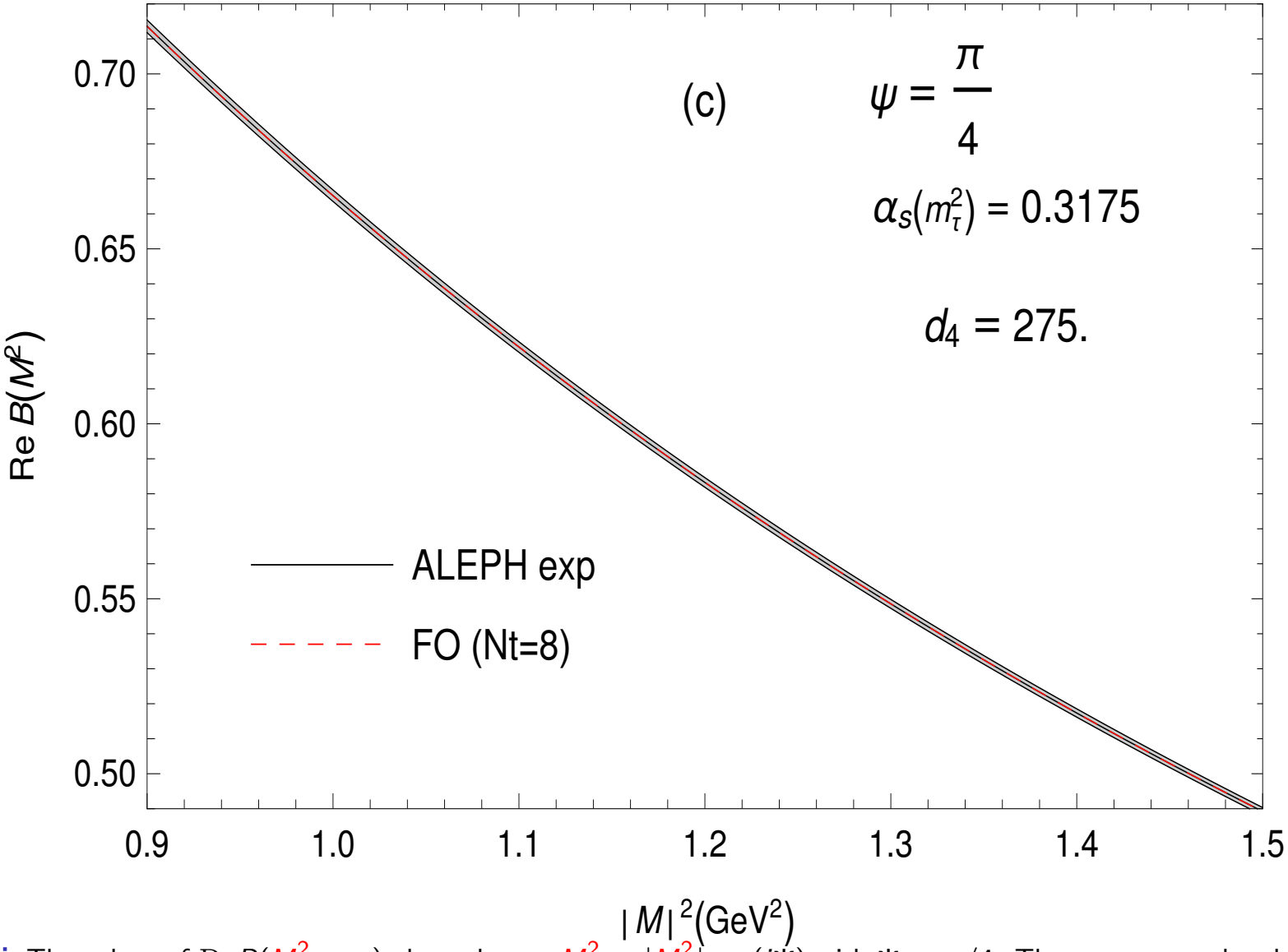

Figure: The values of $\mathrm{Re}B(M^2;\sigma_{\mathrm{m}})$ along the ray $M^2 = |M^2|\exp(i\Psi)$ with $\Psi = \pi/4$. The narrow grey band are the experimental predictions. The red dashed line is the result of the FOPT global fit with truncation index $N_t = 8$.

Results of fitting and Conclusions

The extracted values for α_s are

$$\begin{aligned}
\alpha_s(m_\tau^2)^{(\mathrm{FO})} &= 0.3175 \pm 0.0023(\exp)^{-0.0007}_{+0.0089}(\kappa)^{-0.0081}_{+0.0050}(d_4)^{+0.0033}_{-0.0028}(N_t) \\
&= 0.3175^{+0.0109}_{-0.0089}(N_t = 8^{+2}_{-3}), \\
\alpha_s(m_\tau^2)^{(\mathrm{PV})} &= 0.3193 \pm 0.0024(\exp)^{-0.0013}_{-0.0010}(\kappa)^{-0.0086}_{+0.0035}(d_4) \mp 0.0001(N_t) \\
&\quad \mp 0.0003(\mathrm{amb}) = 0.3193^{+0.0043}_{-0.0091} \quad (N_t = 5^{+2}_{-1}). \qquad (31)
\end{aligned}$$

Experimental uncertaities were obtained by the method of Boito, Peris et al. (2011).
The central values were extracted for the truncation index $N_t = 8, 5$ for the methods FO, and PV, respectively. Renormalisation scale parameter κ varies around $\kappa = 1$ in the interval: $1/2 \leq \kappa \leq 2$.

Results of fitting and Conclusions

The average of the two methods gives

$$\alpha_s(m_\tau^2) = 0.3184^{+0.0044}_{-0.0091} \tag{32}$$

$$\Rightarrow \ \alpha_s(M_Z^2) = 0.1185^{+0.0005}_{-0.0012}. \tag{33}$$

PDG 2022 gives for average of lattice groups: $\alpha_s(M_Z^2) = 0.1182 \pm 0.0008$; and the average of the nonlattice groups: $\alpha_s(M_Z^2) = 0.1176 \pm 0.0010$.

From(31) we see that the theoretical uncertainties due to the uncertainty of the value of d_4 coefficient are the most important. They are larger than the experimental uncertainties. This indicates that in the considered process, the theory remains behind the experiment. Semihadronic τ-decays remain a challenge for perturbative QCD.

Thank you for your attention.

Appendix A: Further details of the results

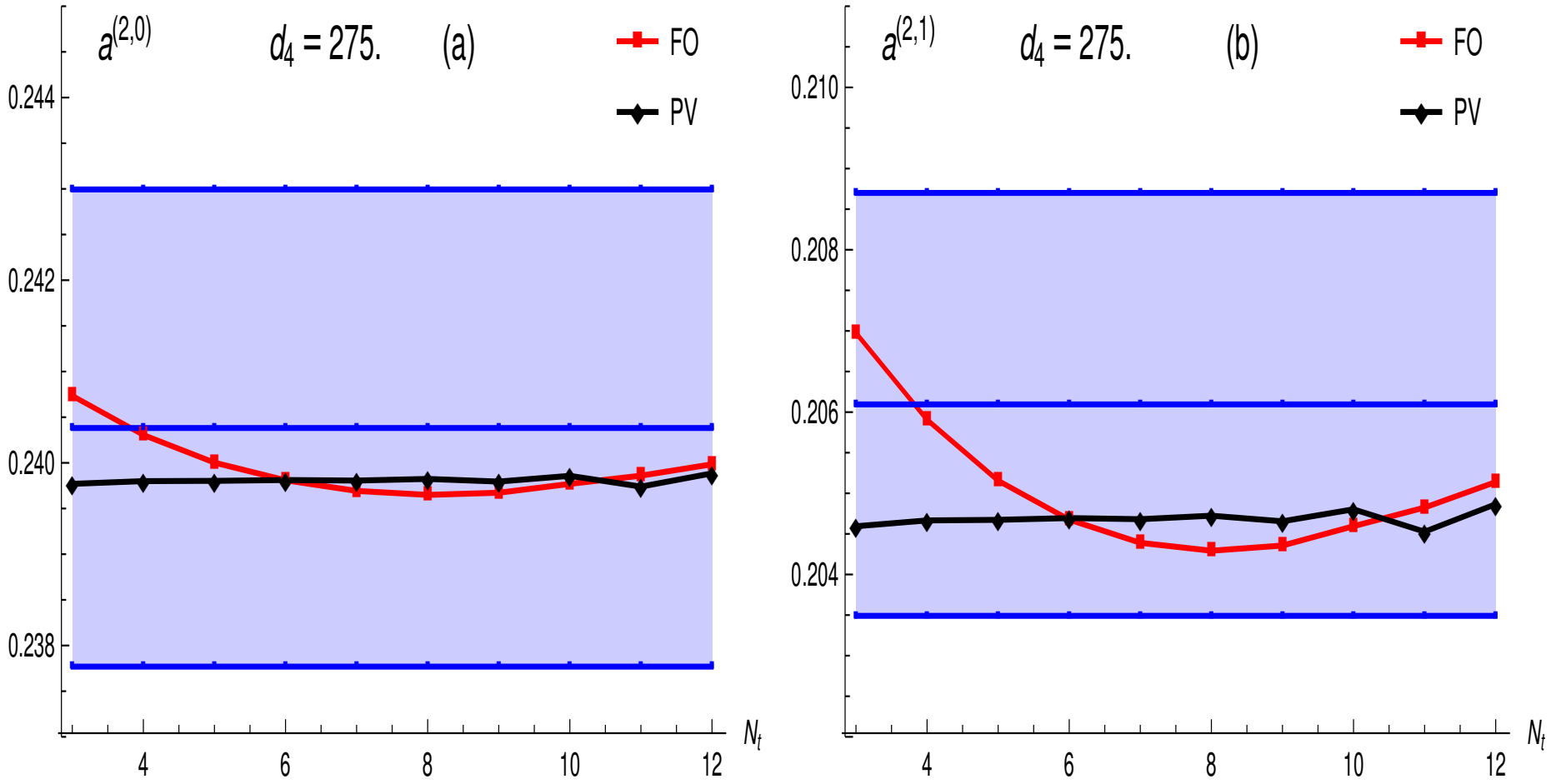

Figure: The moment $a^{(2,0)}(\sigma_{\rm m})$ (a) and $a^{(2,1)}(\sigma_{\rm m})$ (b), as a function of the truncation index N_t, for FO and PV approaches. At each N_t, the corresponding values of the parameters α_s and $\langle O_D \rangle$ obtained from the Borel-Laplace fit were used. The light blue band represents the experimental values (based on the ALEPH data). The best stability under variation of N_t is for $N_t = 8, 5$, respectively.

Appendix A: Further details of the results

Table: The results for $\alpha_s(m_\tau^2)$ and the condensates of the full V+A channel, $\langle O_4 \rangle$ and the effective $D = 6$ condensate $\langle O_6^{(\mathrm{eff})} \rangle = a(\sigma_\mathrm{m})^{k_2} \langle O_6^{(2)} \rangle + a(\sigma_\mathrm{m})^{k_1} \langle O_6^{(1)} \rangle$, both in units of 10^{-3} GeV^6.

method	$\alpha_s(m_\tau^2)$	$\langle O_4 \rangle$	$\langle O_6^{(\mathrm{eff})} \rangle$	N_t	χ^2
FOPT	$0.3175^{+0.0109}_{-0.0089}$	$-3.8^{+3.5}_{-5.6}$	$+2.4^{+1.8}_{-1.9}$	8	1.4×10^{-3}
PV	$0.3193^{+0.0043}_{-0.0091}$	-2.6 ± 0.9	$+2.4^{+1.7}_{-1.0}$	5	0.9×10^{-3}

Appendix A: Further details of the results

1.) If we took, on the other hand, instead of the improved $D = 6$ contributions, the large-β_0 effective leading-order anomalous dimensions of the $D = 6$ operators $[\gamma^{(1)}(O_6^{(2)})/\beta_0 = -1,\ \gamma^{(1)}(O_6^{(1)}) = 0]$ (EPJC, 2022), leads to higher values $\alpha_s(m_\tau^2) = 0.3235^{+0.0138}_{-0.0126}$ $[\alpha_s(M_Z^2) = 0.1191 \pm 0.0016]$.
2.) When $D = 6$ OPE terms are taken as nonrunning (i.e., $\gamma^{(1)}(O_D^{(1)}) = 0$), as we did in (EPJC, 2021), leads to the central value $\alpha_s(m_\tau^2) = 0.3164$ $[\alpha_s(M_Z^2) = 0.1182]$. We note that in our work (EPJC, 2021) we used the central value $d_4 = 338$.

Appendix A: Further details of the results

Table: The values of $\alpha_s(m_\tau^2)$, extracted by various groups applying sum rules and various methods to the ALEPH τ-decay data.

group	sum rule	FO	CI	PV	average
Baikov et al.,2008	$a^{(2,1)} = r_\tau$	0.322 ± 0.020	0.342 ± 0.011	—	0.332 ± 0.016
Beneke&Jamin, 2008	$a^{(2,1)} = r_\tau$	$0.320^{+0.012}_{-0.007}$	—	0.316 ± 0.006	0.318 ± 0.006
Caprini, 2020	$a^{(2,1)} = r_\tau$	—	—	0.314 ± 0.006	0.314 ± 0.006
Davier et al., 2013	$a^{(i,j)}$	0.324	0.341 ± 0.008	—	0.332 ± 0.012
Pich&R.Sánchez, 2016	$a^{(i,j)}$	0.320 ± 0.012	0.335 ± 0.013	—	0.328 ± 0.013
Boito et al., 2014	DV in $a^{(i,j)}$	0.296 ± 0.010	0.310 ± 0.014	—	0.303 ± 0.012
our work, 2021	BL (O_6, O_8)	0.308 ± 0.007		$0.316^{+0.008}_{-0.006}$	0.312 ± 0.007
our work, 2022	BL ($O_6{}^{(1)}, O_6{}^{(2)}$)	$0.323^{+0.013}_{-0.012}$ $0.321^{+0.021}_{-0.030}(\widetilde{\mathrm{FO}})$		$0.327^{+0.027}_{-0.009}$	0.324 ± 0.013
this work	BL ($O_6{}^{(k_1)}, O_6{}^{(k_2)}$)	$0.317^{+0.011}_{-0.009}$		$0.319^{+0.004}_{-0.009}$	$0.318^{+0.004}_{-0.009}$

Appendix B: Borel of Adler function

$$
\begin{aligned}
d(Q^2)_{D=0,\mathrm{pt}} &= d_0 a(\kappa Q^2) + d_1(\kappa)\, a(\kappa Q^2)^2 + \ldots + d_n(\kappa)\, a(\kappa Q^2)^{n+1} + \ldots, \\
d(Q^2)_{D=0,\mathrm{lpt}} &= \widetilde{d}_0 a(\kappa Q^2) + \widetilde{d}_1(\kappa)\, \widetilde{a}_2(\kappa Q^2) + \ldots + \widetilde{d}_n(\kappa)\, \widetilde{a}_{n+1}(\kappa Q^2) + \ldots .
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\widetilde{a}_{n+1}(Q'^2) &\equiv \frac{(-1)^n}{n!\beta_0^n} \left(\frac{d}{d \ln Q'^2} \right)^n a(Q'^2) \qquad (n = 0, 1, 2, \ldots), && (35a) \\
&= a(Q'^2)^{n+1} + \sum_{m \geq 1} k_m(n+1)\, a(Q'^2)^{n+1+m}, && (35b)
\end{aligned}
$$

$$
\Rightarrow\ a(Q'^2)^{n+1} = \widetilde{a}_{n+1}(Q^2) + \sum_{m \geq 1} \widetilde{k}_m(n+1)\, \widetilde{a}_{n+1+m}(Q'^2), \tag{36}
$$

$$
\Rightarrow\ \widetilde{d}_n(\kappa) = d_n(\kappa) + \sum_{s=1}^{n-1} \widetilde{k}_s(n+1-s)\, d_{n-s}(\kappa) \quad (\widetilde{d}_0 = d_0 = 1). \tag{37}
$$

Appendix B: Borel of Adler function

The ansatz for general $\kappa \equiv \mu^2/Q^2$ ($\widetilde{\kappa} \equiv \kappa \exp(\widetilde{K})$):

$$
\begin{aligned}
&\mathcal{B}[\widetilde{d}](u;\kappa) = \\
&= \pi\Bigg\{ \widetilde{d}^{\rm IR}_{2,1}(\widetilde{\kappa})\left[\frac{1}{(2-u)} + \left(-\widetilde{d}^{\rm IR}_{2,0}(\widetilde{\kappa}) + \widetilde{d}^{\rm IR}_{2,-1}(\widetilde{\kappa})(2-u)\right)\ln\left(1-\frac{u}{2}\right)\right] \\
&\quad + \frac{\widetilde{d}^{\rm IR}_{3,2}(\widetilde{\kappa})}{(3-u)^{\kappa_2}} + \frac{\widetilde{d}^{\rm IR}_{3,1}(\widetilde{\kappa})}{(3-u)^{\kappa_1}} + \frac{\widetilde{d}^{\rm UV}_{1,2}(\widetilde{\kappa})}{(1+u)^2}\Bigg\} \\
&= \widetilde{d}_0 + \frac{\widetilde{d}_1(\kappa)}{1!\beta_0}u + \ldots + \frac{\widetilde{d}_n(\kappa)}{n!\beta_0^n}u^n + \ldots .
\end{aligned}
\tag{38}
$$

Appendix B: Borel of Adler function

This generates $\widetilde{d}_n(\kappa)$, thus $d_n(\kappa)$, thus expansion of $\mathcal{B}[d](u)$ has the structure:

$$\frac{1}{\pi}\mathcal{B}[d](u) \equiv \frac{1}{\pi}\left[d_0 + \frac{d_1(\kappa)}{1!\beta_0}u + \ldots + \frac{d_n(\kappa)}{n!\beta_0^n}u^n + \ldots\right]$$

$$= \left\{\frac{d_{2,1}^{\rm IR}(\widetilde{\kappa})}{(2-u)^{1+\widetilde{\gamma}_2}}\left[1+\mathcal{E}_1^{(4)}(2-u)+\mathcal{E}_2^{(4)}(2-u)^2\right]+\mathcal{O}\left((2-u)^{-\widetilde{\gamma}_2+2}\right)\right\}$$

$$\left\{+\frac{d_{3,2}^{\rm IR}(\widetilde{\kappa})}{(3-u)^{\kappa_2+\widetilde{\gamma}_3}}\left[1+\mathcal{E}_1^{(6)}(3-u)+\mathcal{E}_2^{(6)}(3-u)^2\right]+\mathcal{O}\left((3-u)^{-\widetilde{\gamma}_3-\kappa_2+3}\right)\right.$$

$$\left.+\frac{d_{3,1}^{\rm IR}(\widetilde{\kappa})}{(3-u)^{\kappa_1+\widetilde{\gamma}_3}}\left[1+\widetilde{\mathcal{E}}_1^{(6)}(3-u)\right]+\mathcal{O}\left((3-u)^{-\widetilde{\gamma}_3-\kappa_1+2}\right)\right\}$$

$$\left\{+\frac{d_{1,2}^{\rm UV}(\widetilde{\kappa})}{(1+u)^{\overline{\gamma}_1+1}}\left[1+\mathcal{E}_1^{(-2)}(1+u)+\mathcal{E}_2^{(-2)}(1+u)^2\right]+\mathcal{O}\left((1+u)^{-\overline{\gamma}_1+2}\right)\right\} \quad (39$$

where $\widetilde{\gamma}_p = +p\beta_1/\beta_0^2$; $\overline{\gamma}_p = -p\beta_1/\beta_0^2$.

Appendix B: Borel of Adler function

In the inverse Borel transform, Eq. (28), when neglecting the $\mathcal{O}(\ldots)$ terms in the Borel transform Eq. (39), implies that a polynomial correction to the PV Borel integral is needed

$$\delta d(\sigma_{\rm m} e^{i\phi};\kappa)^{[N_t]}_{D=0} = \sum_{n=0}^{N_t-1} (\delta d)_n(\kappa) a(\kappa\sigma_{\rm m} e^{i\phi})^{n+1}, \tag{40}$$

where

Table: The $\overline{\rm MS}$ coefficients d_n and the correction polynomial coefficients δd_n (with $\kappa = 1$).

n:	0	1	2	3	4	5	6	7	8	9	10
d_n:	1	1.64	6.37	49.1	275.	3159.	$1.61 \cdot 10^4$	$3.41 \cdot 10^5$	$3.78 \cdot 10^5$	$6.99 \cdot 10^7$	$-5.83 \cdot 10^8$
$(\delta d)_n$:	-27.2	-45.9	-118.	-304.	-1016.	-3070.	$-1.24 \cdot 10^4$	$1.71 \cdot 10^4$	$5.72 \cdot 10^4$	$1.28 \cdot 10^7$	$6.97 \cdot 10^6$

Apendix C: $\mathcal{F}$ function in $B_{\rm th}(M^2,\sigma_{\rm m})_{D=2k}$ $(k\geq 3)$

The function $\mathcal{F}$ appearing in Eq. (**??**) in Borel-Laplace $D=2k$ part $(k\geq 3)$:

$$\begin{aligned}\mathcal{F}(M^2/\sigma_{\rm m}) &= \frac{2\pi^2 k}{{\sigma_{\rm m}}^k}\langle O^{(2)}_{2k}\rangle\beta_0\Bigg\{\Bigg[\left(1-2\frac{M^2}{\sigma_{\rm m}}+2\left(\frac{M^2}{\sigma_{\rm m}}\right)^2\right)J_k\left(\frac{\sigma_{\rm m}}{M^2}\right)\\ &\quad +2\left(1-\frac{M^2}{\sigma_{\rm m}}\right)J_{k-1}\left(\frac{\sigma_{\rm m}}{M^2}\right)+J_{k-2}\left(\frac{\sigma_{\rm m}}{M^2}\right)\Bigg]\\ &\quad +2\left(\frac{M^2}{\sigma_{\rm m}}\right)^2\exp\left[-\frac{\sigma_{\rm m}}{M^2}\right]\frac{(-1)^k}{k}\Bigg\}\end{aligned} \tag{41}$$

where

$$J_s(A)\equiv\frac{1}{2\pi}\int_{-\pi}^{+\pi}d\phi\ \exp\left[Ae^{i\phi}\right]e^{-is\phi}i\phi. \tag{42}$$

Appendix D: Specific sum rules used

Weight functions $g(Q^2)$ used in the sum rules (5):
The double-pinched Borel-Laplace transforms $B(M^2)$ where M^2 is a complex squared energy parameter:

$$g_{M^2}(Q^2) = \left(1+\frac{Q^2}{\sigma_{\rm m}}\right)^2 \frac{1}{M^2} \exp\left(\frac{Q^2}{M^2}\right) \quad \Rightarrow \tag{43a}$$

$$G_{M^2}(Q^2) = \left\{ \left[\left(1+\frac{Q^2}{\sigma_m}\right)^2 - 2\frac{M^2}{\sigma_m}\left(1+\frac{Q^2}{\sigma_m}\right) + 2\left(\frac{M^2}{\sigma_m}\right)^2\right] \exp\left(\frac{Q^2}{M^2}\right) - 2\left(\frac{M^2}{\sigma_m}\right)^2 \exp\left(-\frac{\sigma_m}{M^2}\right) \right\}, \tag{43b}$$

Appendix D: Specific sum rules used

The two sides of the sum rule (5) are then written as

$$B_{\rm exp}(M^2;\sigma_{\rm m}) = \frac{1}{M^2}\int_0^{\sigma_{\rm m}} d\sigma \left(1-\frac{\sigma}{\sigma_{\rm m}}\right)^2 \exp\left(-\frac{\sigma}{M^2}\right)\omega_{\rm exp}(\sigma), \qquad (44)$$

$$\begin{aligned} B_{\rm th}(M^2;\sigma_{\rm m}) = & \left[\left(1-2\frac{M^2}{\sigma_{\rm m}}\right)+2\left(\frac{M^2}{\sigma_{\rm m}}\right)^2\left(1-\exp\left(-\frac{\sigma_{\rm m}}{M^2}\right)\right)\right] \\ & +\frac{1}{2\pi}\int_{-\pi}^{+\pi} d\phi\Bigg\{\left[\left(1+e^{i\phi}\right)^2-2\frac{M^2}{\sigma_{\rm m}}\left(1+e^{i\phi}\right)+2\left(\frac{M^2}{\sigma_{\rm m}}\right)^2\right]\exp\left(\frac{\sigma_{\rm m}}{M^2}e^{i\phi}\right) \\ & -2\left(\frac{M^2}{\sigma_{\rm m}}\right)^2\exp\left(-\frac{\sigma_{\rm m}}{M^2}\right)\Bigg\} d\left(\sigma_{\rm m}e^{i\phi}\right)_{D=0} \\ & +B_{\rm th}(M^2;\sigma_{\rm m})_{D=4}+\sum_{k\geq 3}B_{\rm th}(M^2;\sigma_{\rm m})_{D=2k}. \end{aligned} \qquad (45)$$

Appendix D: Specific sum rules used

The last terms here are the contributions of the dimension $D = 2k$ condensates of the OPE of the Adler function (9)

$$B_{\rm th}(M^2;\sigma_{\rm m})_{D=4} = \frac{2\pi^2\langle O_4\rangle}{(M^2)^2}\left(1+2\frac{M^2}{\sigma_{\rm m}}\right), \tag{46a}$$

$$B_{\rm th}(M^2;\sigma_{\rm m})_{D=6} = \frac{3\pi}{{\sigma_{\rm m}}^3}\int_{-\pi}^{+\pi} d\phi$$
$$\times\left\{\left[\left(1+e^{i\phi}\right)^2 - 2\frac{M^2}{\sigma_{\rm m}}\left(1+e^{i\phi}\right) + 2\left(\frac{M^2}{\sigma_{\rm m}}\right)^2\right]\exp\left(\frac{\sigma_{\rm m}}{M^2}e^{i\phi}\right)\right.$$
$$\left.-2\left(\frac{M^2}{\sigma_{\rm m}}\right)^2\exp\left(-\frac{\sigma_{\rm m}}{M^2}\right)\right\}\exp^{-i3\phi}\left[a(\sigma_{\rm m}e^{i\phi})^{k_2}\langle O_6^{(2)}\rangle + a(\sigma_{\rm m}e^{i\phi})^{k_1}\langle O_6^{(1)}\rangle\right]. \tag{46b}$$

Appendix D: Specific sum rules used

On the other hand, one can use FESRs with (double-pinched) momenta $a^{(2,n)}$ which are associated with the following weight functions $g^{(2,n)}$ $(n = 0, 1, \ldots)$:

$$g^{(2,n)}(Q^2) = \left(\frac{n+3}{n+1}\right)\frac{1}{\sigma_m}\left(1+\frac{Q^2}{\sigma_m}\right)^2 \sum_{k=0}^{n}(k+1)(-1)^k\left(\frac{Q^2}{\sigma_m}\right)^k, \tag{47a}$$

$$G^{(2,n)}(Q^2) = \left(\frac{n+3}{n+1}\right)\frac{Q^2}{\sigma_m}\left[1-\left(-\frac{Q^2}{\sigma_m}\right)^{n+1}\right]+\left[1-\left(-\frac{Q^2}{\sigma_m}\right)^{n+3}\right]. \tag{47b}$$

Appendix D: Specific sum rules used

The two sides of the sum rule (5) of these FESR moments are then (we subtract unity for convenience)

$$a_{\rm exp}^{(2,n)}(\sigma_m) = \int_0^{\sigma_m} d\sigma\, g^{(2,n)}(-\sigma)\omega_{\rm exp}(\sigma) - 1, \tag{48a}$$

$$a_{\rm th}^{(2,n)}(\sigma_m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, G^{(2,n)}(\sigma_m e^{i\phi})\left[D_{\rm th}(\sigma_m e^{i\phi}) - 1\right]. \tag{48b}$$

We consider the first two moments $a^{(2,0)}$ and $a^{(2,1)}$, up to $D = 6$ terms.