

Double parton distributions in the nucleon from lattice simulations

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XVth Quark Confinement and the Hadron Spectrum

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Work done in collaboration with

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Double Parton Distributions

Two-current matrix elements on the lattice

Lattice Setup and Results

Summary and Outlook

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Introduction

- ▶ Double parton scattering processes (DPS), e.g. Double Drell-Yan (DDY), are important standard model contributions in LHC experiments, especially after high-luminosity upgrade
- ▶ Approximation via Pocket formula:

$$\sigma_{\text{DPS}, i_1 i_2, j_1 j_2} \approx \frac{1}{C} \frac{\sigma_{\text{SPS}, i_1 j_1} \sigma_{\text{SPS}, i_2 j_2}}{\sigma_{\text{eff}}}$$

- ▶ More fundamental description by double parton distributions (DPDs) :

$$\frac{d\sigma_{\text{DPS}, i_1 i_2, j_1 j_2}}{dx_1 dx_2 dx'_1 dx'_2} \propto \int d^2y F_{i_1 i_2}(x_1, x_2, y) F_{j_1 j_2}(x'_1, x'_2, y)$$

- ▶ So far, DPDs unknown from experiments, non-perturbative objects, access via lattice simulations
- ▶ Results for the pion [\[arXiv:1807.03073\]](#), [\[arXiv:2006.14826\]](#)

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This talk: Mellin Moments of DPDs for the nucleon from Lattice QCD

Recently published in [\[arXiv:2106.03451\]](#)

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Points of interest:

- ▶ Size of DPDs and polarization effects
- ▶ Strength of parton-parton correlations
- ▶ Validity of factorization assumptions

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Double Parton Distributions

- ▶ Light cone coordinates for a given 4-vector x^μ : $x^\pm = (x^0 \pm x^3)/\sqrt{2}$, $\mathbf{x} = (x^1, x^2)$
- ▶ Consider a proton rapidly moving in 3-direction, i.e. $p^+ \sim Q \gg \Lambda \sim m$, $\mathbf{p} = \mathbf{0}$, $p^- \sim \Lambda^2/Q$

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Definition of proton DPDs for quarks [\[arXiv:1111.0910\]](https://arxiv.org/abs/1111.0910)

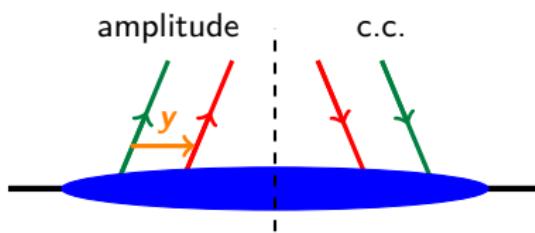
$$\begin{aligned} F_{ab}(x_1, x_2, \mathbf{y}) := & 2p^+ \int dy^- \left[\prod_{j=1,2} \int \frac{dz_j^-}{2\pi} e^{ix_j p^+ z_j^-} \right] \times \\ & \times \tfrac{1}{2} \sum_\lambda \langle p, \lambda | \mathcal{O}_a(y, z_1^-) \mathcal{O}_b(0, z_2^-) | p, \lambda \rangle \Big|_{y^+=0} \end{aligned}$$

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Joint probability to find quark a with momentum $x_1 p^+$ and quark b with momentum $x_2 p^+$ at transverse distance y ($|x_1| + |x_2| \leq 1$)

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Light cone operators

$$\mathcal{O}_a(y, z^-) = \bar{q}(y - \tfrac{z}{2}) \Gamma_a q(y + \tfrac{z}{2}) \Big|_{z=0, z^+=0}$$

- ▶ \bar{q}, q quark operators for certain flavor (**light-like distance z^-**)
- ▶ Γ_a quark polarization

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Twist-2 components: Quark polarizations

operators	twist-2 comp.	polarization
$V_q^\mu = \bar{q} \gamma^\mu q$	$V_q^+ = \mathcal{O}_q$	$q : q^\uparrow + q^\downarrow$ (unpolarized)
$A_q^\mu = \bar{q} \gamma^\mu \gamma_5 q$	$A_q^+ = \mathcal{O}_{\Delta q}$	$\Delta q : q^\uparrow - q^\downarrow$ (longitudinal)
$T_q^{\mu\nu} = \bar{q} i \sigma^{\mu\nu} \gamma_5 q$	$T_q^{+j} = \mathcal{O}_{\delta q}^j$	$\delta q^j : q^{\uparrow,j} - q^{\downarrow,j}$ (transverse)

Double Parton Distributions

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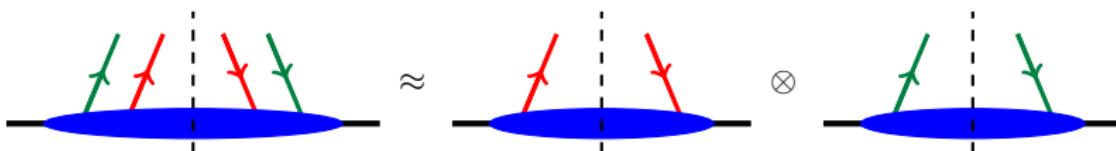
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Factorization assumption I

$$\langle p | \mathcal{O}_a(y, z_1) \mathcal{O}_b(0, z_2) | p \rangle \approx \int \frac{d^2 p' dp'^+}{(2\pi)^3 2p'^+} \langle p | \mathcal{O}_a(y, z_1) | p' \rangle \langle p' | \mathcal{O}_b(0, z_2) | p \rangle \\ \Rightarrow F_{ab}(x_1, x_2, \mathbf{y}) \approx \int d^2 \mathbf{B} f_a(x_1, \mathbf{B} + \mathbf{y}) f_b(x_2, \mathbf{B})$$



Double Parton Distributions

Definition of proton DPDs for quarks [arXiv:1111.0910]

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Factorization assumption II

For the pocket formula [arXiv:1111.0469]:

$$F_{ab}(x_1, x_2, y) \approx f_a(x_1) f_b(x_2) T(y)$$

with flavor independent $T(y)$

Double parton distributions on the lattice

Accessible quantities

$$\langle p | \mathcal{O}_a(y, z_1^-) \mathcal{O}_b(0, z_2^-) | p \rangle$$

$\int_{y^+}^{p^+} dy^- dz_i^- e^{-iz_i x_i p^+} F_{ab}(x_i, \mathbf{y})$

$y^+ = 0, \text{twist-2}$

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$\int_{y^+}^{p^+} dy^- dz_i^- e^{-iz_i^- x_i p^+} F_{ab}(x_i, y)$
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not accessible on the lattice
if $z_i^- > 0$

Double parton distributions on the lattice

Accessible quantities

$$\langle p | \mathcal{O}_a(y, z_1^-) \mathcal{O}_b(0, z_2^-) | p \rangle$$

$\xrightarrow[p^+ \int dy^- dz_i^- e^{-iz_i^- x_i p^+}]{y^+ = 0, \text{twist-2}}$

$$\int dx_i$$

\downarrow

$$\langle p | \mathcal{O}_a(y) \mathcal{O}_b(0) | p \rangle$$

$\xrightarrow[p^+ \int dy^-]{y^+ = 0, \text{twist-2}}$

$z_i^- = 0$

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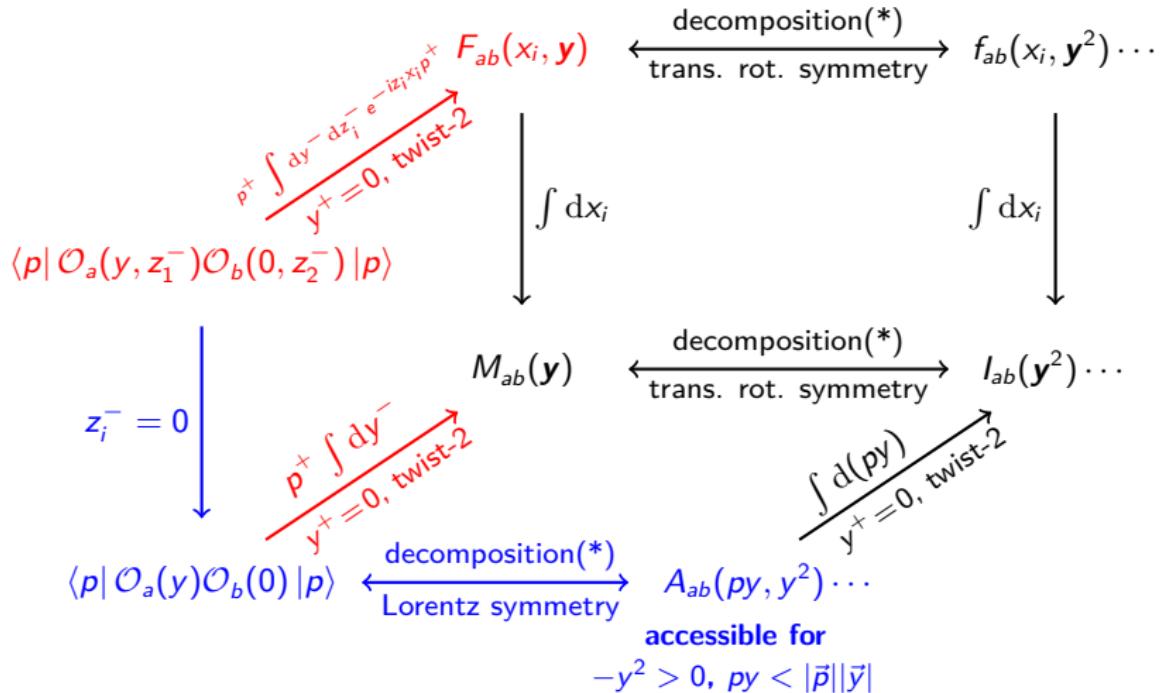
$$\langle p | \mathcal{O}_a(y, z_1^-) \mathcal{O}_b(0, z_2^-) | p \rangle \xrightarrow[p^+ \int dy^- dz_i^- e^{-iz_i^- x_i p^+}]{y^+ = 0, \text{twist-2}} F_{ab}(x_i, y)$$

$$\begin{array}{c} \downarrow \\ z_i^- = 0 \end{array} \quad M_{ab}(y)$$
$$\langle p | \mathcal{O}_a(y) \mathcal{O}_b(0) | p \rangle \xrightarrow[p^+ \int dy^-]{y^+ = 0, \text{twist-2}}$$

accessible if $y^0 = 0$

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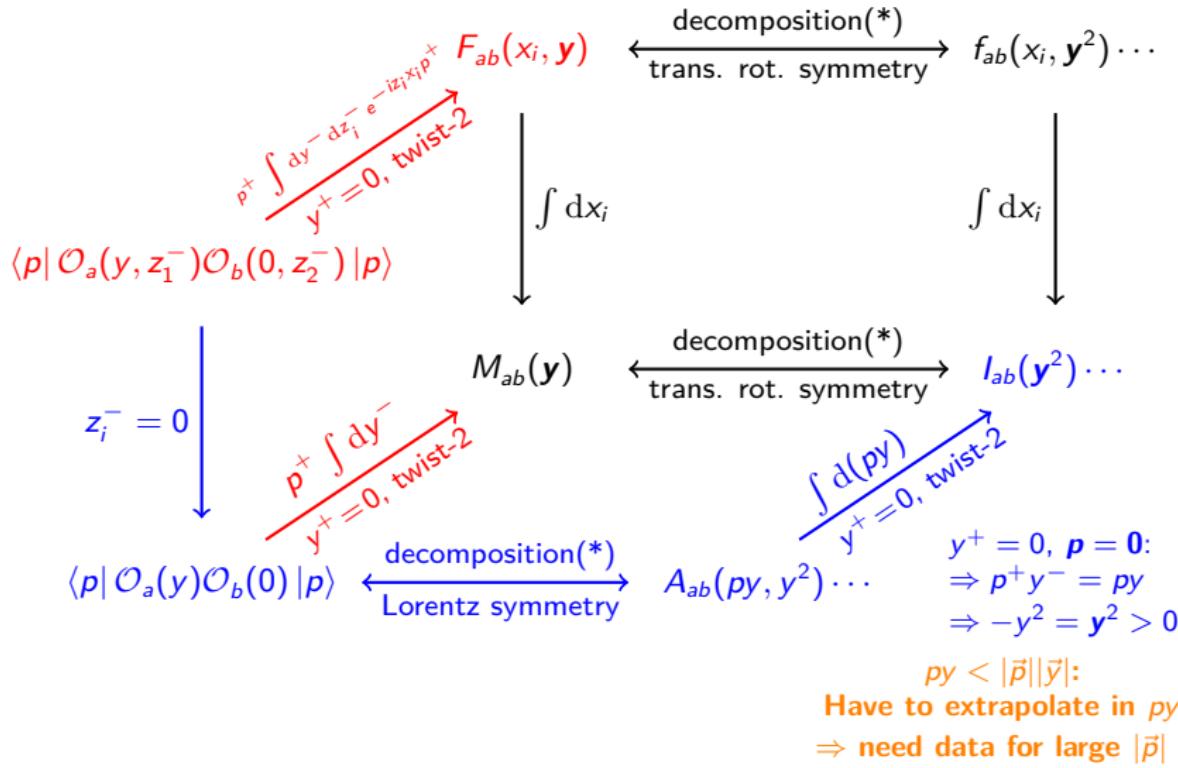
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(*) into basis tensors and scalar functions

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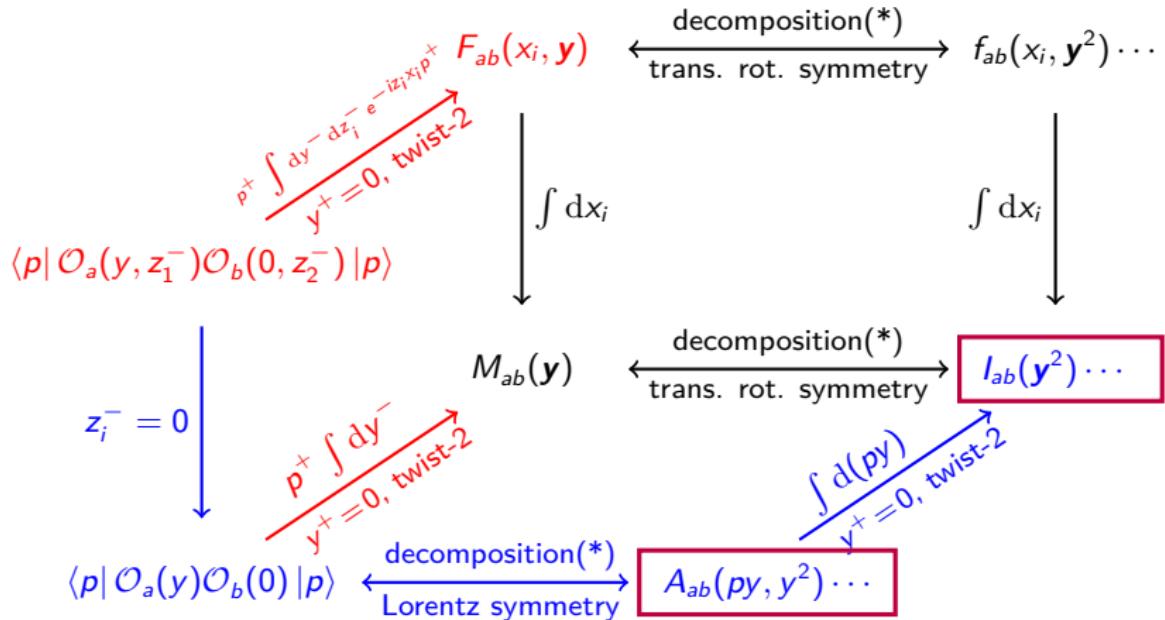
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Results for these quantities

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$$\langle \mathcal{O}[q, \bar{q}, U] \rangle = \frac{1}{Z} \int \left[\prod_{x \in \Lambda} d\bar{q}(x) dq(x) dU(x) \right] \mathcal{O}[q, \bar{q}, U] e^{-S_F[\bar{q}, q, U] - S_G[U]}$$

$$Z = \int \left[\prod_{x \in \Lambda} d\bar{q}(x) dq(x) dU(x) \right] e^{-S_F[\bar{q}, q, U] - S_G[U]}$$

- ▶ Reduce spacetime to a lattice
- ▶ Finite volume \Rightarrow IR regularization
- ▶ Finite lattice spacing \Rightarrow UV regularization
- ▶ Evaluate the fermionic part (Grassmann variables) using Wick's theorem
 \Rightarrow Wick contractions (graphs)
- ▶ Euclidean spacetime: $e^{iS} \rightarrow e^{-S}$ suitable weight for Monte Carlo integration
- ▶ Evaluate gauge integral by Monte Carlo integration \Rightarrow gauge ensembles of N configuration, statistical error $\propto N^{-\frac{1}{2}}$:

$$\int \left[\prod_x dU(x) \right] \det\{\mathcal{D}[U]\} e^{-S[U]} \mathcal{O}[q, \bar{q}, U] \xrightarrow{\text{ensemble}} \sum_{U \sim P(U)} \mathcal{O}[q, \bar{q}, U],$$

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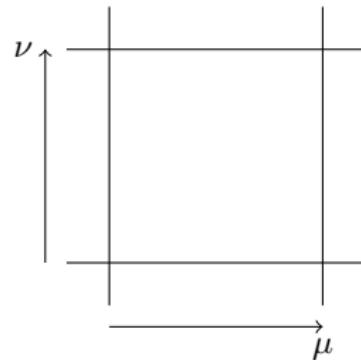
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$$\int \left[\prod_x dU(x) \right] \det\{\mathcal{D}[U]\} e^{-S[U]} \mathcal{O}[q, \bar{q}, U] \xrightarrow{\text{ensemble}} \sum_{U \sim P(U)} \mathcal{O}[q, \bar{q}, U],$$

Lattice QCD

Reduce spacetime \mathbb{R}^4 to finite lattice with spacing a , extensions $L^3 \times T$:

- ▶ put fermions on the grid points
- ▶ replace derivatives by symmetric differential quotient and integrals by sums
- ▶ restore gauge invariance (gauge links $U_\mu(x) \sim e^{iaA_\mu(x)}$)
- ▶ add pure gauge part, the plaquette, $\beta = 3g^{-2}$



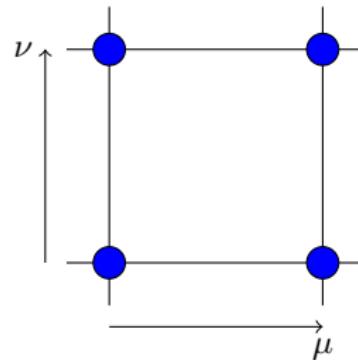
$$S[q, \bar{q}, U] = \int d^4x \bar{q}(x) \mathcal{D}q(x)$$

$$\mathcal{D} = i\gamma_\mu \partial^\mu - m\mathbb{1}$$

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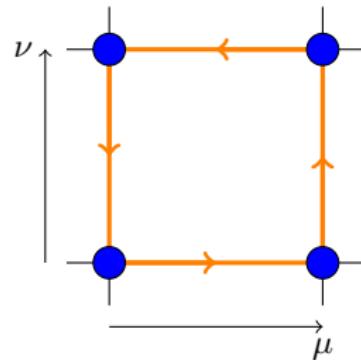
$$S[q, \bar{q}, U] = a^4 \sum_{x,y \in \Lambda_4} \bar{q}(x) \mathcal{D}(x|y) q(y)$$

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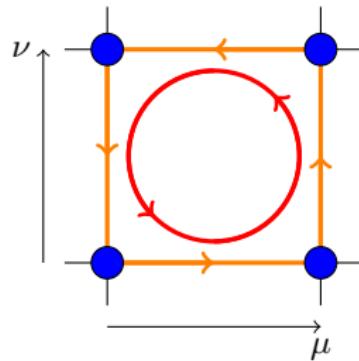
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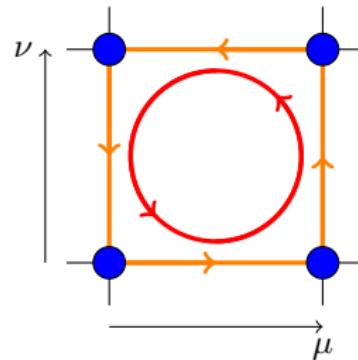
$$S[q, \bar{q}, U] = a^4 \sum_{x, y \in \Lambda_4} \bar{q}(x) \mathcal{D}(x|y) q(y) + \frac{\beta}{3} \sum_{x \in \Lambda} \sum_{\mu < \nu} \text{Retr}\{\mathbb{1} - U_{\mu\nu}(x)\}$$

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⇒ Add additional terms to reduce discretization artifacts (not unique, in this work Wilson/Sheikholeslami-Wohlert fermions + Lüscher-Weiss gauge action)

Two-current matrix elements on the lattice

In Euclidean spacetime:

Access via 4-point functions

$$\frac{1}{2} \sum_{\lambda} \langle p, \lambda | \mathcal{O}_{a_1}(y) \mathcal{O}_{a_2}(0) | p, \lambda \rangle \Big|_{y^0=0} = 2V \sqrt{m^2 + \vec{p}^2} \left. \frac{C_{4\text{pt}}^{\vec{p}, ij}(t, \tau, \vec{y})}{C_{2\text{pt}}^{\vec{p}}(t)} \right|_{0 \ll \tau \ll t}$$

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$$C_{4\text{pt}}^{\vec{p},ij}(t, \tau, \vec{y}) = \sum_{\vec{x}} e^{-i\vec{p}(\vec{z}' - \vec{z})} \left\langle \text{tr} \left\{ P_+ \mathcal{P}^{\vec{p}}(\vec{z}', t) \mathcal{O}_i^{q_1 q_2}(0, \tau) \mathcal{O}_j^{q_3 q_4}(\vec{y}, \tau) \bar{\mathcal{P}}^{\vec{p}}(\vec{z}, 0) \right\} \right\rangle$$

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and Proton interpolators:

$$\mathcal{P}^{\vec{p}}(\vec{x}, t) = \epsilon_{abc} u_a(x) [u_b^T(x) C \gamma_5 d_c(x)] \Big|_{x^4=t}$$

$$\bar{\mathcal{P}}^{\vec{p}}(\vec{x}, t) = \epsilon_{abc} [\bar{u}_a(x) C \gamma_5 \bar{d}_b^T(x)] \bar{u}_c(x) \Big|_{x^4=t}$$

Two-current matrix elements on the lattice

Wick contractions

$$C_{1,q_1 \dots q_4}^{ij} = \text{Diagram showing two horizontal lines with arrows from left to right. The top line has a red dot labeled } \mathcal{O}_i^{q_1 q_2} \text{ at its center. The bottom line has a red dot labeled } \mathcal{O}_j^{q_3 q_4} \text{ at its center. Both lines are surrounded by teal circles at both ends.}$$

$$S_{1,q}^{ij} = \text{Diagram showing two horizontal lines with arrows from left to right. The top line has a red dot labeled } \mathcal{O}_i^{qq} \text{ at its center. The bottom line has a red dot labeled } \mathcal{O}_j \text{ at its center. The top line is surrounded by teal circles at both ends. A small loop is attached to the bottom line between the two dots.}$$

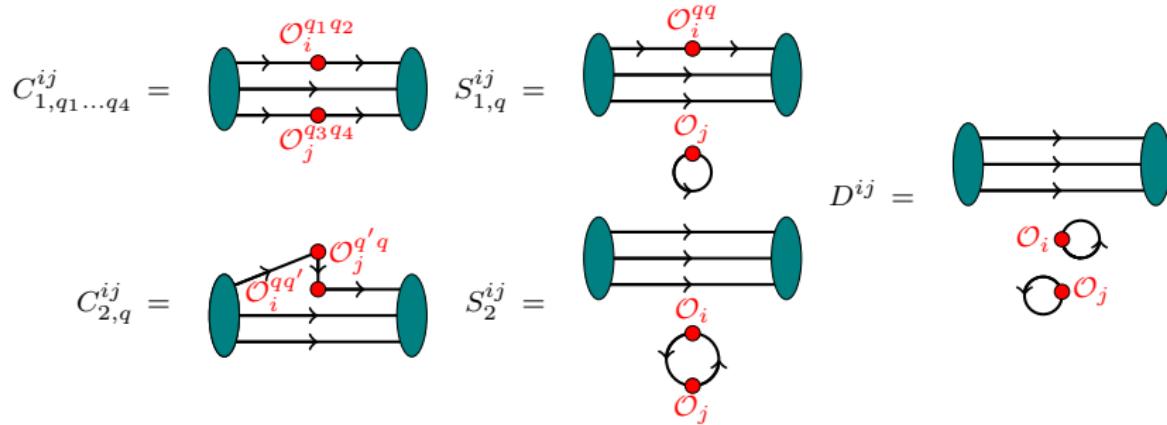
$$D^{ij} = \text{Diagram showing two horizontal lines with arrows from left to right. The top line has a red dot labeled } \mathcal{O}_i \text{ at its center. The bottom line has a red dot labeled } \mathcal{O}_j \text{ at its center. Both lines are surrounded by teal circles at both ends. A small loop is attached to each line between the two dots.}$$

$$C_{2,q}^{ij} = \text{Diagram showing two horizontal lines with arrows from left to right. The top line has a red dot labeled } \mathcal{O}_i^{qq'} \text{ at its center. The bottom line has a red dot labeled } \mathcal{O}_j^{q'q} \text{ at its center. The top line is surrounded by teal circles at both ends. A small loop is attached to the bottom line between the two dots. The labels } \mathcal{O}_i \text{ and } \mathcal{O}_j \text{ are swapped relative to the first diagram.}$$

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Two-current matrix elements on the lattice

Wick contractions



Physical matrix elements

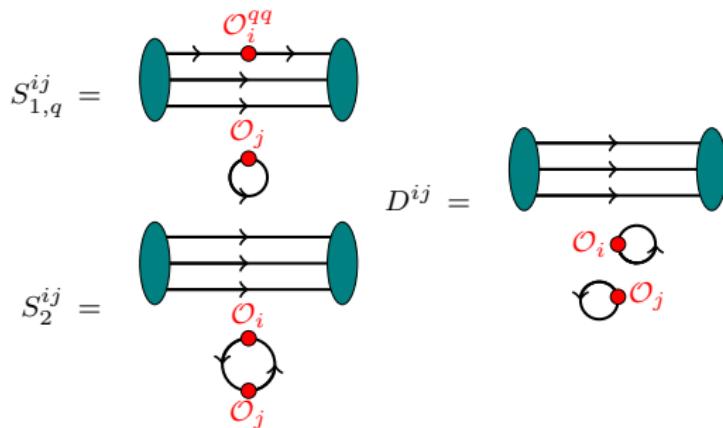
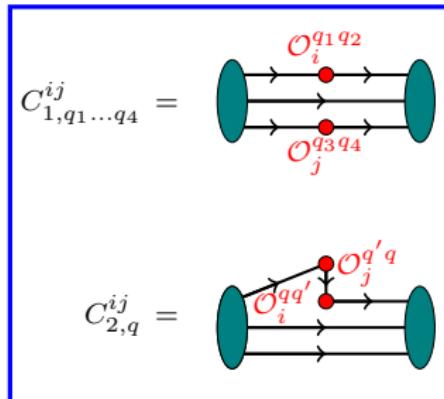
$$\langle p | \mathcal{O}_i^{uu}(\vec{0}) \mathcal{O}_j^{dd}(\vec{y}) | p \rangle = C_{1,uudd}^{ij,\vec{p}}(\vec{y}) + S_{1,u}^{ij,\vec{p}}(\vec{y}) + S_{1,d}^{ji,\vec{p}}(-\vec{y}) + D^{ij,\vec{p}}(\vec{y})$$

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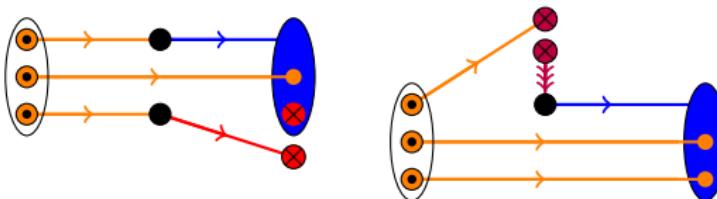
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Technical Details



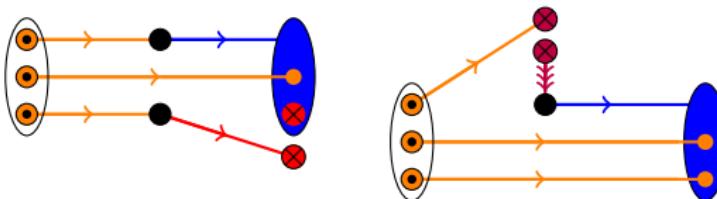
○ → point source / propagator

✖ → stochastic source / propagator / with HPE

● → sequential source / propagator with constituents

- ▶ APE smearing [*Nucl. Phys. B251 (1985)*]
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Stochastic propagator: $\mathcal{D}\psi^\ell = \eta^\ell$, $N_{\text{stoch}} = 2$ (C_1), or $N_{\text{stoch}} = 96$ (C_2)
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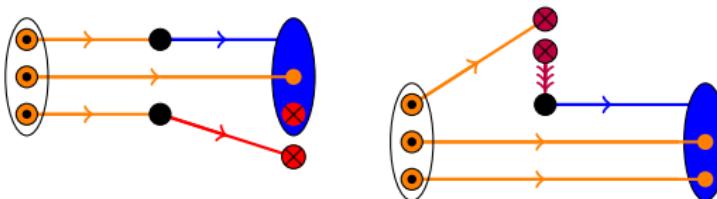
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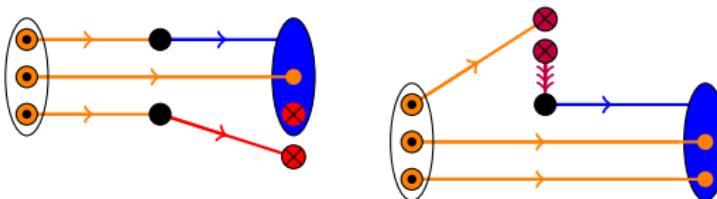
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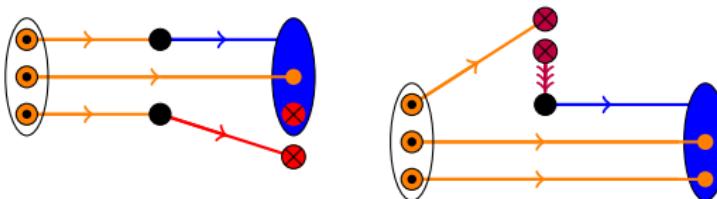
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Content

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Double Parton Distributions

Two-current matrix elements on the lattice

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Summary and Outlook

Lattice Setup

CLS ensembles ($n_f = 2 + 1$, Wilson fermions, order- a improved [[arXiv:1411.3982](#)]), start with H102, 990 configs used:

id	β	$a[\text{fm}]$	$L^3 \times T$	$\kappa_{I/s}$	$m_{\pi/K} [\text{MeV}]$	$m_\pi L$	conf.
H102	3.4	0.0856	$32^3 \times 96$	0.136865 0.136549339	355 441	4.9	2037

- ▶ $t_{\text{src}} = 48a$ (point sources at random spatial position)
- ▶ $t = t_{\text{snk}} - t_{\text{src}} = \begin{cases} 12a & \vec{p} = \vec{0} \\ 10a & \vec{p} \neq \vec{0} \end{cases}$
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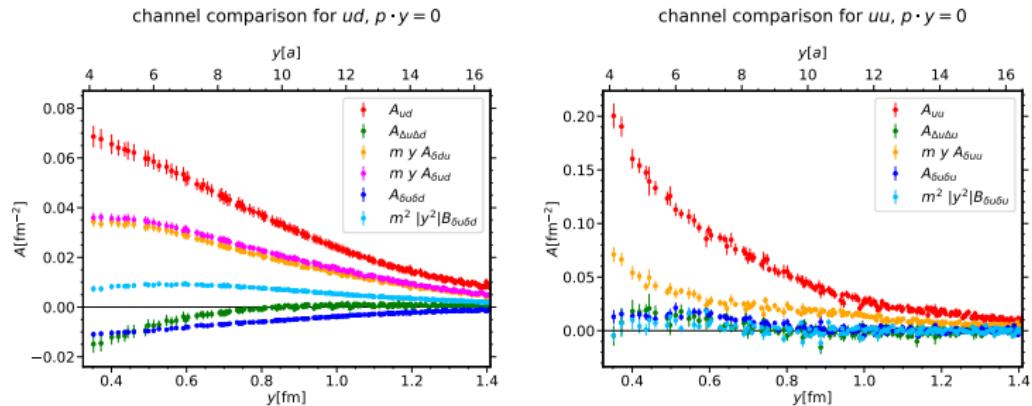
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Renormalization for $\beta = 3.4$, including conversion to $\overline{\text{MS}}$ at $\mu = 2 \text{ GeV}$
[[arXiv:2012.06284](#)]:

	V	A	T
Z	0.7128	0.7525	0.8335

Results: Polarization dependence

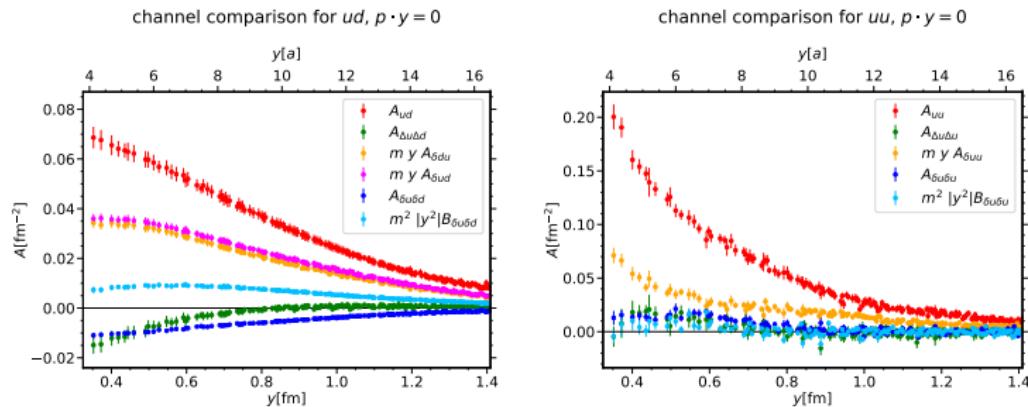
Invariant functions $A(py = 0, y^2)$, connected graphs only (notation $y = \sqrt{-y^2}$, $y^2 = y^\mu y_\mu$):



- ▶ Signal of good quality for most channels
- ▶ ud : Clear contributions from all polarized channels (large for δud , δdu)
- ▶ uu : Polarization effects suppressed, but visible for δuu
- ▶ Moments: Similar conclusions

Results: Polarization dependence

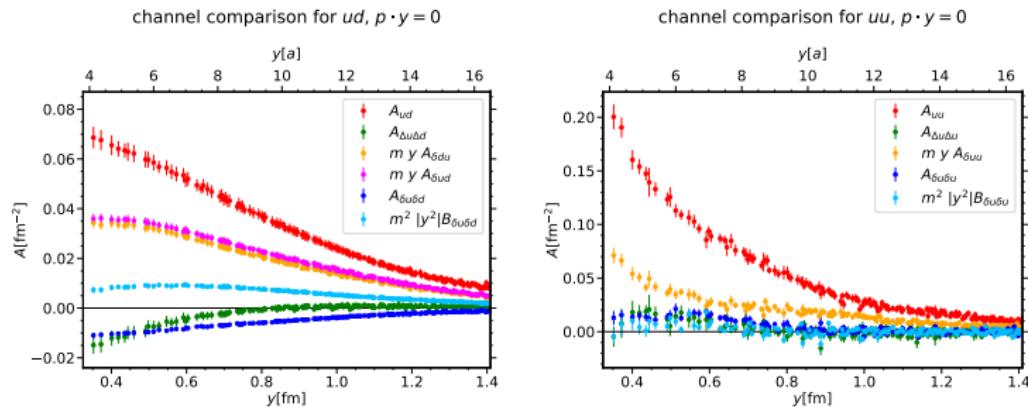
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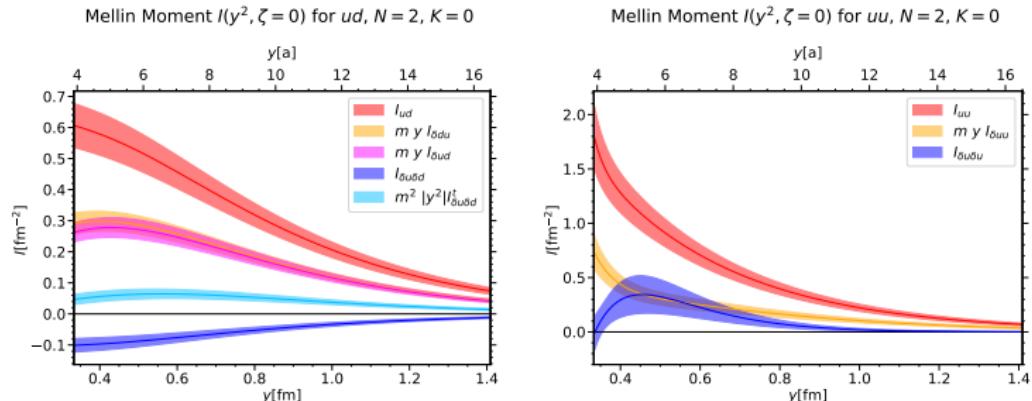
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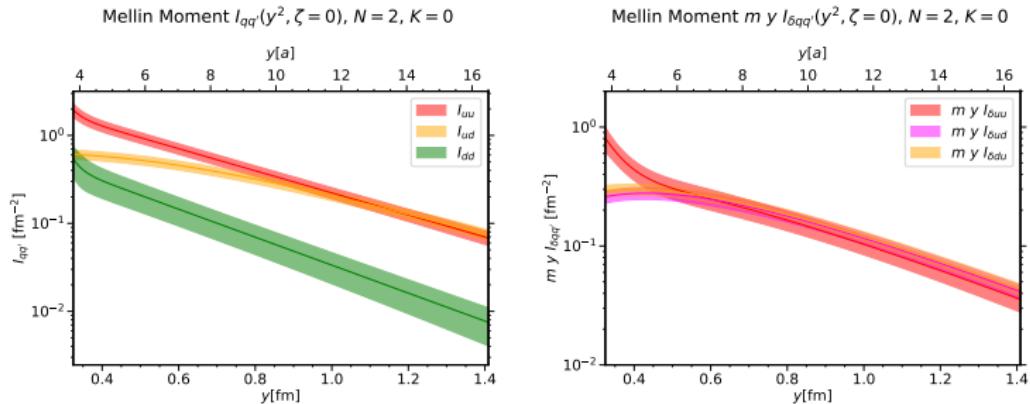
Results: Polarization dependence

From **naive** fit ansatz in py and $\int d(py) \Rightarrow$ DPD moments $I(y^2)$ (notation $y = |\mathbf{y}|$):



- ▶ Signal of good quality for most channels
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- ▶ uu : Polarization effects suppressed, but visible for δuu
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Results: Flavor dependence



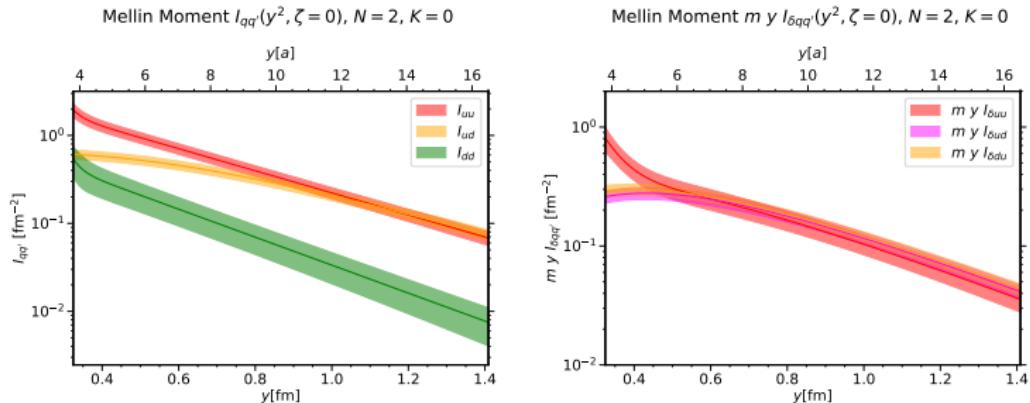
- ▶ Clear flavor dependence observable
- ▶ Reminder: Assumption for the pocket formula:

$$F_{ab}(x_1, x_2, y) = f_a(x_1)f_b(x_2)T(y) \quad \Rightarrow \quad I_{ab}(y^2) = C_{ab}T(y^2)$$

with flavor independent $T(y)$

- ▶ Clearly not fulfilled

Results: Flavor dependence



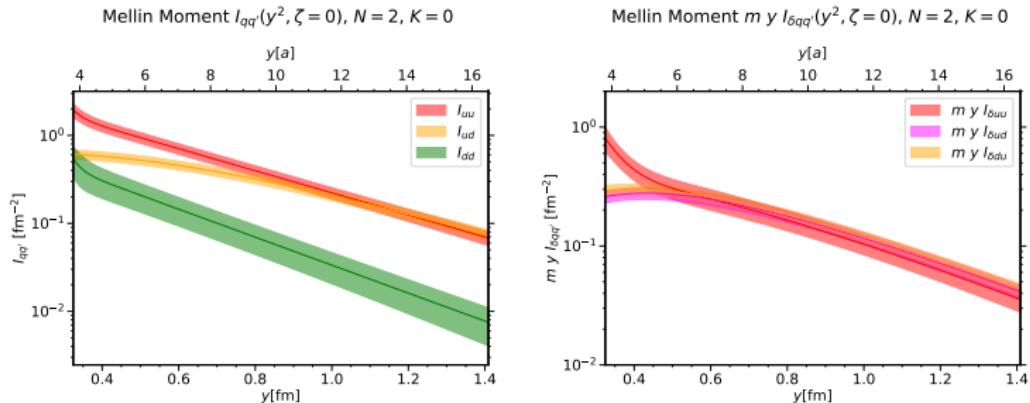
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Results: Flavor dependence



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with **flavor independent** $T(\mathbf{y})$

- ▶ **Clearly not fulfilled**

Factorization tests

Factorization in terms of impact parameter distributions $f_q(x, \mathbf{b})$:

$$F_{qq'}(x_1, x_2, \mathbf{y}) \approx \int d^2\mathbf{b} \ f_q(x_1, \mathbf{b} + \mathbf{y}) \ f_{q'}(x_2, \mathbf{b})$$

Factorization tests

For the Mellin moments

$$I_{qq'}(\mathbf{y}) \approx \int \frac{dr}{2\pi} \ r J_0(ry) \left[F_1^q(-\mathbf{r}^2) F_1^{q'}(-\mathbf{r}^2) + \frac{\mathbf{r}^2}{4m^2} F_2^q(-\mathbf{r}^2) F_2^{q'}(-\mathbf{r}^2) \right]$$

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⇒ Obtain form factors F_1 , F_2 from the lattice [T. Wurm, priv. comm.]

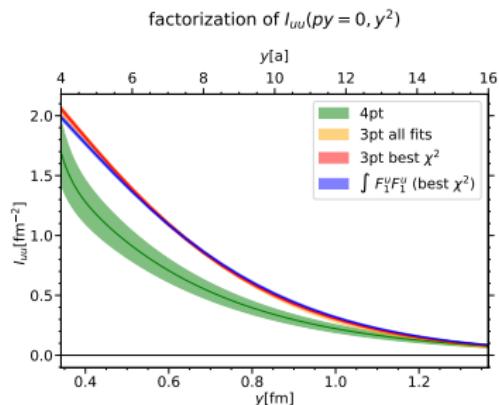
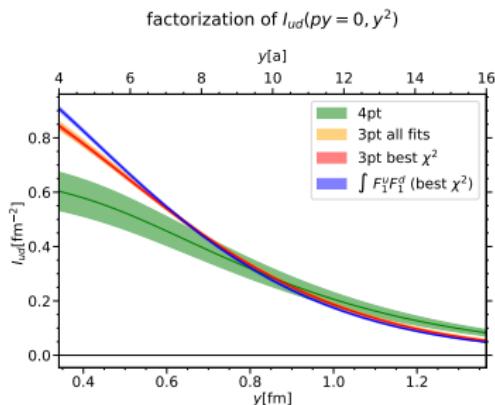
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Results for I_{ud} and I_{uu} :



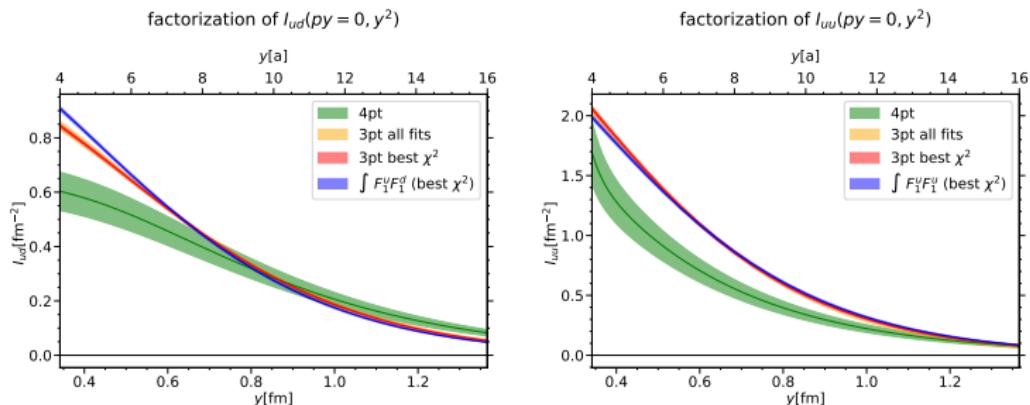
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Results for I_{ud} and I_{uu} :



Comparable size but deviations are visible

Content

Introduction

Double Parton Distributions

Two-current matrix elements on the lattice

Lattice Setup and Results

Summary and Outlook

Summary and Outlook

Achieved/Observed:

- ▶ Calculated nucleon four-point function on the lattice in order to obtain two-current matrix elements
⇒ parameterized by Lorentz invariant functions
- ▶ Extracted DPD Mellin moments for specific quark polarizations / flavors
- ▶ Polarization effects visible, large for $\delta u d$ and $\delta d u$
- ▶ Clear flavor dependence of shape in y (\not{z} pocket formula)
- ▶ Factorization in terms of form factors / GPDs yields correct order of magnitude, deviations visible

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Future work / currently in progress:

- ▶ Include (flavor) interference effects (data currently available)
- ▶ Repeat analysis for further ensembles (⇒ physical limit)
- ▶ Consider derivatives (higher Mellin moments)
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Thank you for your attention!

Backup Slides

Parameterization of DPDs

Decomposition in terms of rotational invariant functions $f(x_1, x_2, \mathbf{y}^2)$:

$$F_{qq'}(x_1, x_2, \mathbf{y}) = f_{qq'}(x_1, x_2, \mathbf{y}^2)$$

$$F_{\Delta q \Delta q}(x_1, x_2, \mathbf{y}) = f_{\Delta q \Delta q'}(x_1, x_2, \mathbf{y}^2)$$

$$F_{q \Delta q'}(x_1, x_2, \mathbf{y}) = F_{\Delta q q'}(x_1, x_2, \mathbf{y}) = 0$$

$$F_{q \delta q'}^j(x_1, x_2, \mathbf{y}) = \epsilon^{j\ell} y^\ell m f_{q \delta q'}(x_1, x_2, \mathbf{y}^2)$$

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$$F_{\delta q \delta q'}^{jk}(x_1, x_2, \mathbf{y}) = \delta^{jk} f_{\delta q \delta q'}(x_1, x_2, \mathbf{y}^2) + (2y^j y^k - \delta^{jk} \mathbf{y}^2) m^2 f_{\delta q \delta q'}^t(x_1, x_2, \mathbf{y}^2)$$

Parameterization of DPDs

Decomposition in terms of rotational invariant functions $I(\mathbf{y}^2)$:

$$M_{qq'}(\mathbf{y}) = I_{qq'}(\mathbf{y}^2)$$

$$M_{\Delta q \Delta q'}(\mathbf{y}) = I_{\Delta q \Delta q'}(\mathbf{y}^2)$$

$$M_{q \Delta q'}(\mathbf{y}) = M_{\Delta q q'}(\mathbf{y}) = 0$$

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Parameterization of two-current matrix elements

Decomposition in terms of Lorentz invariant functions $A(py, y^2), B(py, y^2), \dots$, blue: twist-2 contributions:

$$\langle p | V_q^{\{\mu}(0)V_{q'}^{\nu\}}(y) | p \rangle =$$

$$= (2p^\mu p^\nu - \frac{m^2}{2}g^{\mu\nu})A_{q'q}(py, y^2) + (2p^{\{\mu}y^{\nu\}} - \frac{py}{2}g^{\mu\nu})m^2B_{q'q}(py, y^2) \\ + (2y^\mu y^\nu - \frac{y^2}{2}g^{\mu\nu})m^4C_{q'q}(py, y^2) + g^{\mu\nu}D_{q'q}(py, y^2)$$

$$\langle p | A_q^{\{\mu}(0)A_{q'}^{\nu\}}(y) | p \rangle =$$

$$= (2p^\mu p^\nu - \frac{m^2}{2}g^{\mu\nu})A_{\Delta q'\Delta q}(py, y^2) + (2p^{\{\mu}y^{\nu\}} - \frac{py}{2}g^{\mu\nu})m^2B_{\Delta q'\Delta q}(py, y^2) \\ + (2y^\mu y^\nu - \frac{y^2}{2}g^{\mu\nu})m^4C_{\Delta q'\Delta q}(py, y^2) + g^{\mu\nu}D_{\Delta q'\Delta q}(py, y^2)$$

$$\langle p | T_q^{\mu\nu}(0)V_{q'}^\rho(y) | p \rangle + \frac{2}{3}g_{\lambda\sigma}g^{\rho[\mu}\langle p | T_q^{\nu]\lambda}(0)V_{q'}^\sigma(y) | p \rangle =$$

$$= (4y^{[\mu}p^{\nu]}p^\rho + \frac{4m^2}{3}g^{\rho[\mu}y^{\nu]} - \frac{4py}{3}g^{\rho[\mu}p^{\nu]})m A_{q'\delta q}(py, y^2) \\ + (4y^{[\mu}p^{\nu]}y^\rho + \frac{4py}{3}g^{\rho[\mu}y^{\nu]} - \frac{4y^2}{3}g^{\rho[\mu}p^{\nu]})m^3B_{q'\delta q}(py, y^2)$$

$$\frac{1}{2}\langle p | T_q^{\mu\nu}(0)T_{q'}^{\rho\sigma}(y) | p \rangle + \frac{1}{2}\langle p | T_q^{\rho\sigma}(0)T_{q'}^{\mu\nu}(y) | p \rangle =$$

$$= -8p^{[\nu}g^{\mu][\rho}p^{\sigma]}A_{\delta q'\delta q}(py, y^2) - (16y^{[\mu}p^{\nu]}y^{[\rho}p^{\sigma]} - 8y^2 p^{[\nu}g^{\mu][\rho}p^{\sigma]})m^2B_{\delta q'\delta q}(py, y^2) \\ - (4p^{[\nu}g^{\mu][\rho}y^{\sigma]} + 4y^{[\nu}g^{\mu][\rho}p^{\sigma]})m^2C_{\delta q'\delta q}(py, y^2) - 8y^{[\nu}g^{\mu][\rho}y^{\sigma]}m^4D_{\delta q'\delta q}(py, y^2) \\ + 2g^{\mu[\rho}g^{\sigma]\nu}m^2E_{\delta q'\delta q}(py, y^2)$$

Fit ansatz for invariant functions

Skewed DPDs (additional phase with skewness ζ):

$$F_{ab}(x_1, x_2, \zeta, y) := 2p^+ \int dy^- e^{-i\zeta p^+ y^-} \left[\prod_{i=1,2} \frac{dz_i^-}{2\pi} e^{ix_i p^+ z_i} \right] \langle p | \mathcal{O}_a(y, z_1) \mathcal{O}_b(0, z_2) | p \rangle$$

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Symmetries and support region:

$$F(x_1, x_2, \zeta, \mathbf{y}) = F(x_1, x_2, -\zeta, \mathbf{y}) \quad |x_i \pm \zeta/2| \leq 1 \quad |\zeta| \leq 1$$

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Ansatz:

$$I(\zeta, \mathbf{y}^2) \propto \sum_n \zeta^{2n} \Theta(1 - \zeta^2) \quad \Rightarrow \quad A(p\mathbf{y}, y^2) = A(0, y^2) \sum_n a_n(y^2) h_n(p\mathbf{y})$$

$$h_n(x) := \frac{1}{2} \int_{-1}^1 d\zeta e^{ix\zeta} \zeta^{2n} = \sin(x)s_n(x) + \cos(x)c_n(x)$$

$$s_n(x) := \sum_{m=0}^n \frac{(2n)!(-1)^m}{(2n-2m)!x^{1+2m}} \quad c_n(x) := \sum_{m=0}^{n-1} \frac{(2n)!(-1)^m}{(2n-2m-1)!x^{2+2m}}$$

Fit ansatz for invariant functions

Skewed DPDs (additional phase with skewness ζ):

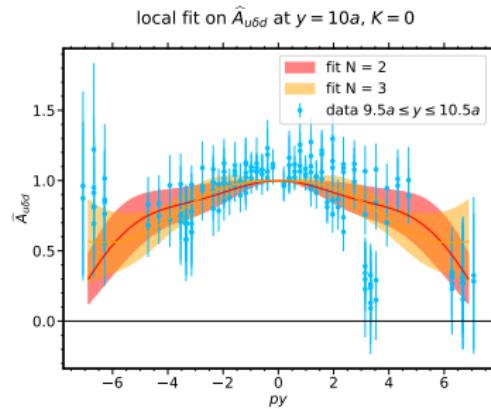
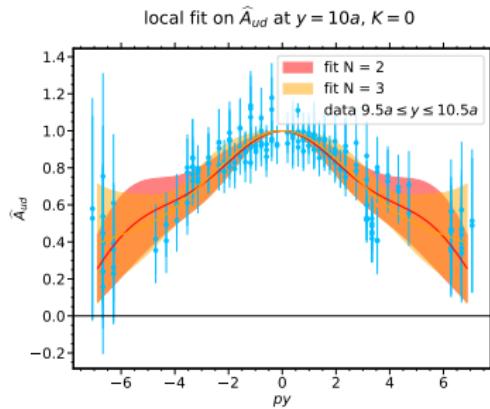
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Fit ansatz for invariant functions

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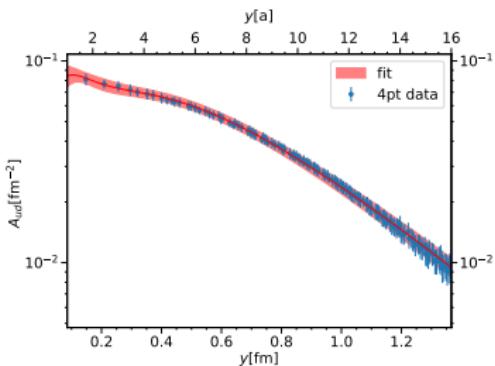
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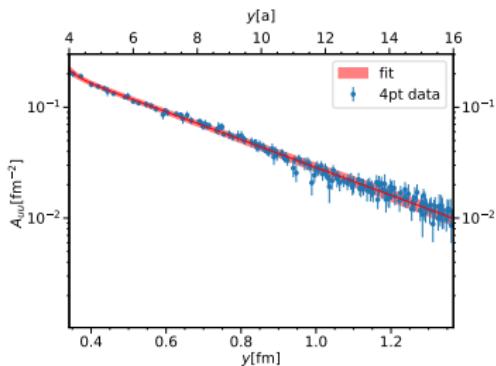
y^2 -dependence:

$$A(0, y^2) = \sum_{i=1,2} A_i (\eta_i y)^{\delta} e^{-\eta_i (y-y_0)}$$

double-exponential fit on $A_{ud}(py = 0, y^2)(\log)$



double-exponential fit on $A_{uu}(py = 0, y^2)(\log)$



Fit ansatz for invariant functions

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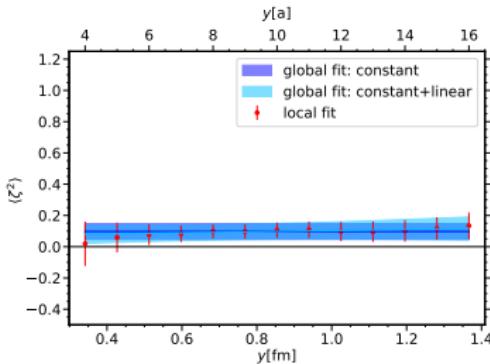
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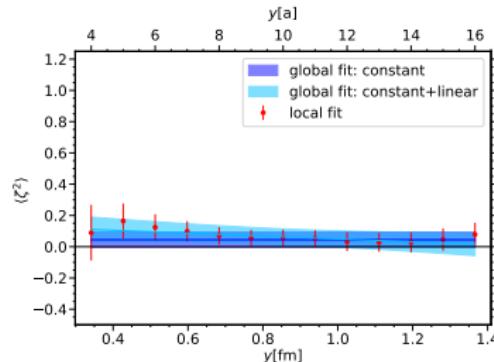
Ansatz $a_m(y^2)$:

$$a_m(y^2) = \sum_k c'_{mk} \sqrt{-y^2}^k \quad (Tc')_{nk} = c_{nk} \quad \frac{\partial^{2n} A(py, y^2)}{\partial (py)^{2n}} \Big|_{py=0} = A(0, y^2) \sum_k c_{nk} \sqrt{-y^2}^k$$

$\langle \zeta^2 \rangle$ for I_{ud} , $N = 2$



$\langle \zeta^2 \rangle$ for $I_{u\bar{d}}$, $N = 2$



Fit ansatz for invariant functions

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Total ansatz (red: fit parameters)

$$A(py, y^2) = \sum_{i=1,2} A_i e^{-\eta_i (y - y_0)} \sum_{n,m=0}^N \sum_{k=0}^K c'_{nk} \sqrt{-y^2}^{k+\delta} \eta_i^\delta h_n(py)$$

$$I(y^2) = \pi \sum_{i=1,2} A_i e^{-\eta_i (y - y_0)} \sum_{k=0}^K c'_{0k} \sqrt{-y^2}^{k+\delta} \eta_i^\delta$$

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- ▶ $c'_{nk} = T_{nm}^{-1} c_{mk}$
- ▶ Notice: $c_{00} \equiv 1$ and $c_{01} \equiv 0$ (only influence $A(0, y^2)$)
- ▶ In this work: $(N, K) = (2, 0), (2, 1), (3, 0)$

Fit ansatz for invariant functions

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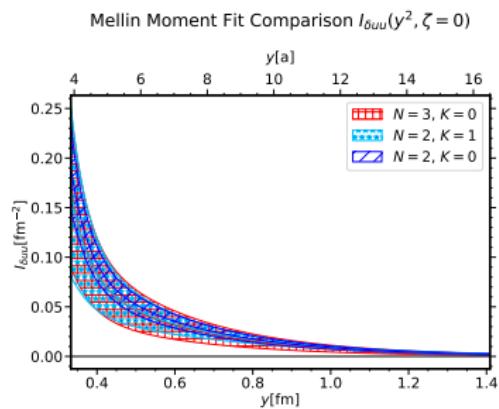
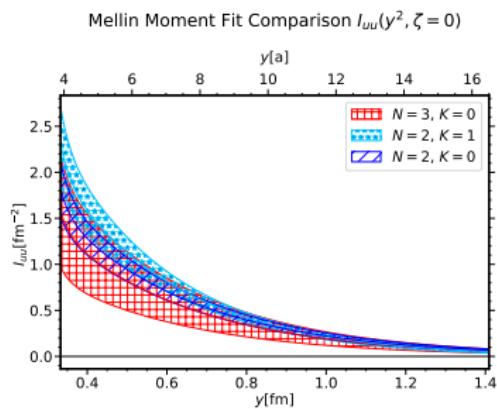
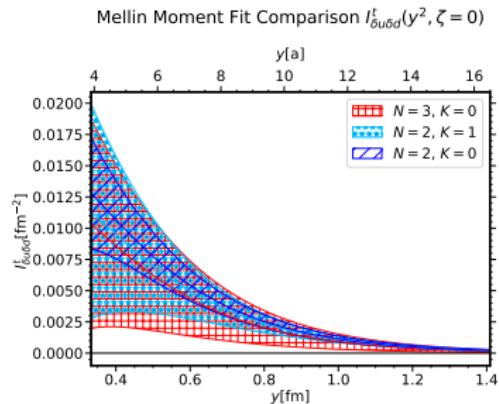
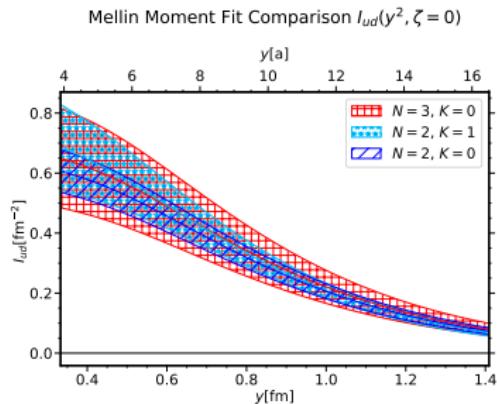
$$F(x_1, x_2, \zeta, \mathbf{y}) = F(x_1, x_2, -\zeta, \mathbf{y}) \quad |x_i \pm \zeta/2| \leq 1 \quad |\zeta| \leq 1$$

Total ansatz (red: fit parameters)

$$A(py, y^2) = \sum_{i=1,2} A_i e^{-\eta_i (y - y_0)} \sum_{n,m=0}^N \sum_{k=0}^K c'_{nk} \sqrt{-y^2}^{k+\delta} \eta_i^\delta h_n(py)$$
$$I(y^2) = \pi \sum_{i=1,2} A_i e^{-\eta_i (y - y_0)} \sum_{k=0}^K c'_{0k} \sqrt{-y^2}^{k+\delta} \eta_i^\delta$$

Caution: Preliminary ansatz! We are currently exploring more sophisticated models based on parton splitting at small y

Mellin moments: Fit ansatz dependence



DPD number sum rule

For $x_1 > 0$ (otherwise $F_{qq'}(x_1, \dots) \rightarrow -F_{\bar{q}q'}(-x_1, \dots)$)

The number sum rule [Gaunt, Stirling '10; Diehl, Plößl, Schäfer '19]

$$\begin{aligned} \int_{-1}^1 dx_2 \int_{b_0/\mu} d^2 \mathbf{y} F_{qq'}(x_1, x_2, \mathbf{y}; \mu) = \\ = (N_{q'} + \delta_{q\bar{q}'} - \delta_{qq'}) f_q(x_1; \mu) + \mathcal{O}(\alpha_s(\mu)) + \mathcal{O}((b_0 \Lambda / \mu)^2) \end{aligned}$$

with $b_0 = 2e^{-\gamma}$ and $\mu = 2$ GeV ($\gamma \approx 0.577$, splitting singularity $\sim \alpha_s / \mathbf{y}^2$)

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Implies for I_{ud}

$$\int_{b_0/\mu} d^2 \mathbf{y} I_{ud}(\mathbf{y}^2) = 2 + \mathcal{O}(\alpha_s^2(\mu)) + \mathcal{O}((b_0 \Lambda / \mu)^2)$$

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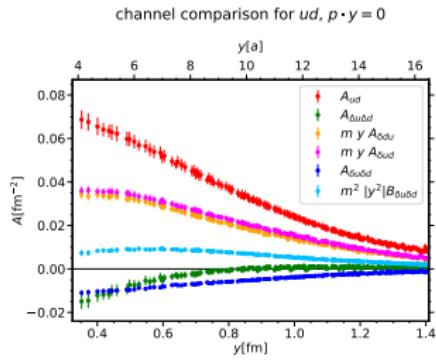
From our data:

N	K	χ^2/dof	integral
2	0	0.47	1.93(23)
3	0	0.46	2.07(51)
2	1	0.46	1.98(24)

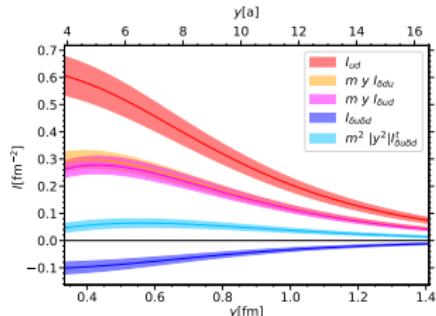
Results for the pion

Comparison of A_{ab} and I_{ab} for ud :

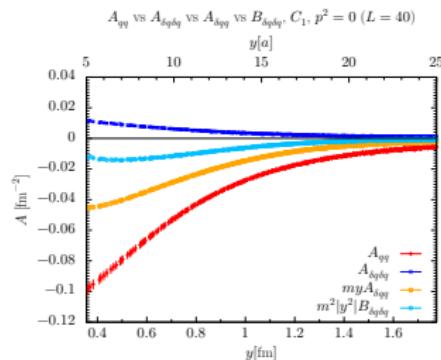
proton (p)



Mellin Moment $I(y^2, \zeta = 0)$ for ud , $N = 2, K = 0$



pion π^+



Mellin Moment $I_{\delta\delta}(y^2, \zeta = 0)$ for ud , $N = 1, M = 1$

