XVth Quark Confinement & the Hadron Spectrum Conference

Non-perturbative insights into the spectral properties of finite-temperature correlation functions

(based on: 2201.12180, 2202.09142, 2207.14718)

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CRC-TR 211
Strong-interaction matter under extreme conditions

Local QFT beyond the vacuum

"Local QFT" → Define QFTs using a core set of physically-motivated assumptions, e.g. causality, Poincaré invariance, positive energy, ...

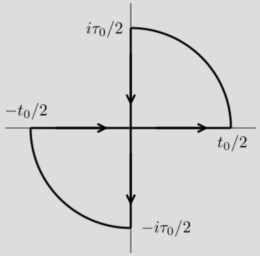
- This approach has led to many fundamental *non-perturbative* insights:

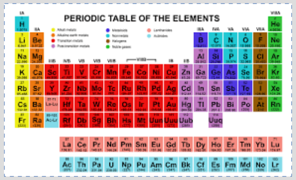
  → Relationship between Minkowski and Euclidean QFTs

  → *CPT* is a symmetry of *any* QFT

  → Connection between spin & particle statistics

  → Existence of dispersion relations, etc.





But... this framework only describes QFTs in the vacuum state

→ **Can one apply a similar approach to regimes where $T > 0$ or $\mu \neq 0$?**

Yes! *Important progress was made by J. Bros and D. Buchholz*

→ See: [*Z. Phys. C* 55 (1992), *Ann. Inst. H.Poincare Phys.Theor.* 64 (1996), *Nucl. Phys. B* 429 (1994), Nucl. Phys. B 627 (2002)]

Non-perturbative implications

- By demanding fields to be local ($[\Phi(x),\Phi(y)]=0$ for $(x\text{-}y)^2< 0$) this imposes significant constraints on the structure of correlation functions

$\rightarrow$ For $T=1/\beta > 0$, the scalar spectral function has the representation:

$$\rho(\omega,\vec{p}) := \mathcal{F}\left[\langle\Omega_\beta|\,[\phi(x),\phi(y)]\,|\Omega_\beta\rangle\right] = \int_0^\infty ds \int \frac{d^3\vec{u}}{(2\pi)^2}\,\epsilon(\omega)\,\delta\big(\omega^2-(\vec{p}-\vec{u})^2-s\big)\,\widetilde{D}_\beta(\vec{u},s)$$

Note: this is a ***non-perturbative*** representation!

"Thermal spectral density"

- In the limit of vanishing temperature one recovers the well-known *Källén-Lehmann* spectral representation:

$$\rho(\omega,\vec{p}) \xrightarrow{\beta\to\infty} 2\pi\epsilon(\omega)\int_0^\infty ds\;\delta(p^2-s)\,\rho(s)$$

e.g. $\rho(s)=\delta(s\text{-}m^2)$ for a massive free theory

Important question: *what does the thermal spectral density $\widetilde{D}_\beta(\boldsymbol{u},s)$ look like?*

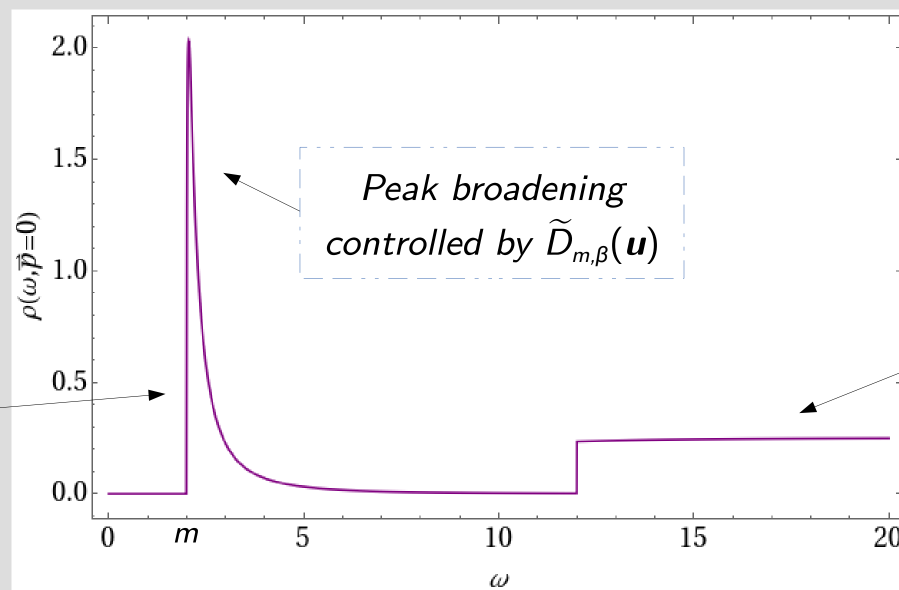
Non-perturbative implications

- A natural decomposition [Bros, Buchholz, *NPB* 627 (2002)] is:

$$\widetilde{D}_{\beta}(\vec{u},s) = \widetilde{D}_{m,\beta}(\vec{u})\,\delta(s-m^2) + \widetilde{D}_{c,\beta}(\vec{u},s)$$

"Damping factor"

Causes $T=0$ mass pole m to be ***screened*** *by thermal effects*



Continuous component

Fixes T-dependence of continuous spectral contributions

→ Damping factors hold the key to understanding in-medium effects!

In-medium observables from Euclidean data

- In many instances, $T > 0$ *Euclidean* data is used to calculate observables, e.g. spectral functions from $C_\Gamma(\tau, \vec{x}) = \langle O_\Gamma(\tau, \vec{x})\, O_\Gamma(0, \vec{0})\rangle_T$

$$\widetilde{C}_\Gamma(\tau, \vec{p}) = \int_0^\infty \frac{d\omega}{2\pi} \frac{\cosh\left[\left(\frac{\beta}{2} - |\tau|\right)\omega\right]}{\sinh\left(\frac{\beta}{2}\omega\right)} \rho_\Gamma(\omega, \vec{p})$$

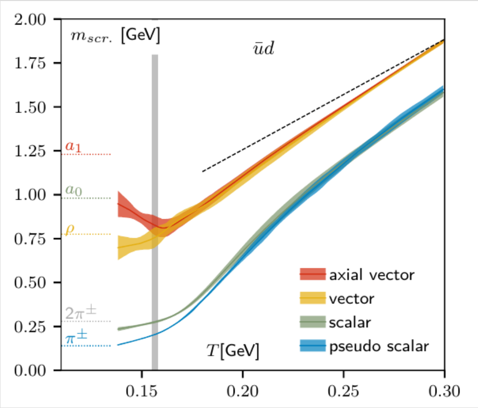
Determine $\rho_\Gamma(\omega, \mathbf{p})$ given $C_\Gamma(\tau, \mathbf{p})$

$\rightarrow$ ***Problem is ill-conditioned, need additional information!***

- A quantity of particular interest in lattice studies is the *spatial* correlator

$$C_\Gamma(x_3) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau\, C_\Gamma(\tau, \vec{x}) = \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} e^{ip_3 x_3} \int_0^\infty \frac{d\omega}{\pi\omega} \rho_\Gamma(\omega, p_1 = p_2 = 0, p_3)$$

- Large-x_3 behaviour $C_\Gamma(x_3) \sim exp(-m_{scr}|x_3|)$ used to extract *screening masses* $m_{scr}(T)$



[HotQCD collaboration, *Phys. Rev. D* 100 (2019)]

Locality constraints: spectral functions from lattice data

- Locality implies the following connection between the spatial correlator and thermal spectral density [P.L., 2201.12180; P.L, O. Philipsen, 2207.14718]

$$C(x_3) = \frac{1}{2}\int_0^\infty ds \int_{|x_3|}^\infty dR\, e^{-R\sqrt{s}} D_\beta(R,s)$$

→ Lightest T=0 states dominate:

$$C(x_3) \approx \frac{1}{2}\sum_{i=1}^{n} \int_{|x_3|}^\infty dR\, e^{-m_i R} D_{m_i,\beta}(R)$$

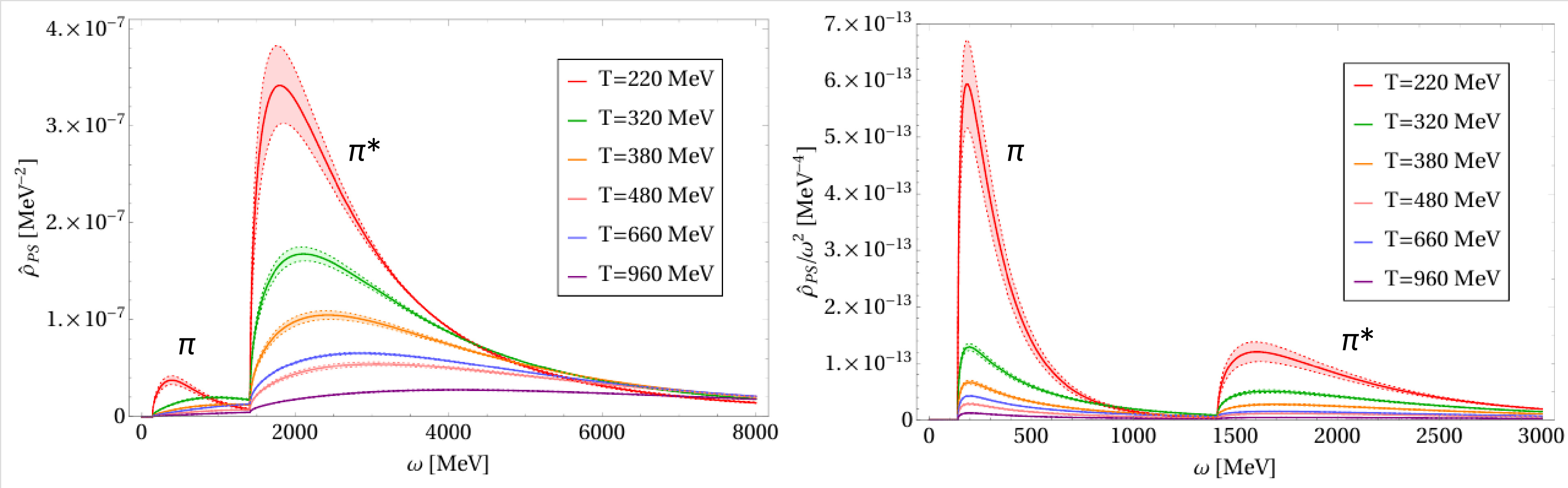
- Once the damping factors of all contributing states are known, one can use these to compute their contribution to $\rho(\omega,\boldsymbol{p})$

- In QCD, perhaps the simplest spatial correlator example is that of the light quark pseudo-scalar meson operator $\mathcal{O}^a_{\mathrm{PS}} = \overline{\psi}\gamma_5 \frac{\tau^a}{2}\psi$

Goal: Use lattice data from [Rohrhofer et al. *PRD* **100** (2019)] (N_f=2 with chiral fermions and physical masses) to compute the spectral function $\rho_{PS}(\omega,\boldsymbol{p})$

Locality constraints: spectral functions from lattice data

- **Step 1**: Perform fits to the spatial correlator data $C_{PS}(x_3)$ to obtain the functional dependence at different temperatures (A exp(-Bx_3)+C exp(-Dx_3) ansatz describes the data very well)
- **Step 2**: Calculate the corresponding damping factors from $C_{PS}(x_3)$ (for π and π^*)
- **Step 3**: Use $D_{m,\beta}(\boldsymbol{x})$ to compute $\rho_{PS}(\omega,\boldsymbol{p})$ via the spectral representation



The π and π^* dominate the spectral function at these T, and the π has a pronounced peak in some range $T > T_{pc}$ [P.L, O. Philipsen, 2207.14718]

Locality constraints: spectral functions from lattice data

- Exponential contributions to spatial correlators imply particle states with exponential damping factors → $D_{m_i,\beta}(\vec{x}) = \alpha_i \, e^{-\gamma_i |\vec{x}|}$

$$\rho_{\mathrm{PS}}(\omega,\vec{p}) = \epsilon(\omega) \Bigg[\theta(\omega^2 - m_\pi^2) \, \frac{4\,\alpha_\pi \gamma_\pi \sqrt{\omega^2 - m_\pi^2}}{(|\vec{p}|^2 + m_\pi^2 - \omega^2)^2 + 2(|\vec{p}|^2 - m_\pi^2 + \omega^2)\gamma_\pi^2 + \gamma_\pi^4} + \theta(\omega^2 - m_{\pi^*}^2) \, \frac{4\,\alpha_{\pi^*} \gamma_{\pi^*} \sqrt{\omega^2 - m_{\pi^*}^2}}{(|\vec{p}|^2 + m_{\pi^*}^2 - \omega^2)^2 + 2(|\vec{p}|^2 - m_{\pi^*}^2 + \omega^2)\gamma_{\pi^*}^2 + \gamma_{\pi^*}^4} \Bigg]$$

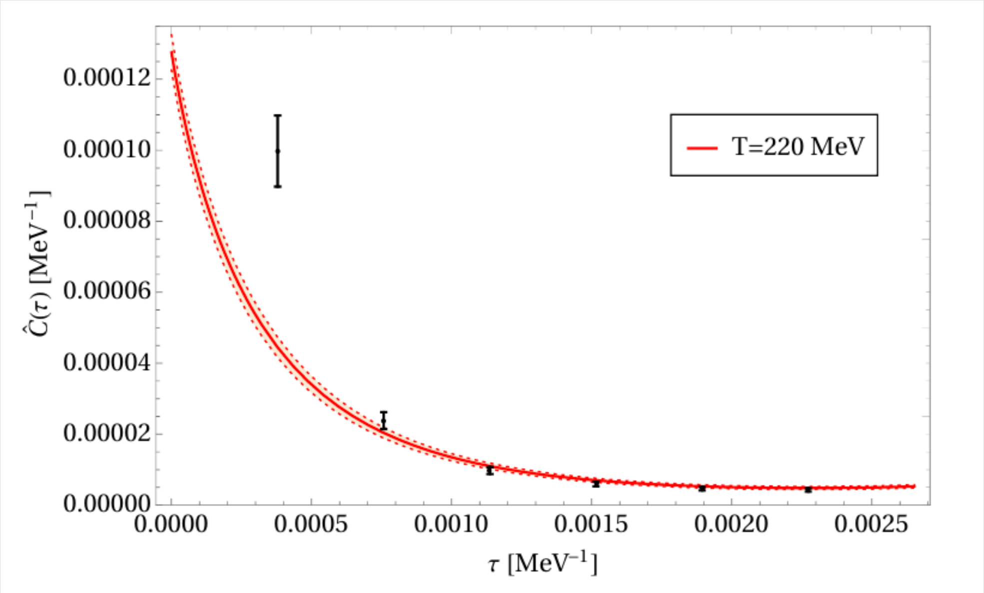
- This in turn fixes the form of the particle screening masses:

$$m_i^{\mathrm{scr}}(T) = m_i + \gamma_i(T), \quad i = \pi,\, \pi^*$$

- $\gamma_i(T)$ controls the width of the peaks, and $\gamma_i(T) \to 0$ for $T \to 0$
- This happens at different rates for π and π^* → *sequential melting!*

A non-perturbative test of the spectral function

- Since we have the full analytic structure of $\rho_{PS}(\omega,\mathbf{p})$, one can use this to *predict* the form of the corresponding *temporal* correlator $C_{PS}(\tau,\mathbf{p}{=}0)$
- $C_{PS}(\tau,\mathbf{p}{=}0)$ has a very different $\rho_{PS}(\omega,\mathbf{p})$ dependence $\rightarrow$ *highly non-trivial test!*
- We used the temporal data from [Rohrhofer et al. *PLB* **802** (2020)], which has the same lattice parameters as the spatial correlator study [Rohrhofer et al. *PRD* **100** (2019)]



- Temporal prediction matches $T{=}220$ MeV data well for large τ, and then undershoots $\rightarrow$ Makes sense: higher excited state contributions (π^{**} etc.) missing

Summary & outlook

- Local QFT is an analytic framework that attempts to address the fundamental question "*what is a QFT?*"

- The framework can be extended to $T > 0$, and this has important implications, including:

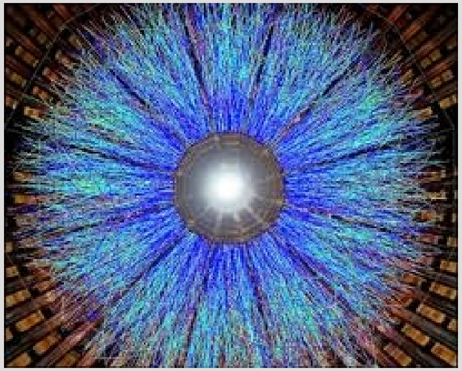
    → Spectral representations for thermal correlation functions

    → Ability to extract real-time observables from Euclidean data

    → Interpretation of screening masses

- So far only real scalar fields $\Phi(x)$ with $T > 0$ considered, but this approach can be extended (higher spin, $\mu \neq 0$). *Work in progress!*

→ This framework provides a way of obtaining **non-perturbative** insights into the phase structure of QFTs, and the resulting in-medium phenomena



[Brookhaven National Lab]

Backup: *Local QFT*

- In the 1960s, A. Wightman and R. Haag pioneered an approach which set out to answer the fundamental question "*what is a QFT?*"
- The resulting approach, *Local QFT*, defines a QFT using a core set of physically motivated axioms

Axiom 1 (Hilbert space structure). *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a, \alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}_+^\uparrow}$.*

Axiom 2 (Spectral condition). *The spectrum of the energy-momentum operator P^μ is confined to the closed forward light cone $\overline{V}^+ = \{p^\mu \mid p^2 \geq 0,\, p^0 \geq 0\}$, where $U(a,1) = e^{iP^\mu a_\mu}$.*

Axiom 3 (Uniqueness of the vacuum). *There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.*

Axiom 4 (Field operators). *The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type (κ)) which have components $\varphi_l^{(\kappa)}(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.*

Axiom 5 (Relativistic covariance). *The fields $\varphi_l^{(\kappa)}(x)$ transform covariantly under the action of $\overline{\mathscr{P}_+^\uparrow}$:*

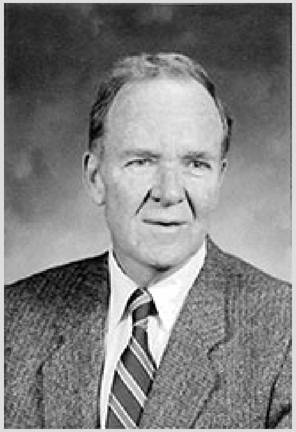
$$U(a,\alpha)\varphi_i^{(\kappa)}(x)U(a,\alpha)^{-1} = S_{ij}^{(\kappa)}(\alpha^{-1})\varphi_j^{(\kappa)}(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}_+^\uparrow}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}_+^\uparrow}$.

Axiom 6 (Local (anti-)commutativity). *If the support of the test functions f, g of the fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$ are space-like separated, then:*

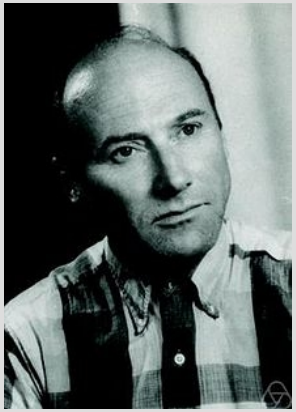
$$[\varphi_l^{(\kappa)}(f), \varphi_m^{(\kappa')}(g)]_\pm = \varphi_l^{(\kappa)}(f)\varphi_m^{(\kappa')}(g) \pm \varphi_m^{(\kappa')}(g)\varphi_l^{(\kappa)}(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$.



A. Wightman

[R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that* (1964).]



R. Haag

[R. Haag, *Local Quantum Physics*, Springer-Verlag (1992).]

Backup: *Local QFT beyond the vacuum*

- **Idea**: Look for a generalisation of the standard axioms that is compatible with $T > 0$, and approaches the vacuum case for $T \to 0$

Axiom 1 (**Hilbert space structure**). *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a, \alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}_+^\uparrow}$.*

→ H_β is defined for fixed $\beta = 1/T$

Axiom 2 (**Spectral condition**). *The spectrum of the energy-momentum operator P^μ is confined to the closed forward light cone $\overline{V}^+ = \{p^\mu \mid p^2 \geq 0,\, p^0 \geq 0\}$, where $U(a, 1) = e^{iP^\mu a_\mu}$.*

→ Replaced by the KMS condition

$$\langle \Omega_\beta | \phi(x_1) \cdots \phi(x_k) \phi(x_{k+1}) \cdots \phi(x_n) | \Omega_\beta \rangle$$
$$= \langle \Omega_\beta | \phi(x_{k+1}) \cdots \phi(x_n) \phi(x_1 + i(\beta, \vec{0})) \cdots \phi(x_k + i(\beta, \vec{0})) | \Omega_\beta \rangle$$

Axiom 3 (**Uniqueness of the vacuum**). *There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.*

→ Instead, thermal background state $|\Omega_\beta\rangle$

Axiom 4 (**Field operators**). *The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type (κ)) which have components $\varphi_l^{(\kappa)}(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.*

✓ Fields are still distributions

Axiom 5 (**Relativistic covariance**). *The fields $\varphi_l^{(\kappa)}(x)$ transform covariantly under the action of $\overline{\mathscr{P}_+^\uparrow}$:*

$$U(a, \alpha) \varphi_i^{(\kappa)}(x) U(a, \alpha)^{-1} = S_{ij}^{(\kappa)}(\alpha^{-1}) \varphi_j^{(\kappa)}(\Lambda(\alpha) x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}_+^\uparrow}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}_+^\uparrow}$.

→ The fields no longer transform under general unitary Lorentz transformations

Axiom 6 (**Local (anti-)commutativity**). *If the support of the test functions f, g of the fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$ are space-like separated, then:*

$$[\varphi_l^{(\kappa)}(f), \varphi_m^{(\kappa')}(g)]_\pm = \varphi_l^{(\kappa)}(f) \varphi_m^{(\kappa')}(g) \pm \varphi_m^{(\kappa')}(g) \varphi_l^{(\kappa)}(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$.

✓ Locality is unaffected by the properties of the background state. ***This is important!***

Backup: *Damping factors from Euclidean FRG data*

- Locality constraints also imply that particle damping factors $D_{m,\beta}(\boldsymbol{x})$ can be directly calculated from Euclidean momentum space data [P.L., 2201.12180]

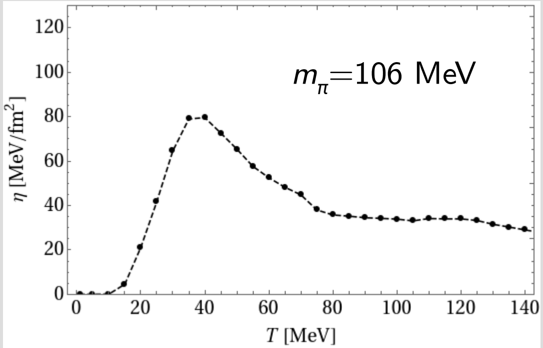
$$D_{m,\beta}(\vec{x}) \sim e^{|\vec{x}|m} \int_0^\infty \frac{d|\vec{p}|}{2\pi}\; 4|\vec{p}| \sin(|\vec{p}||\vec{x}|)\; \widetilde{G}_\beta(0,|\vec{p}|).$$

p-space Euclidean propagator

Holds for large separation $|\boldsymbol{x}|$

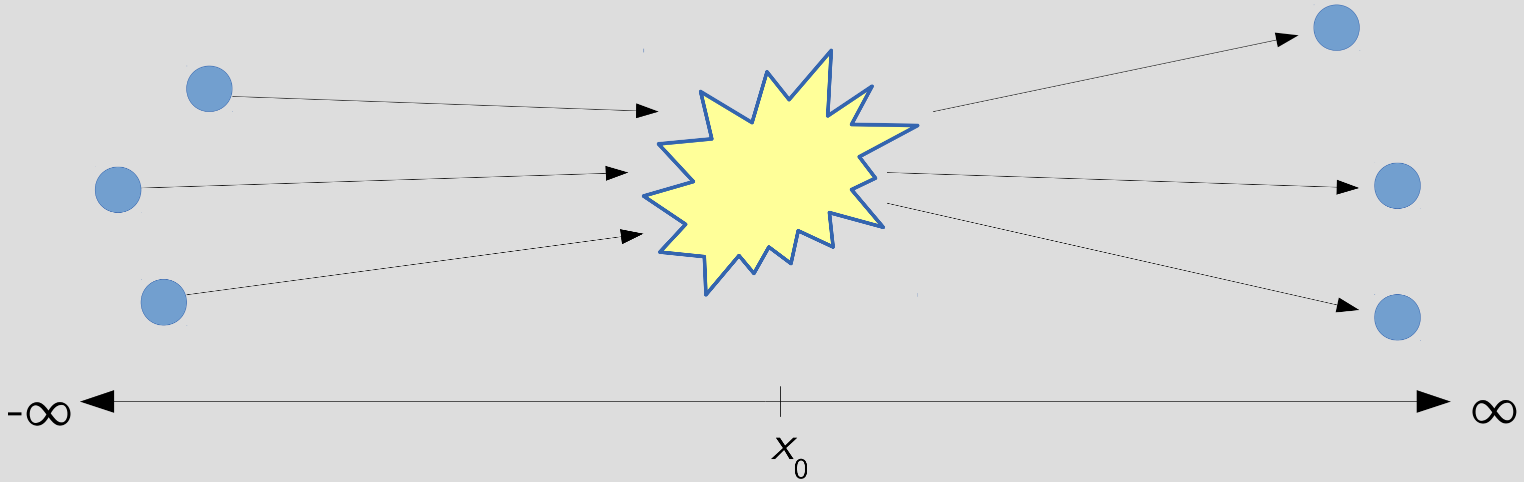
- In [P.L., R.-A. Tripolt, 2202.09142] pion propagator data from the quark-meson model (FRG calculation) was used to compute the damping factor at different values of T via the analytic relation above
- Fits to the resulting data were consistent with the form: $D_{m_\pi,\beta}(\vec{x}) = \alpha_\pi\, e^{-\gamma_\pi |\vec{x}|}$
- $D_{m,\beta}(\boldsymbol{x})$ can then be used as input for calculations, e.g. shear viscosity

Similar qualitative features to results from chiral perturbation theory



Backup: *Damping factors from asymptotic dynamics*

- Since all observable quantities are computed using correlation functions, which are characterised by damping factors, one can use these to gain new insights into the properties of QFTs when $T > 0$
- It has been proposed [Bros, Buchholz, *NPB* 627 (2002)] that these quantities are controlled by the large-time x_0 dynamics of the theory



Important: Interactions with the thermal background persist, even for large x_0

$\rightarrow$ ***Need to take this into account in definition of scattering states!***

Backup: *Damping factors from asymptotic dynamics*

- Idea: thermal scattering states are defined by imposing an asymptotic field condition [*NPB* 627 (2002)]:

Asymptotic fields Φ_0 are assumed to satisfy dynamical equations, but only at large x_0

In Φ^4 theory

"Asymptotic coupling"

$$(\partial^2 + m^2)\phi_0(x) + \frac{\lambda}{3!}\phi_0^3(x) \xrightarrow{|x_0|\to\infty} 0$$

"Asymptotic mass"

- Given that the thermal spectral density has the decomposition

$$\widetilde{D}_\beta(\vec{u}, s) = \widetilde{D}_{m,\beta}(\vec{u})\,\delta(s - m^2) + \widetilde{D}_{c,\beta}(\vec{u}, s)$$

it follows that: **1.** The continuous contribution to $\langle\Omega_\beta|\phi(x)\phi(y)|\Omega_\beta\rangle$ is **suppressed** for large x_0

2. The particle damping factor $\widetilde{D}_{m,\beta}(\boldsymbol{u})$ is **uniquely fixed** by the asymptotic field equation

- This means that the non-perturbative thermal effects experienced by particle states are entirely controlled by the asymptotic dynamics!

Backup: Φ^4 theory for $T > 0$

- Applying the asymptotic field condition for Φ^4 theory, the resulting damping factors have the form [*NPB* 627 (2002)]:

→ **For $\boldsymbol{\lambda} < 0$:** $$D_{m,\beta}(\vec{x}) = \frac{\sin(\kappa|\vec{x}|)}{\kappa|\vec{x}|}$$

→ **For $\boldsymbol{\lambda} > 0$:** $$D_{m,\beta}(\vec{x}) = \frac{e^{-\kappa|\vec{x}|}}{\kappa_0|\vec{x}|}$$

where κ is defined with $r = m/T$:

$$\kappa = T\sqrt{|\lambda|}K(r), \qquad K(r) = \sqrt{\int \frac{d^3\hat{\vec{q}}}{(2\pi)^3 2\sqrt{|\hat{\vec{q}}|^2 + r^2}} \frac{1}{e^{\sqrt{|\hat{\vec{q}}|^2+r^2}} - 1}}$$

→ *The parameter κ has the interpretation of a thermal width: $\kappa \to 0$ for $T \to 0$*, or equivalently *κ^{-1} is mean-free path*

- Now that one has the exact dependence of $D_{m,\beta}(\boldsymbol{x})$ on the external physical parameters, in this case T, m and λ, one can use this to calculate observables ***analytically***

Backup: Φ^4 theory for $T > 0$

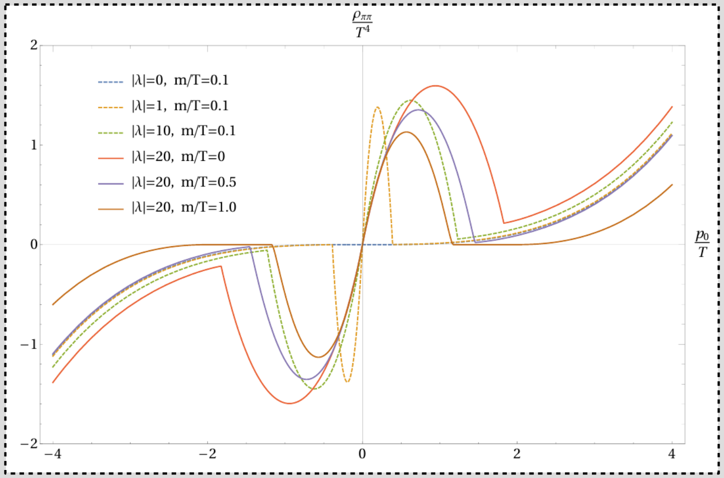
- Of particular interest is the *shear viscosity* η, which measures the resistance of a medium to sheared flow

  $\rightarrow$ This quantity can be determined from the spectral function of the spatial traceless energy-momentum tensor

$$\rho_{\pi\pi}(p_0) = \lim_{\vec{p}\to 0} \mathcal{F}\left[\langle\Omega_\beta| \left[\pi^{ij}(x), \pi_{ij}(y)\right] |\Omega_\beta\rangle\right](p)$$

  ... and η is recovered via the *Kubo relation* $\eta = \dfrac{1}{20} \lim_{p_0\to 0} \dfrac{d\rho_{\pi\pi}}{dp_0}$

- Using $D_{m,\beta}(\boldsymbol{x})$ for $\lambda < 0$, the EMT spectral function $\rho_{\pi\pi}$ has the form:



- The presence of interactions causes resonant peaks to appear $\rightarrow$ peaked when $p_0 \sim \kappa = 1/\ell$
- For $\lambda \to 0$ the free-field result is recovered, as expected
- The dimensionless ratio m/T controls the magnitude of the peaks

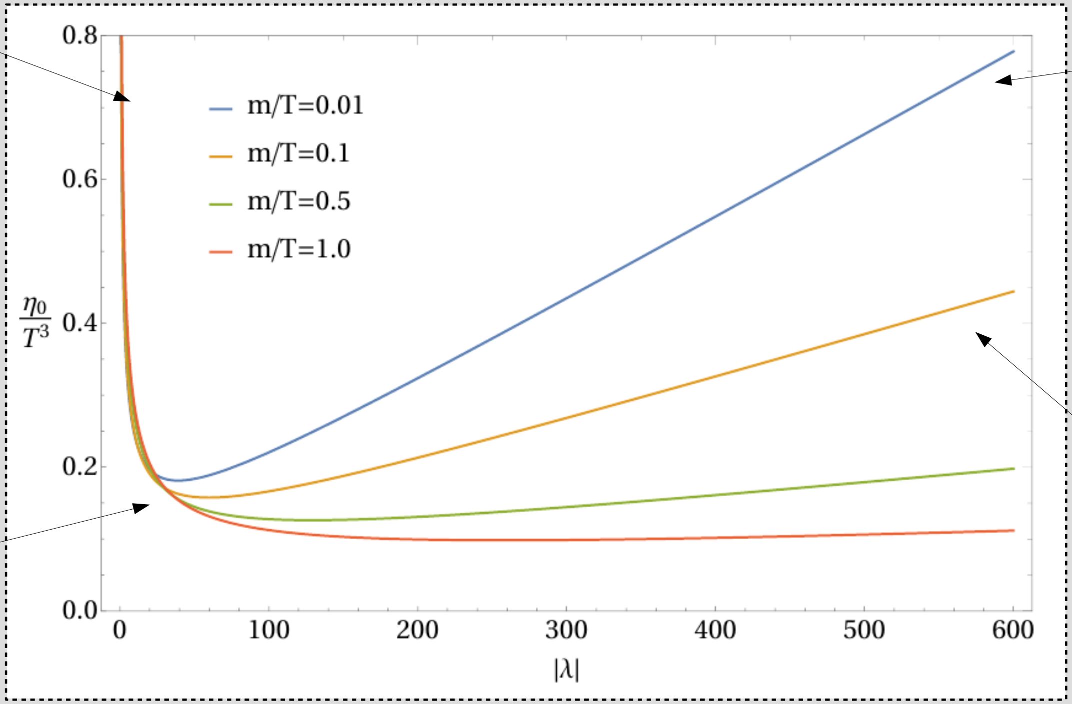
Backup: Φ^4 theory for $T > 0$

- Applying Kubo's relation, the shear viscosity η_0 arising from the asymptotic states can be written [P.L., R.-A. Tripolt, J. M. Pawlowski, D. H. Rischke, PRD 104, 065010 (2021)]

$$\eta_0 = \frac{T^3}{15\pi}\left[\frac{\mathcal{K}_3\left(\frac{m}{T},0,\infty\right)}{\sqrt{|\lambda|}} + \sqrt{|\lambda|}\,\mathcal{K}_1\left(\frac{m}{T},0,\infty\right) + \frac{\mathcal{K}_4\left(\frac{m}{T},\sqrt{|\lambda|}K\left(\frac{m}{T}\right),\sqrt{|\lambda|}K\left(\frac{m}{T}\right)\right)}{4|\lambda|}\right]$$

Dominant component for small $|\lambda|$

Global minima



For large $|\boldsymbol{\lambda}|$, $\boldsymbol{\eta_0} \sim |\boldsymbol{\lambda}|$

Magnitude of large $|\boldsymbol{\lambda}|$ growth controlled by m/T

→ For fixed coupling, η_0/T^3 is entirely controlled by functions of m/T

Backup: Φ^4 theory for $T > 0$

- What about the case $\lambda > 0$? $\rightarrow$ η_0 *diverges!*

  Why? – *The particle damping factor* $D_{m,\beta}(\boldsymbol{u})$ *does not decay rapidly enough at large momenta*

- This characteristic is related to the "bad" UV behaviour of the quartic interaction, i.e. the triviality of Φ^4 appears to have an impact beyond $T\!=\!0$!

- In [PRD 104, 065010 (2021)] it was shown more generally that the finiteness of η_0 is related to the existence of thermal equilibrium

  If the KMS condition holds $\implies$ η_0 is finite

- This procedure demonstrates that asymptotic dynamics can be used to explore the non-perturbative properties of QFTs when T>0

  $\rightarrow$ *Can also calculate other observables, e.g. transport coefficients, entropy density, pressure, etc.*

Backup: *spectral representations*

- For thermal asymptotic states, the spectral function $\rho_{\pi\pi}$ has the form

$$\rho_{\pi\pi}(p_0) = \sinh\left(\frac{\beta}{2}p_0\right) \int \frac{d^3\vec{q}}{(2\pi)^4} \frac{2}{3}|\vec{q}|^4 \int_{-\infty}^{\infty} dq_0 \, \frac{\widetilde{C}_\beta(q_0,\vec{q})\, \widetilde{C}_\beta(p_0-q_0,\vec{q})}{\sinh\left(\frac{\beta}{2}q_0\right) \sinh\left(\frac{\beta}{2}(p_0-q_0)\right)}$$

... which after applying the generalised KL representation, together with the Kubo relation, implies

$$\eta_0 = \frac{T^5}{240\pi^5} \int_0^\infty ds \int_0^\infty dt \int_0^\infty d|\vec{u}| \int_0^\infty d|\vec{v}| \, |\vec{u}||\vec{v}| \, \widetilde{D}_\beta(\vec{u},s)\, \widetilde{D}_\beta(\vec{v},t)$$
$$\times \Bigg[4\left[1+\epsilon(|\vec{u}|-|\vec{v}|)\right] \left\{ \frac{|\vec{v}|}{T} \mathcal{I}_3\left(\frac{\sqrt{t}}{T},0,\infty\right) + \frac{|\vec{v}|^3}{T^3} \mathcal{I}_1\left(\frac{\sqrt{t}}{T},0,\infty\right) \right\}$$
$$+ \left\{ \mathcal{I}_4\left(\frac{\sqrt{t}}{T}, \frac{|\vec{v}|}{T}, \frac{s-t+(|\vec{u}|+|\vec{v}|)^2}{2(|\vec{u}|+|\vec{v}|)T}\right) + \epsilon(|\vec{u}|-|\vec{v}|)\, \mathcal{I}_4\left(\frac{\sqrt{t}}{T}, \frac{|\vec{v}|}{T}, \frac{s-t+(|\vec{v}|-|\vec{u}|)^2}{2(|\vec{v}|-|\vec{u}|)T}\right) \right\} \Bigg]$$

- The model dependence of η_0 factorises, and is controlled by the thermal spectral density $D_\beta(\mathbf{u},s)$

Backup: *Euclidean spectral relations*

- One can use the assumptions of local QFT at finite T to put constraints on the the structure of Euclidean correlation functions

→ From the KMS condition and locality:

$$\mathcal{W}_E(\tau,\vec{x}) = \frac{1}{\beta}\sum_{N=-\infty}^{\infty} w_N(\vec{x}) e^{\frac{2\pi i N}{\beta}\tau}$$

- The Fourier coefficients of the Euclidean two-point function are then related to the thermal damping factors as follows [P.L., 2201.12180]:

$$w_N(\vec{x}) = \frac{1}{4\pi|\vec{x}|}\left[D_m(\vec{x}) e^{-|\vec{x}|\sqrt{m^2+\omega_N^2}} + \int_0^\infty ds\, e^{-|\vec{x}|\sqrt{s+\omega_N^2}} D_c(\vec{x},s)\right]$$

→ *The continuous component $D_c(\mathbf{x},s)$ is exponentially suppressed!*

- $\omega_N = 2\pi N T$ are the Matsubara frequencies. For $N=0$ this leads to:

$$\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau\, \mathcal{W}_E(\tau,\vec{x}) \sim \frac{1}{4\pi|\vec{x}|} D_{m,\beta}(\vec{x})\, e^{-|\vec{x}|m}$$