Coupled-channel dispersive analysis of {ππ, KK} scattering and its application to ee→J/ψππ(KK) and $(g\text{-}2)_\mu$

Oleksandra Deineka*

in coll. with *Igor Danilkin, Marc Vanderhaeghen
Martin Hoferichter and Peter Stoffer

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THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

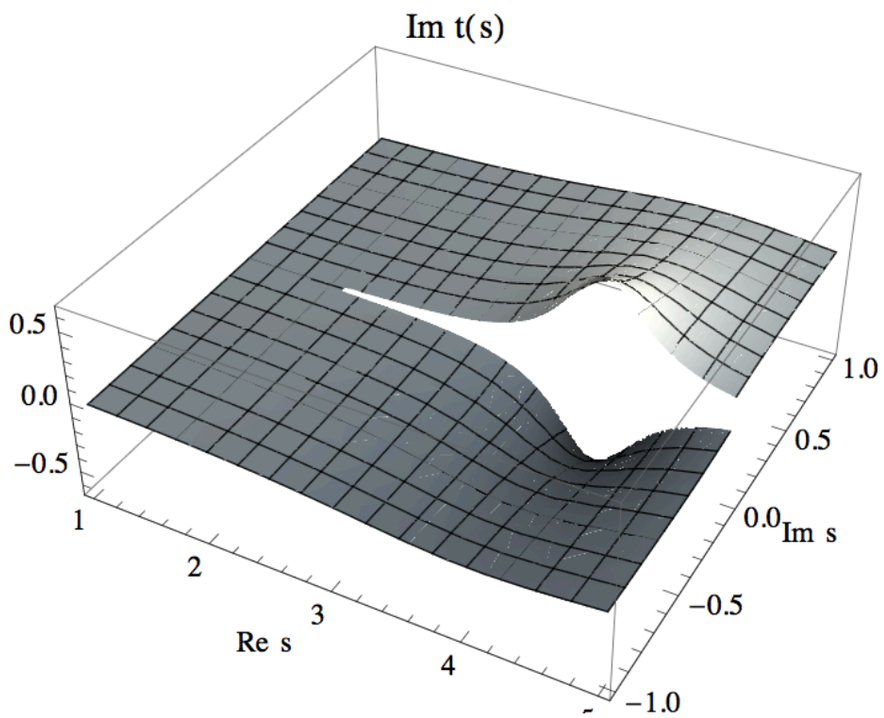
JOHANNES GUTENBERG UNIVERSITÄT MAINZ

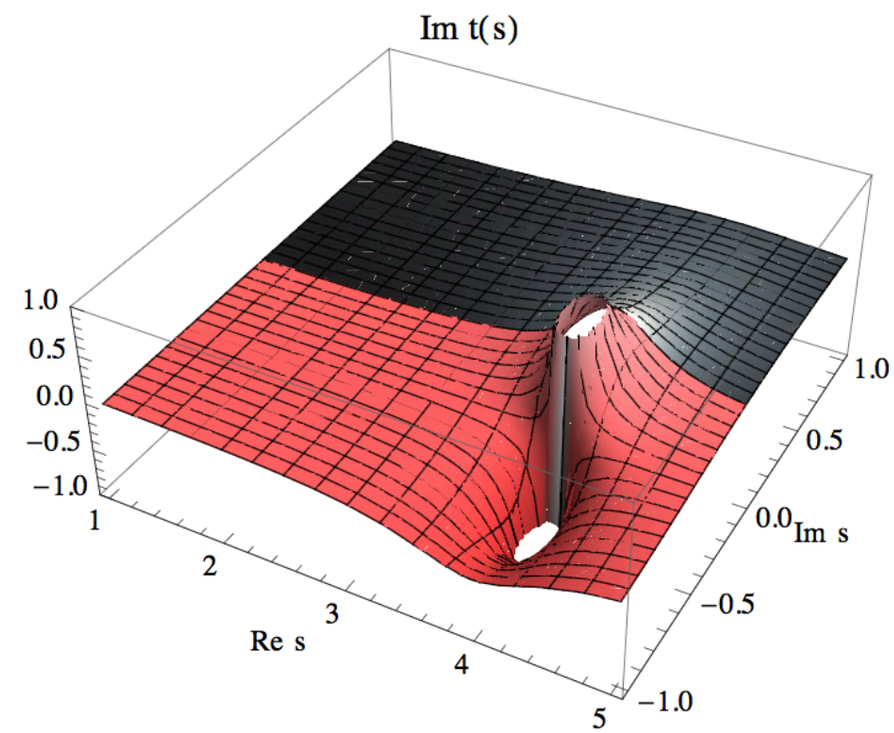
JG|U

Introduction and Motivation

- New era of hadron spectroscopy, motivated by recent discoveries of unexpected exotic hadron resonances: LHCb, BESIII, COMPASS, Belle, …

- To correctly identify resonance parameters one has to search for poles in the complex plane





- Particularly important when
  - ➢ there is an interplay between several inelastic channels
  - ➢ the pole is lying very deep in the complex plane

Introduction and Motivation

- Call for a framework which complies with the main principles of the S-matrix theory:
  - ➢ Unitarity
  - ➢ Analyticity
  - ➢ Crossing symmetry

Roy (Roy-Steiner) equations

- Practical application of **Roy-like equations** is **limited**:
  - ➢ requires experimental knowledge of many partial waves in direct and crossed channels
  - ➢ finite truncation limits results to a given kinematical region
  - ➢ coupled-channel treatment is very complicated

$$t_J^I(s) = k_J^I(s) + \sum_{I'=0}^{2} \sum_{J'=0}^{\infty} \int_{4m_\pi^2}^{\infty} ds' \, K_{JJ'}^{II'}(s,s') \, \mathrm{Im} t_{J'}^{I'}(s')$$

subtraction polynomial

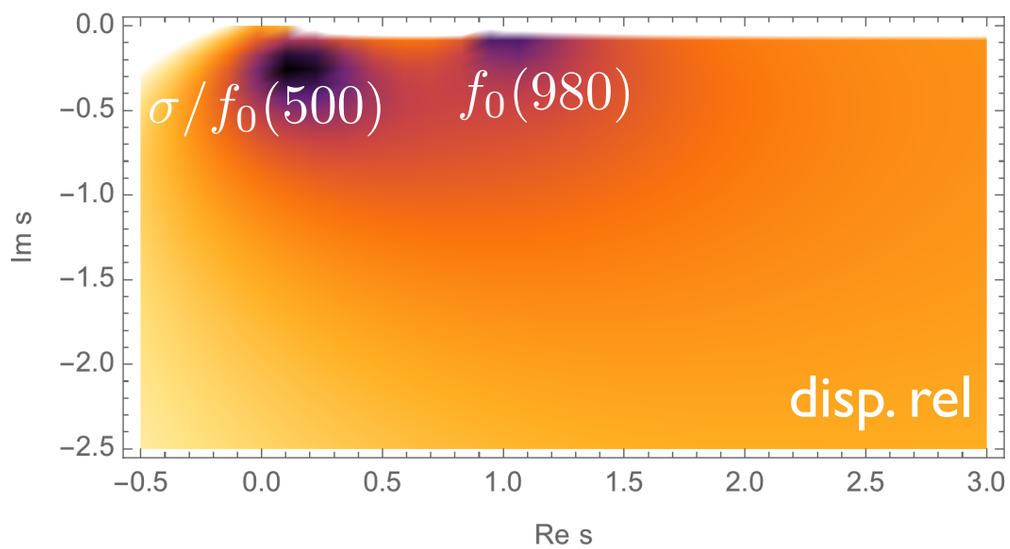
kernel functions known analytically

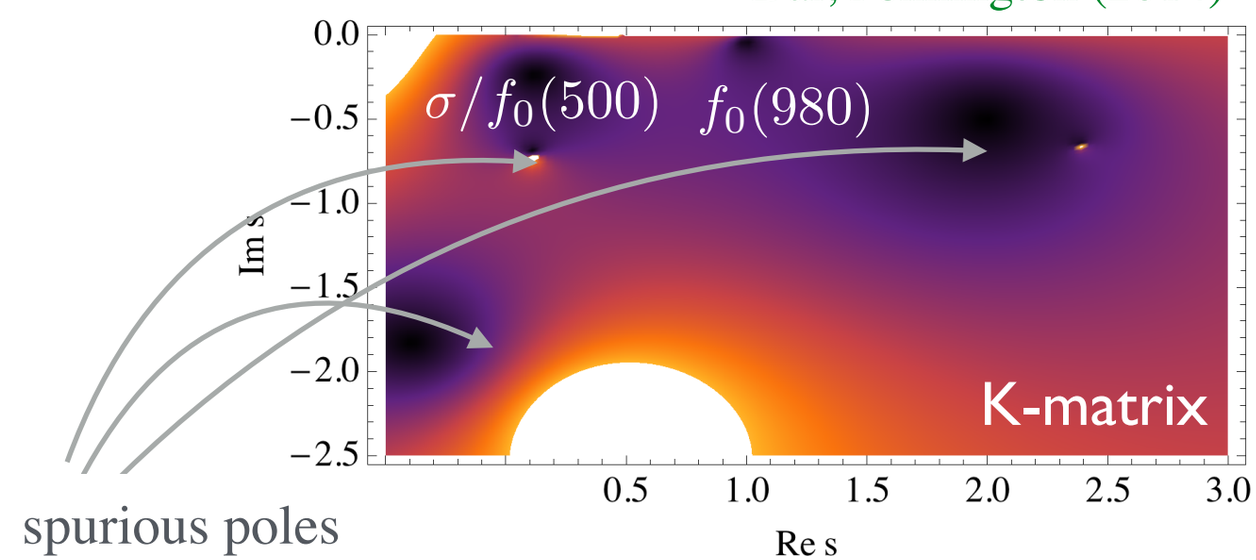
Roy (1971)
Colangelo et al. (2001)
Caprini et al. (2006)
Garcia-Martin et al. (2011)

Introduction and Motivation

- It is a common practice to **ignore** all S-matrix constraints or implement just **unitarity**:
  - ➢ Sum of Breit-Wigner parameterisations
  - ➢ Bethe-Salpeter like equations
  - ➢ K-matrix
  - ➢ ….

Dai, Pennington (2014)





- Alternatively, once can consider a coupled-channel **p.w. dispersion relation** which respects both **uniarity** and **analyticity**. It can be solved using **N/D ansazt**

Chew, Mandelstam (1960)
Luming (1964)
Johnson, Warnock (1981)

$$t_{ab}(s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\text{Disc } t_{ab}(s')}{s'-s} + \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{\text{Disc } t_{ab}(s')}{s'-s} = \sum_c D^{-1}_{ac}(s)\, N_{cb}(s)$$

Introduction and Motivation

- Why do we solve **p.w. dispersion relation** for the **coupled-channel** {ππ,KK} scattering?

  ➢ Well studied systems (amplitudes are known from Roy (Roy-Steiner) analyses) Caprini et al. (2006) Garcia-Martin et al. (2011)

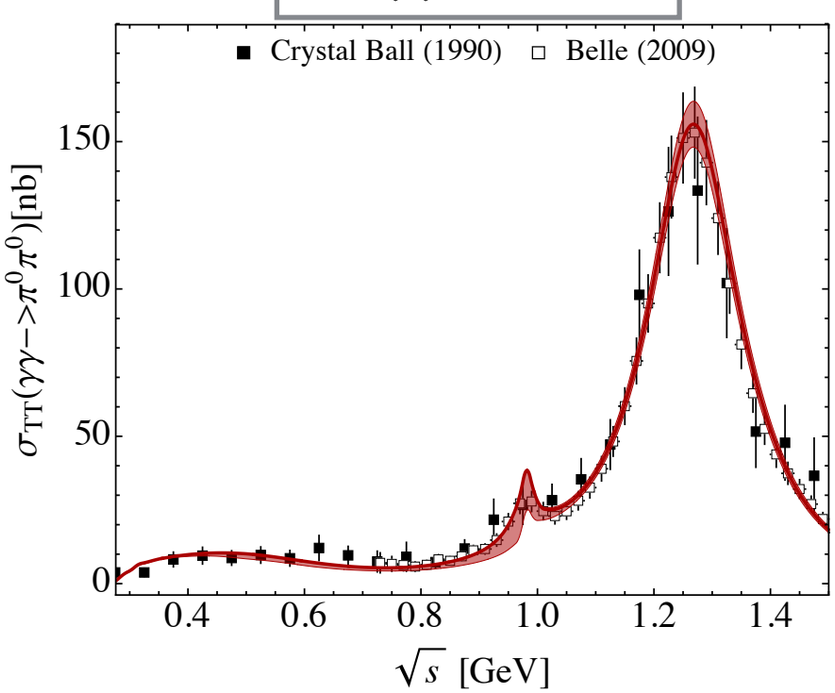
  ➢ Recent **lattice** studies of {ππ,KK} scattering for $m_\pi \neq$ physical Hadron Spectrum Coll.(2017, 2019)

  ➢ The system of {ππ,KK} shows up very often as a part of FSI in many hadronic reactions:

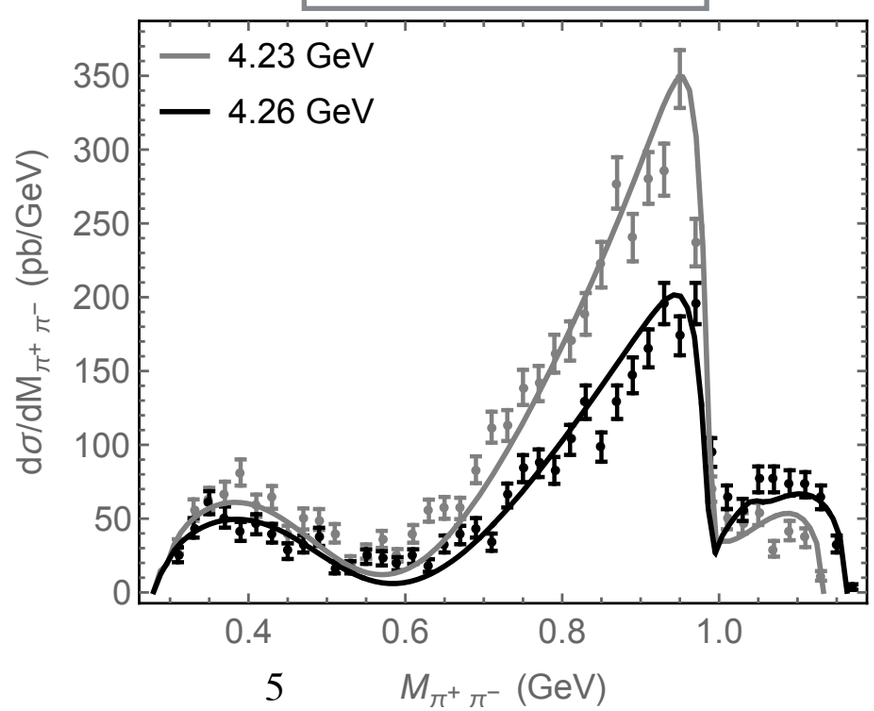
  $\gamma^*\gamma^* \to \{\pi\pi, KK\}$, $e^+e^- \to J/\psi\{\pi\pi, KK\}$, $\varphi \to \gamma\{\pi\pi, KK\}, \ldots$

- In practical applications, the FSI are implemented with the help of so-called **Omnes function**, which does not have left-hand cuts. It arises naturally from the *N/D* approach as an inverse of the *D*-function

$\gamma\gamma \to \pi^0\pi^0$



$e^+e^- \to J/\psi\pi\pi$



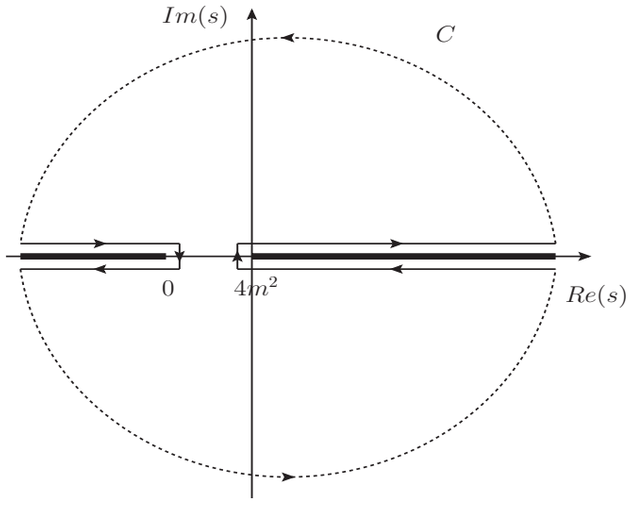
I.D, Deineka, Vanderhaeghen (2019, 2020)
I.D, Molnar, Vanderhaeghen (2019, 2020)
I.D, Hoferichter, Stoffer (2021)

$f_0(500)$ & $f_0(980)$ to $(g-2)_\mu$

p.w. dispersion relation

- Based on maximal analyticity principle on can write **p.w. dispersion relation**

$$t_{ab}(s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\text{Disc } t_{ab}(s')}{s'-s} + \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{\text{Disc } t_{ab}(s')}{s'-s}$$



- **Unitarity relation** for the p.w. amplitude
  ➢ guarantees that the p.w. amplitudes behave asymptotically no worse **than a constant**

$$\text{Disc}\, t_{ab}(s) = \sum_c t_{ac}(s)\, \rho_c(s)\, t^*_{cb}(s)$$

$$-\frac{1}{2\rho_1} \leq \text{Re}\, t_{11}(s) \leq \frac{1}{2\rho_1}, \quad 0 < \text{Im}\, t_{11}(s) \leq \frac{1}{\rho_1}$$

...

- In accordance with **unitarity bound** we subtract once the dispersion relation

$$t_{ab}(s) = \underbrace{t_{ab}(0) + \frac{s}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\text{Disc } t_{ab}(s')}{s'-s}}_{U_{ab}(s)} + \frac{s}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t_{ac}(s')\, \rho_c(s')\, t^*_{cb}(s')}{s'-s}$$

N/D method

- Once-subtracted p.w. dispersion relation

$$t_{ab}(s) = \underbrace{t_{ab}(0) + \frac{s}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\text{Disc } t_{ab}(s')}{s'-s}}_{U_{ab}(s)} + \frac{s}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$

can be solved using N/D method with input from $U_{ab}(s)$ **above threshold**

Chew, Mandelstam (1960)
Luming (1964)
Johnson, Warnock (1981)

$$t_{ab}(s) = \sum_c D^{-1}_{ac}(s)\, N_{cb}(s)$$

$$N_{ab}(s) = U_{ab}(s) + \frac{s}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{N_{ac}(s')\,\rho_c(s')\,(U_{cb}(s') - U_{cb}(s))}{s'-s}$$

$$D_{ab}(s) = \delta_{ab} - \frac{s}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{N_{ab}(s')\,\rho_b(s')}{s'-s} = \Omega^{-1}_{ab}(s)$$

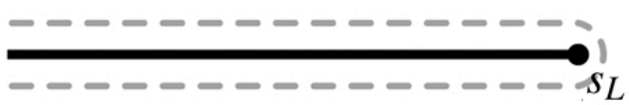
the obtained N/D solution **fulfils** the p.w. dispersion relation

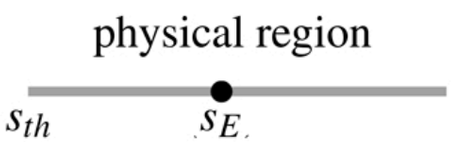
- Using the known analytical structure of left-hand cuts, one can approximate $U_{ab}(s)$ as an expansion in a **conformal mapping variable** $\xi(s)$

Gasparyan, Lutz (2010)

$$U_{ab}(s) = \sum_{n=0}^{\infty} C_{ab,n}\,(\xi_{ab}(s))^n$$

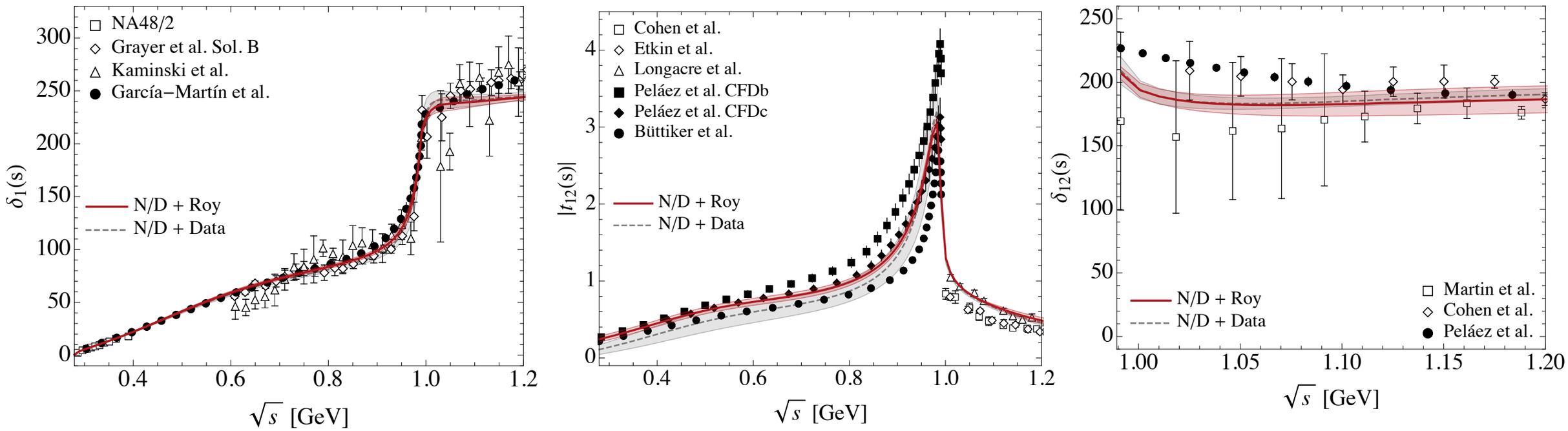
unknown coefficients fitted to data





$\xi(s_E) = 0$

Coupled-channel analysis {ππ, KK}



Input: experimental data/Roy analysis + threshold parameters NNLO (a, b) + Adler zero NLO

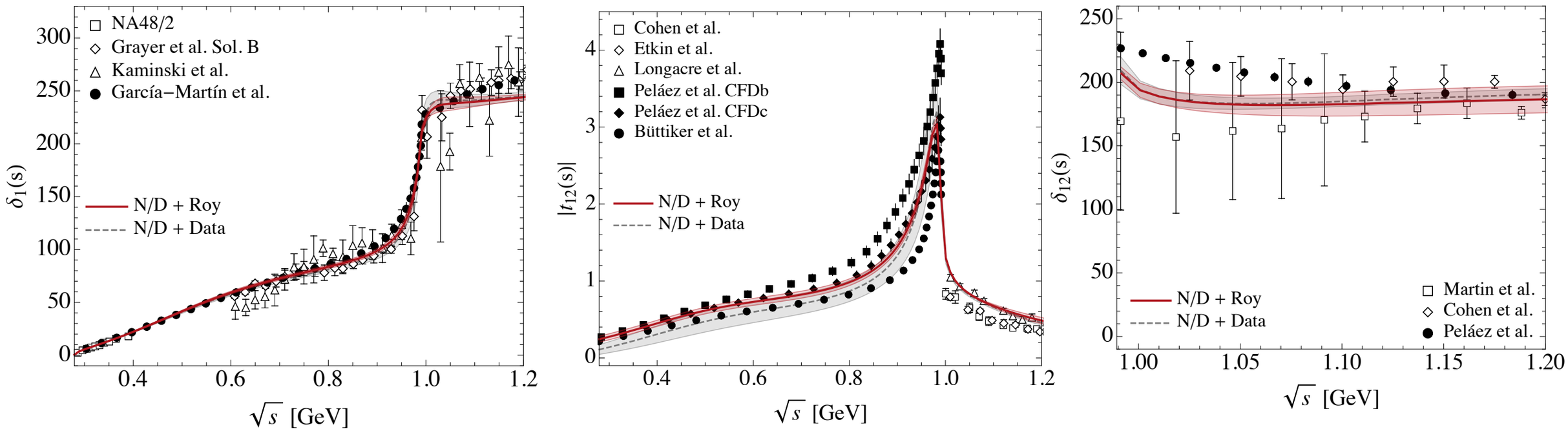
$$t_{ab}(s) = \begin{pmatrix} \frac{\eta(s)\, e^{2\,i\,\delta_1(s)} - 1}{2\,i\,\rho_1(s)} & |t_{12}(s)|\, e^{\delta_{12}(s)} \\ |t_{12}(s)|\, e^{\delta_{12}(s)} & \frac{\eta(s)\, e^{2\,i\,\delta_2(s)} - 1}{2\,i\,\rho_2(s)} \end{pmatrix}_{ab}$$

$$\eta(s) = \sqrt{1 - 4\,\rho_1(s)\,\rho_2(s)\,|t_{12}(s)|^2}$$

$$\delta_{12}(s) = \delta_1(s) + \delta_2(s)\,\theta(s > 4m_K^2)$$

In the **two-channel approximation** one needs to make the choice of which experimental data/Roy analysis include in the fit

Coupled-channel analysis {ππ, KK}

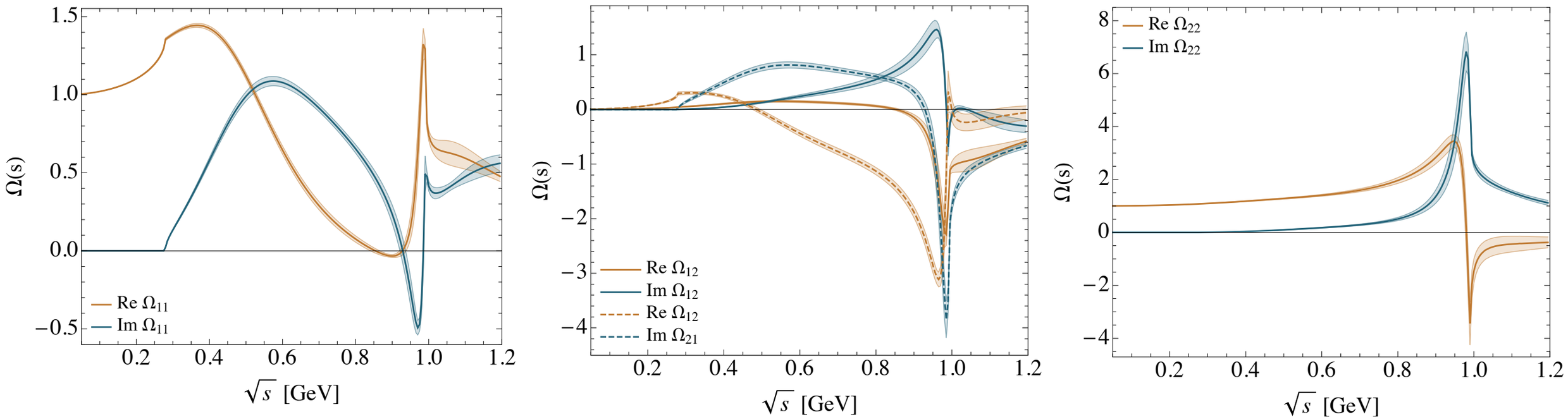


Input: experimental data/Roy analysis + threshold parameters NNLO (a, b) + Adler zero NLO

	Our results		Roy-like analyses	
	pole position, MeV	couplings, GeV	pole position, MeV	couplings, GeV
$\sigma/f_0(500)$	$458(10)^{+7}_{-15} - i\,256(9)^{+5}_{-8}$	$\pi\pi : 3.33(8)^{+0.12}_{-0.20}$ $K\bar{K} : 2.11(17)^{+0.27}_{-0.11}$	$449^{+22}_{-16} - i\,275(15)$	$\pi\pi : 3.45^{+0.25}_{-0.29}$ $K\bar{K} : -$
fit to Exp	$454(12)^{+6}_{-7} - i\,262(12)^{+8}_{-12}$			
$f_0(980)$	$993(2)^{+2}_{-1} - i\,21(3)^{+2}_{-4}$	$\pi\pi : 1.93(15)^{+0.07}_{-0.12}$ $K\bar{K} : 5.31(24)^{+0.04}_{-0.24}$	$996^{+7}_{-14} - i\,25^{+11}_{-6}$	$\pi\pi : 2.3(2)$ $K\bar{K} : -$
fit to Exp	$990(7)^{+2}_{-4} - i\,17(7)^{+4}_{-1}$			

Caprini et al. (2006)
Garcia-Martin et al. (2011)
Moussallam (2011)

Omnes matrix {ππ, KK}



Omnes function fulfils the unitarity relation on the right-hand cut and analytic everywhere else.
For the case of no bound states or CDD poles:

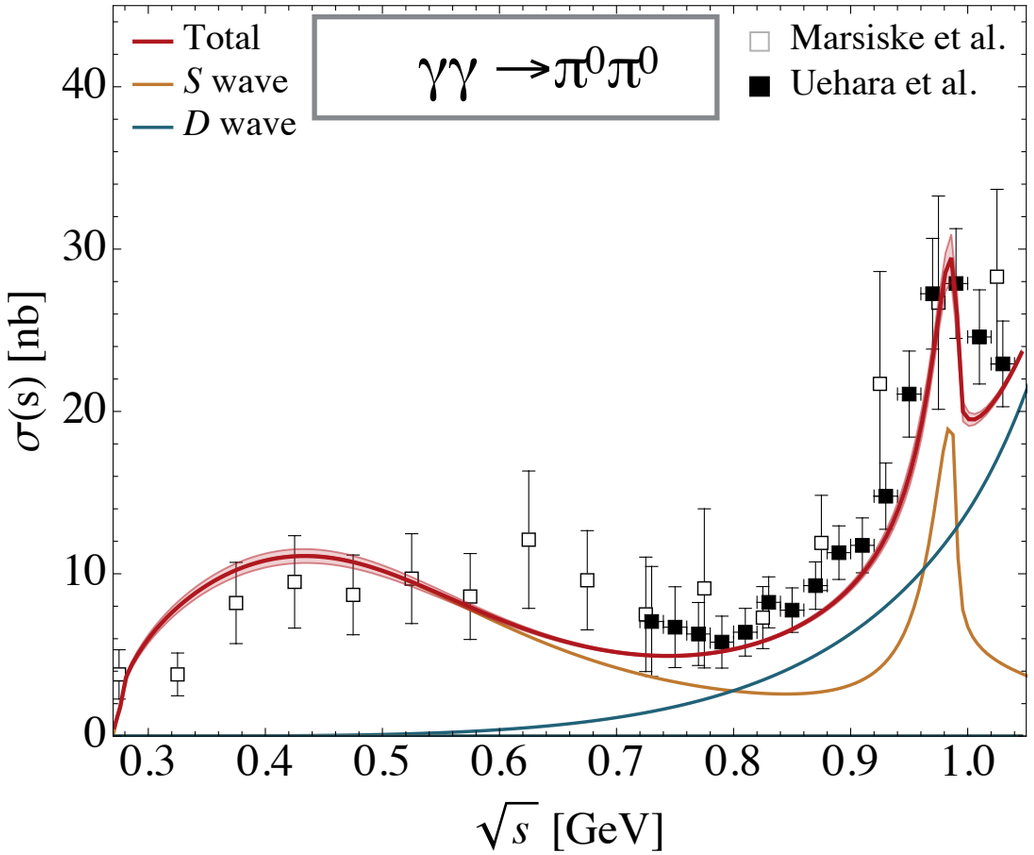
$$\Omega_{ab}(s) = D^{-1}_{ab}(s)$$

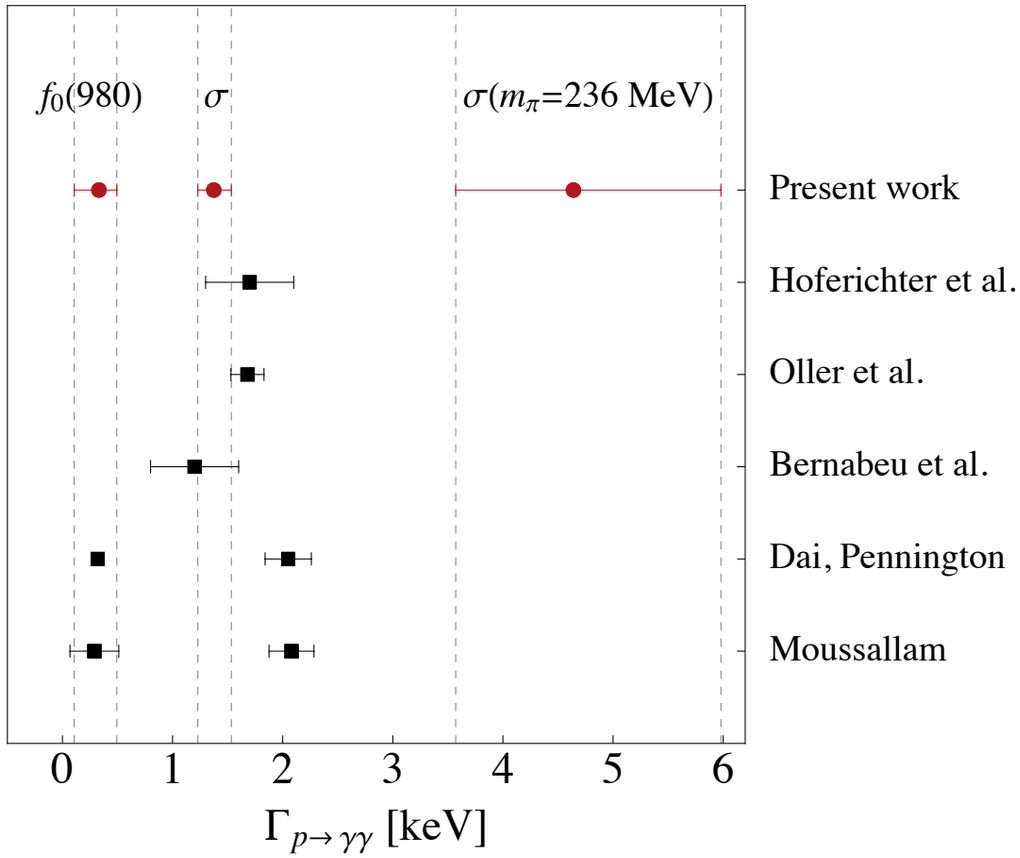
which automatically satisfies a once-subtracted dispersion relation (i.e. Ω(s) is asymptotically bounded)

$$\Omega_{ab}(s) = \delta_{ab} + \frac{s}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t^*_{ac}(s')\, \rho_c(s')\, \Omega_{cb}(s')}{s' - s}$$

different from
Donoghue et al. (1990)
Moussallam (2000)

Applications of the Omnes matrix





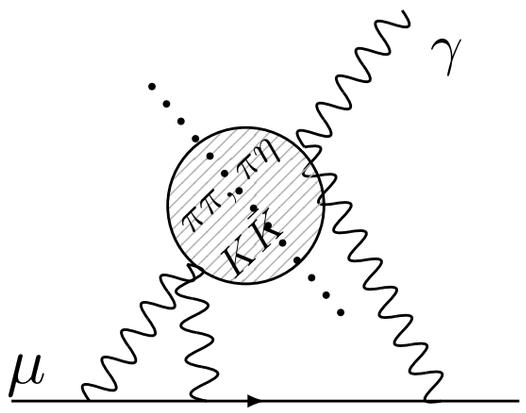
If in the coupled-channel system {γγ, ππ, KK} we neglect γ γ intermediate states in the unitarity relation: (3 × 3) dispersion relation reduces down to (2×1) which requires **hadronic Omnes (2×2) matrix** as input

$$\begin{pmatrix} t_{12}(s) \\ t_{13}(s) \end{pmatrix} = \underbrace{\begin{pmatrix} U_{12}(s) \\ U_{13}(s) \end{pmatrix}}_{\text{Born}} + D^{-1}(s) \left[-\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\text{Disc}\, D(s')}{s'-s} \begin{pmatrix} U_{12}(s') \\ U_{13}(s') \end{pmatrix} \right]$$

Alternatively, this equation can be obtained by writing a dispersion relation for $(t(s) - U(s))D^{-1}(s)$

Muskhelishvili (1953) Omnès (1958)

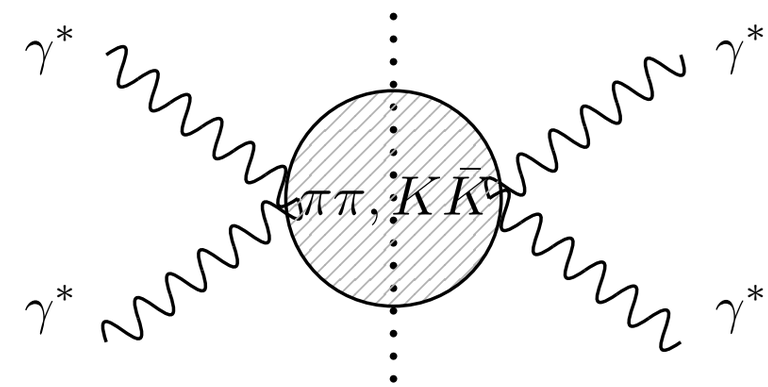
Contribution to (g-2)



Important ingredients:

$\gamma^*\gamma^* \to \pi\pi, K\bar{K}, ...$

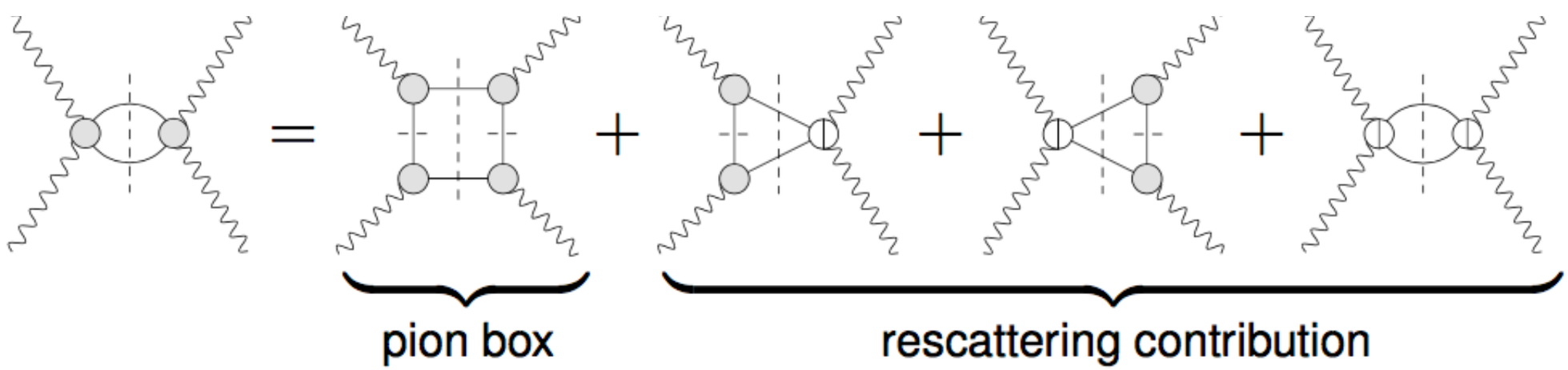
$q^2 = -Q^2 < 0$ space-like γ^*



$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\, Q_1^3\, Q_2^3 \sum_{i=1}^{12} T_i(Q_1,Q_2,Q_3)\, \bar{\Pi}_i(Q_1,Q_2,Q_3),$$

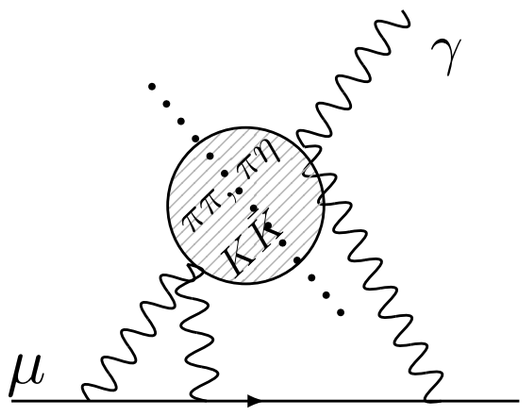
Colangelo et al. (2014-2017)

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\, \Pi_i \qquad \bar{\Pi}_i \text{ linear combination of } \Pi_i$$



$$a_\mu^{\pi\pi,KK}[\mathrm{box}] = -16.4(2) \times 10^{-11}$$

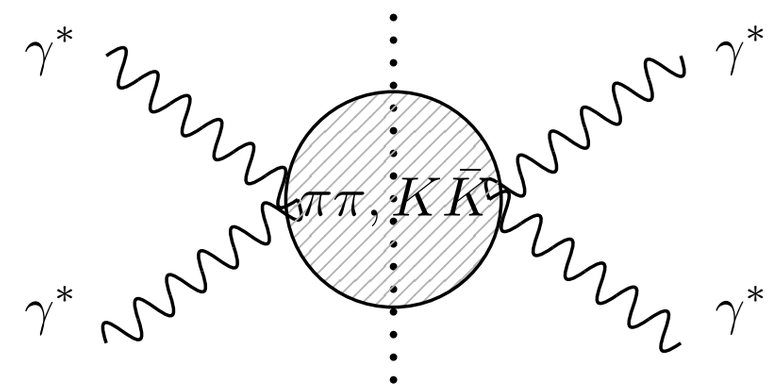
Contribution to (g-2)



Important ingredients:

$\gamma^*\gamma^* \to \pi\pi, K\bar{K}, ...$

$q^2 = -Q^2 < 0$ space-like γ^*



$$a_\mu^{\rm HLbL} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\, Q_1^3\, Q_2^3 \sum_{i=1}^{12} T_i(Q_1,Q_2,Q_3)\, \bar{\Pi}_i(Q_1,Q_2,Q_3),$$

Rescattering contribution ($\bar{h} \equiv h - h^{\rm Born}$) in the S-wave

$$\bar{\Pi}_3^{J=0} = \frac{1}{\pi}\int_{4m_\pi^2}^\infty ds' \frac{-2}{\lambda_{12}(s')(s'+Q_3^2)^2}\Big(4s'\,{\rm Im}\bar{h}^{(0)}_{++,++}(s') - (s'-Q_1^2+Q_2^2)(s'+Q_1^2-Q_2^2)\,{\rm Im}\bar{h}^{(0)}_{00,++}(s')\Big)$$

$$\bar{\Pi}_9^{J=0} = \frac{1}{\pi}\int_{4m_\pi^2}^\infty ds' \frac{4}{\lambda_{12}(s')(s'+Q_3^2)^2}\Big(2\,{\rm Im}\bar{h}^{(0)}_{++,++}(s') - (s'+Q_1^2+Q_2^2)\,{\rm Im}\bar{h}^{(0)}_{00,++}(s')\Big)$$

+crossed

Unitarity $\gamma^*\gamma^* \to \gamma^*\gamma^*$ $\gamma^*\gamma^* \to \pi\pi$ $\gamma^*\gamma^* \to KK$

$$\mathrm{Im}\bar{h}^{(0)}_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s) = \frac{1}{2}\,\bar{h}^{(0)}_{\lambda_1\lambda_2}(s)\,\rho_\pi(s)\,\bar{h}^{(0)*}_{\lambda_3\lambda_4}(s) + \frac{1}{2}\,\bar{k}^{(0)}_{\lambda_1\lambda_2}(s)\,\rho_K(s)\,\bar{k}^{(0)*}_{\lambda_3\lambda_4}(s)$$

Contribution to (g-2)

- Using S-wave elastic helicity amplitudes on $\gamma^*\gamma^* \to \pi\pi$, $f_0(500)$ contribution was calculated previously

$$a_\mu^{\rm HLbL}[\text{S-wave}, I=0]_{\rm rescattering} = -9.3(1) \times 10^{-11}$$

Colangelo et al. (2014-2017)

- Extending to KK channel allowed us to access energies up to ~1.2 GeV ($f_0(500)$ + $f_0(980)$ contributions)

$$a_\mu^{\rm HLbL}[\text{S-wave}, I=0]_{\rm rescattering} = -9.8(1) \times 10^{-11}$$

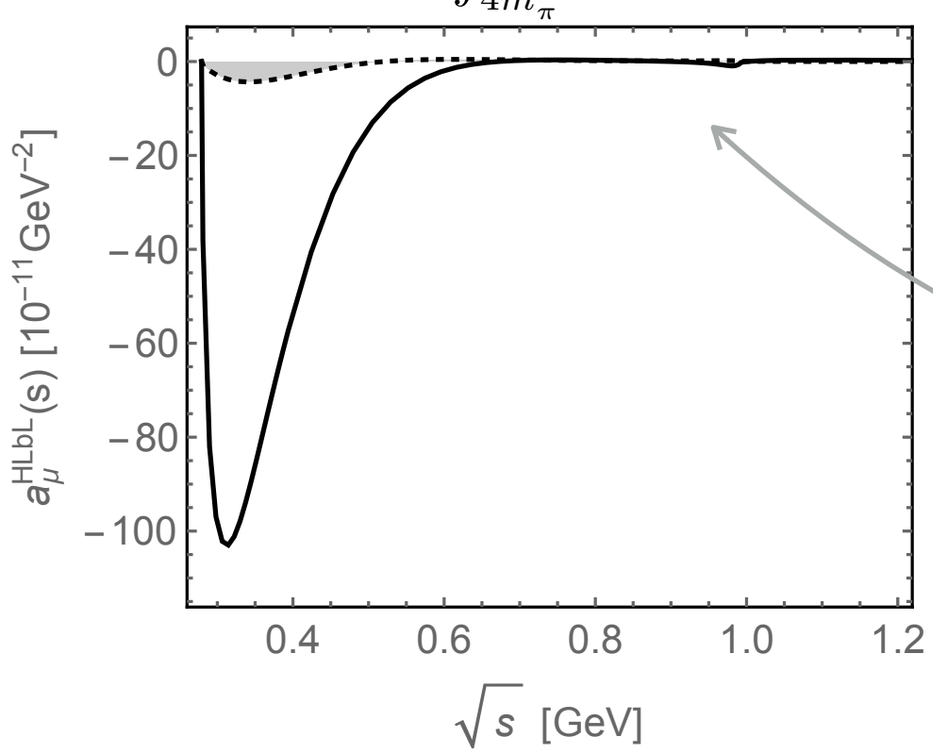
I.D, Hoferichter, Stoffer (2021)

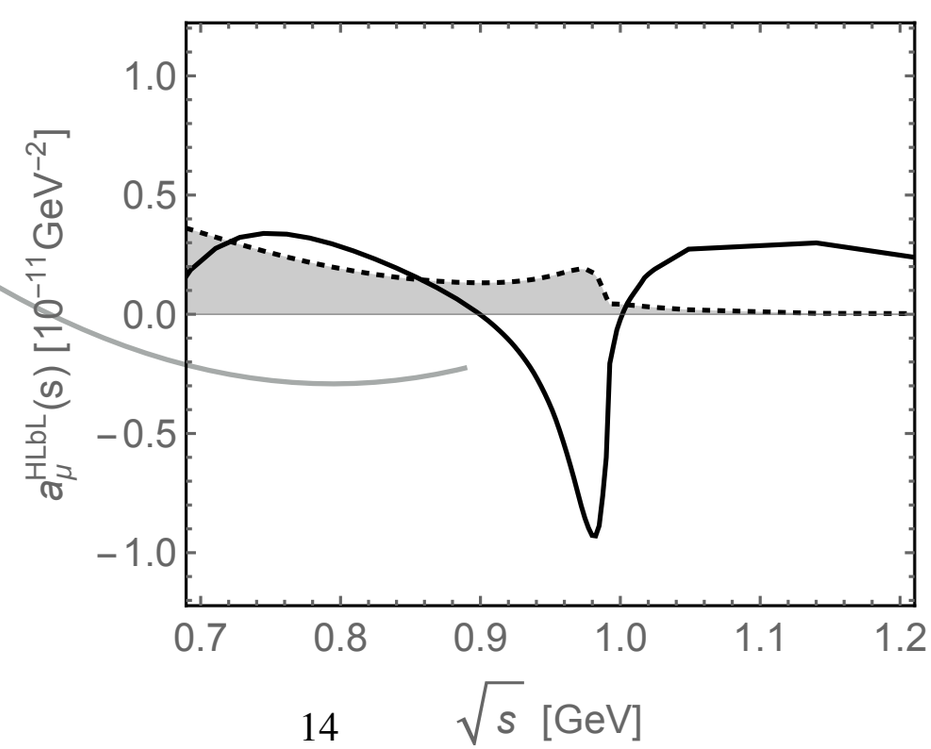
What is the contribution just from $f_0(980)$?

- One can define it as an integral over the deficit in shape

$$a_\mu^{\rm HLbL} = \int_{4m_\pi^2}^{\infty} ds' a_\mu^{\rm HLbL}(s')$$

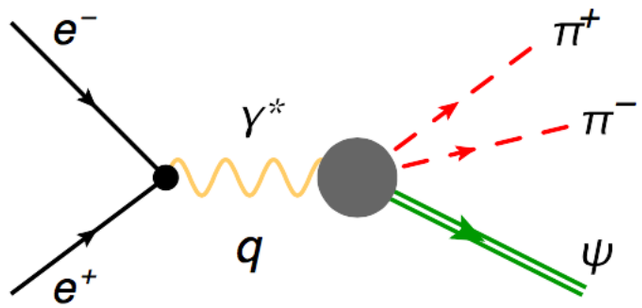
$$a_\mu^{\rm HLbL}[f_0(980)]_{\rm rescattering} = -0.2(1) \times 10^{-11}$$





shaded area is a sum rule violation → result is largely basis independent

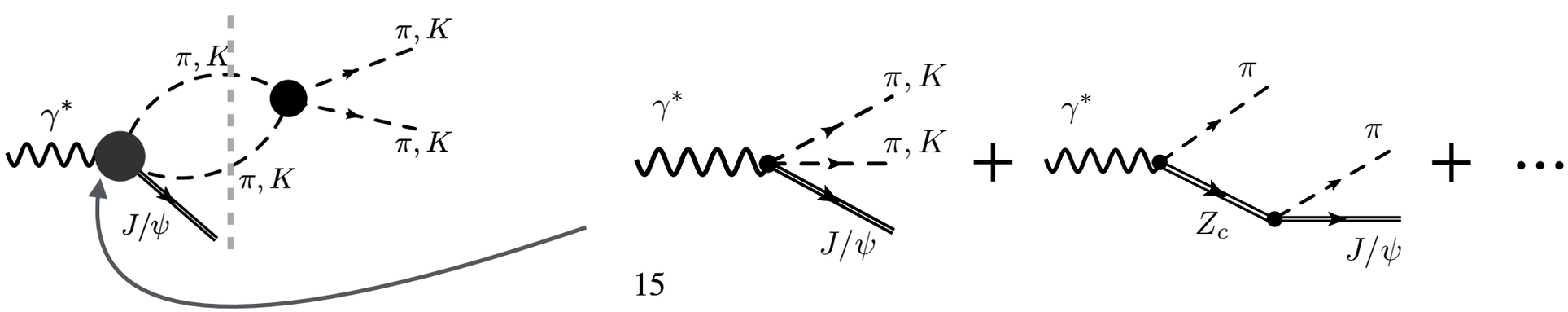
Application to $e^+e^- \to \pi\pi(KK)J/\psi$



- For fixed e^+e^- c.m. energy we perform a **simultaneous** description of $\pi\pi(KK)$ and $\pi\psi$ invariant mass distributions

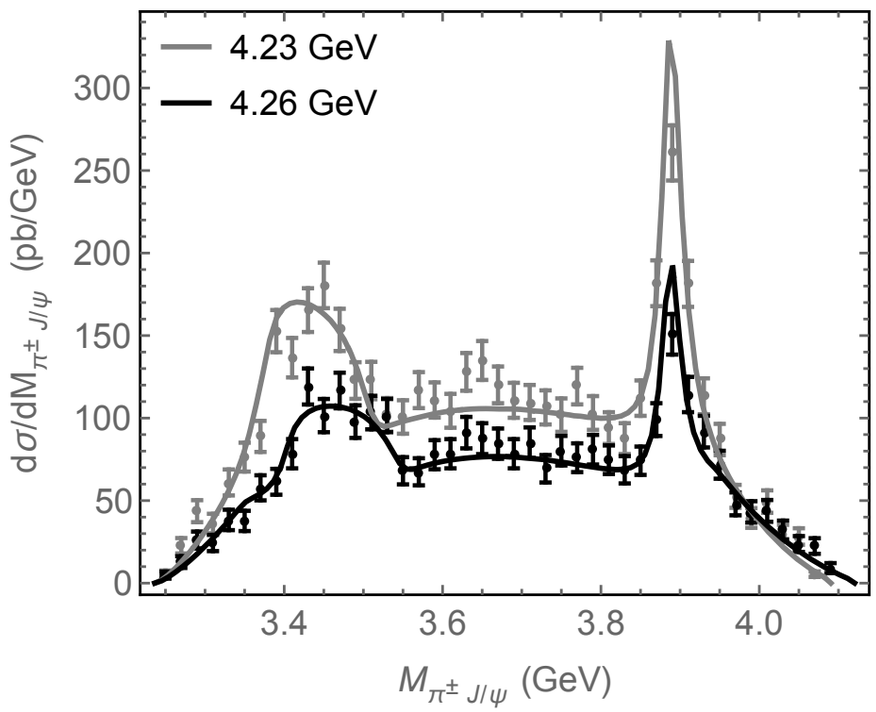
- Take **Zc(3900) as explicit d.o.f.** (i.e. minimum assumptions about its the nature)
  ➢a strange partner of Zc [recently observed Zcs(3985)] cannot be seen as peak in the KJ/ψ invariant mass distribution at 4.23 GeV and 4.26 GeV c.m. energies

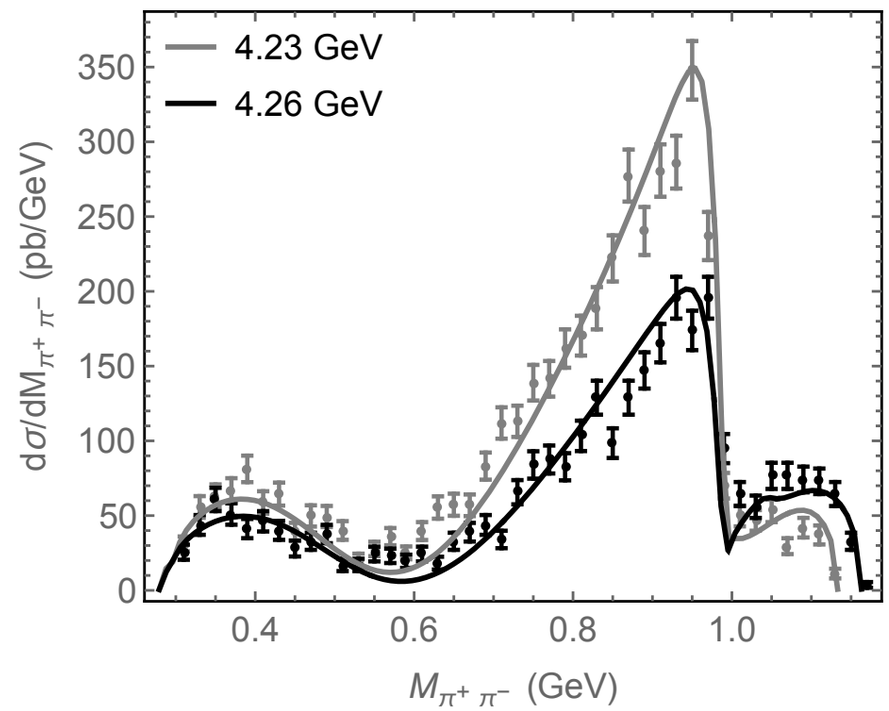
- Consider **$\pi\pi$(KK) FSI** in **S and D waves**
  ➢Direct interaction of two pions
  ➢Rescattering through Zc as a left-hand cut (so-called crossed-channel 3-body effect)
  ➢Other left-hand cuts absorbed in the subtraction constants



Results for $e^+e^- \to \pi\pi(KK)J/\psi$

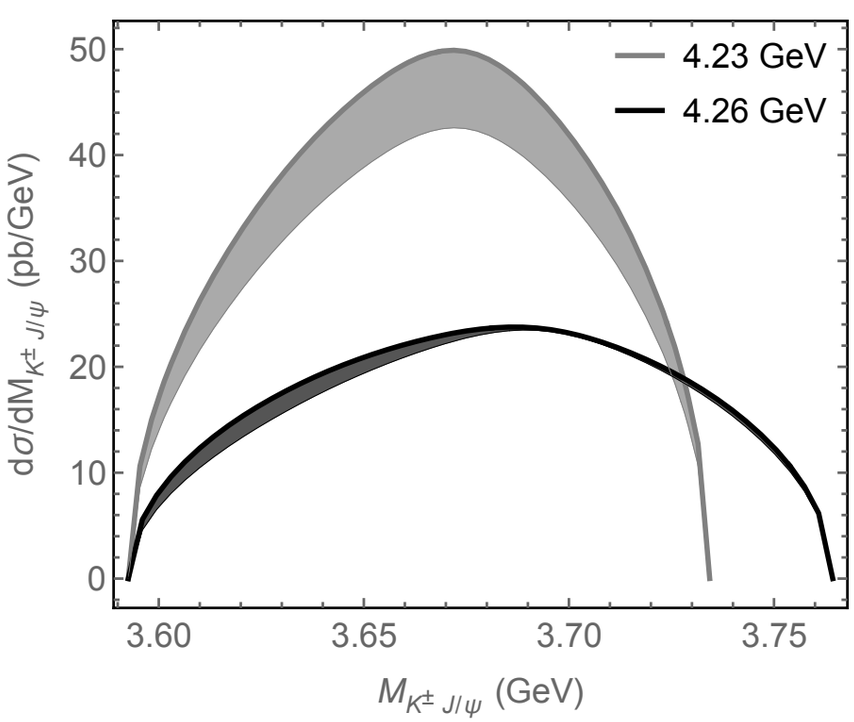
➢ Fit to invariant mass distributions and $\sigma(e^+e^- \to KKJ/\psi)$ total cross section (4 subtraction constants in CC)

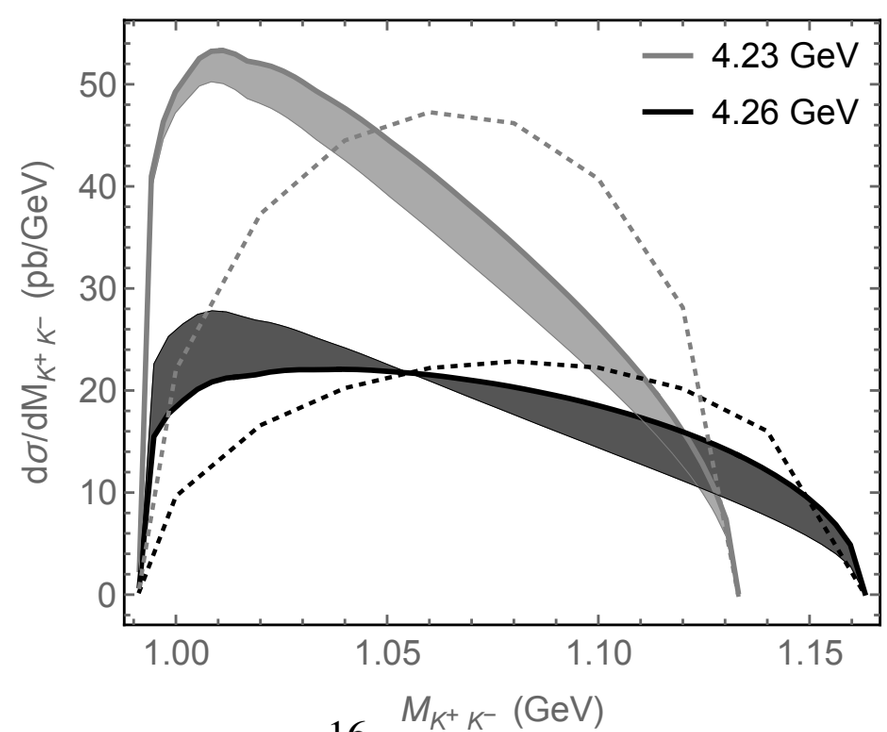




Obtained fit parameters do not vary much between q=4.23 and 4.26 GeV

Prediction





We observe a rapid rise just above threshold due to **f_0(980) resonance**

Summary and outlook

- We presented a data driven analysis of {ππ,KK} scattering using the once-subtracted **p.w. dispersion relation**, in which left-hand cuts were accounted for using conformal expansion that converges uniformly in the resonance region

  ➢ Obtained coupled-channel {ππ, KK} Omnes matrix (which we constrained by the recent Roy-like results) has already been implemented in the analysis of

  $e^+e^- \to J/\psi\{\pi\pi,KK\}$ and $f_0(980)$ to $(g-2)_\mu$

- p.w. dispersion relations is a **good alternative** to widely used unitarization techniques like Breit-Wigner, K-matrix, Bethe-Salpeter equations, …

  ➢ Can be applied to a vast **experimental** or **lattice data** which possesses a broad (or coupled-channel) resonance that does not have a genuine QCD nature ($a_0(980)$, X(6900), Tcc^+, …)

  ➢ can be matched to EFT

Results for $e^+e^- \to \pi\pi J/\psi$