Emergence of Pion and Proton parton distributions

J. Rodríguez-Quintero

FÍSICA MATEMÁTICAS COMPUTACIÓN

In collaboration with: D. Binosi, F. De Soto, L. Chang, Z-F. Cui, M. Ding, J.M. Morgado, J. Papavassiliou, K. Raya, C.D. Roberts, S. Schmidt.

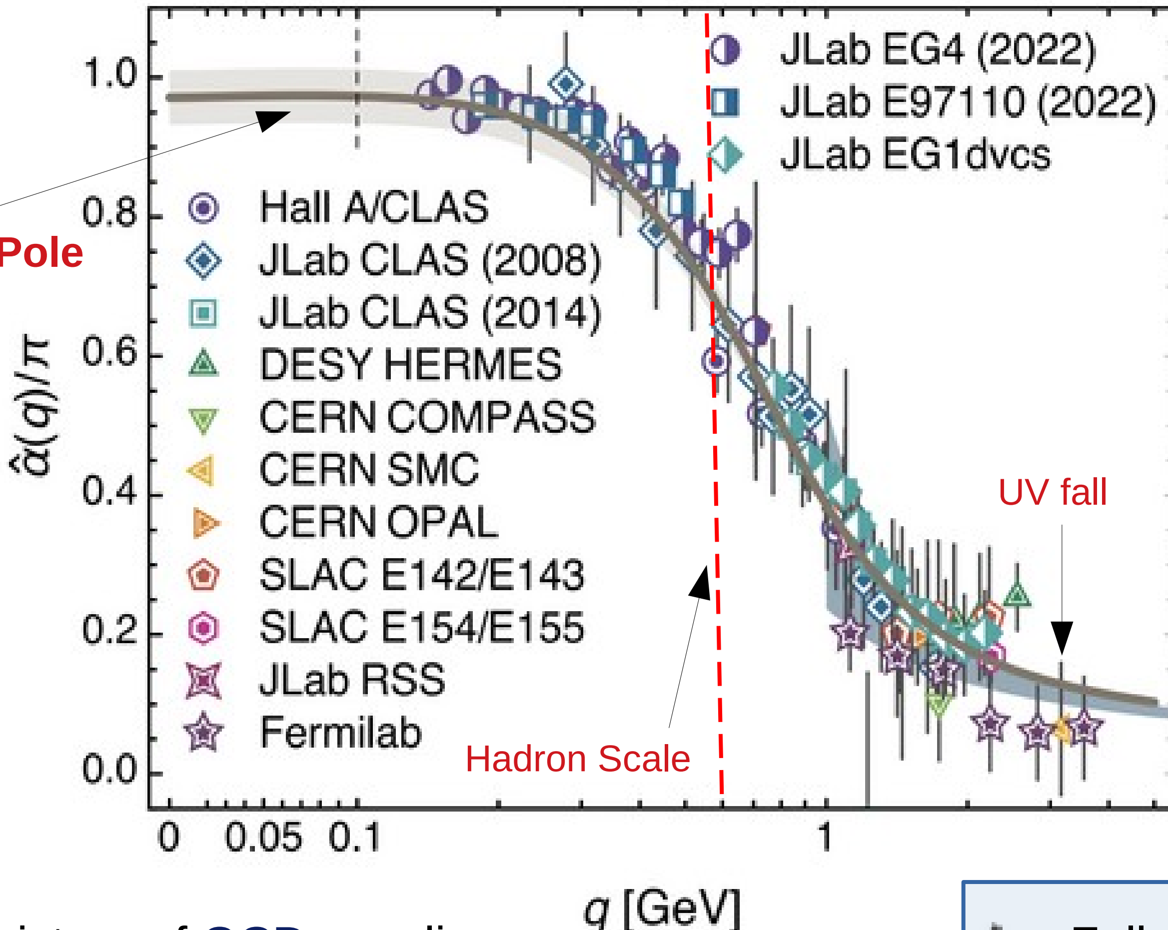
Stavanger (Norway), ConfXV 2022, August 1st - 6th, 2022.

QCD: Basic Facts

- **Confinement** and the **EHM** are tightly connected with **QCD's running coupling**.

'Effective Charge' (figure: D. Binosi's courtesy!)

No Landau Pole
Enhancement



$$\mathcal{L}_{QCD} = \sum_{j=u,d,s,\ldots} \bar{q}_j[\gamma_\mu D_\mu + m_j]q_j + \tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu},$$

$$D_\mu = \partial_\mu + ig\tfrac{1}{2}\lambda^a A^a_\mu,$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu - \underline{g f^{abc} A^b_\mu A^c_\nu},$$

Modern picture of **QCD** coupling.

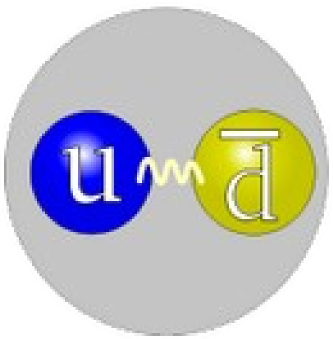
Combined continuum + QCD lattice analysis

ζ_H : Fully **dressed valence** quarks express all hadron's properties

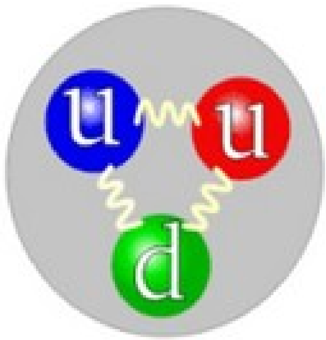
Why bother about pions?

- **Pions** and **kaons** emerge as (pseudo)-**Goldstone** bosons of **DCSB**.

(besides being 'simple' bound states)



$m_\pi \approx 0.140$ GeV



$m_p \approx 0.940$ GeV

- Their study is crucial to understand the **EHM** and the ***hadron structure***:

  - Dominated by **QCD** dynamics

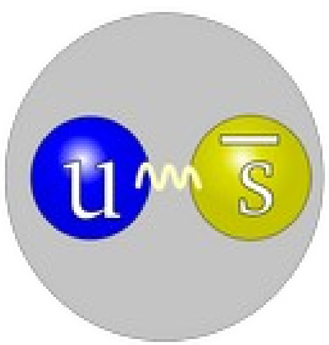
    Simultaneously explains the mass of the proton and the *masslessness* of the pion

  - Interplay between **Higgs** and **strong** mass generating mechanisms.

'Higgs' masses

$m_{u/d} \approx 0.004$ GeV

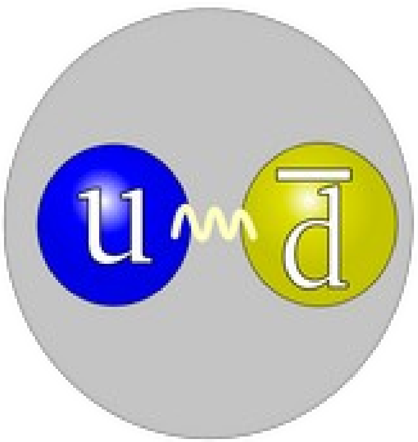
$m_s \approx 0.095$ GeV

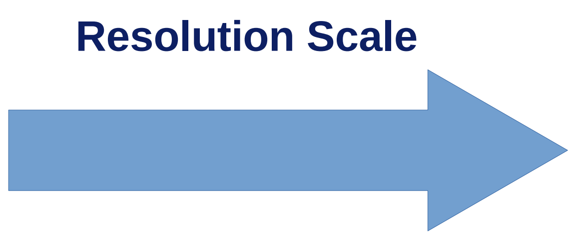


$m_K \approx 0.490$ GeV

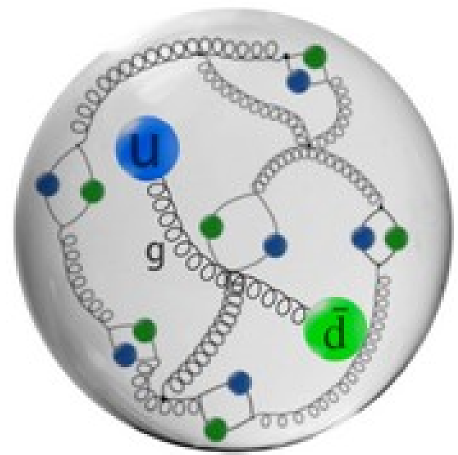
Parton distributions: energy scales

ζ_H





$\zeta > \zeta_H$



- **Fully-dressed valence quarks**

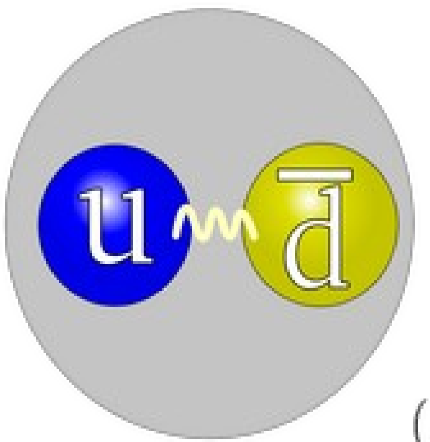
(quasiparticles)

- **Unveiling of glue and sea d.o.f.**

(partons)

Parton distributions: energy scales

ζ_H

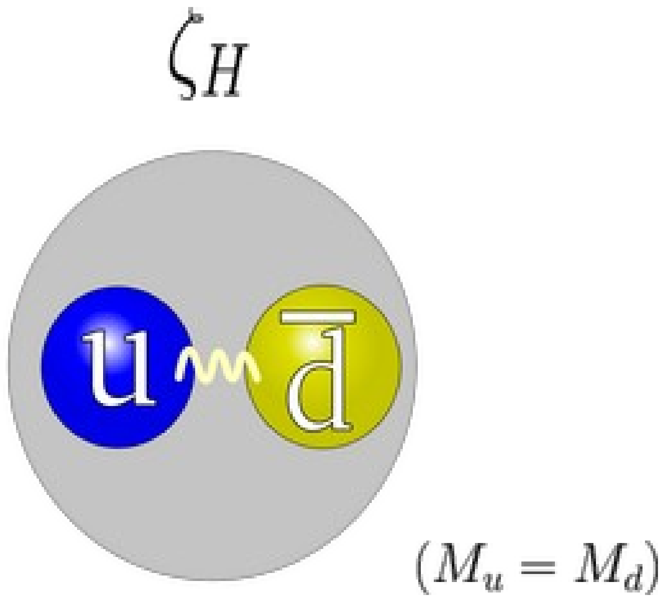


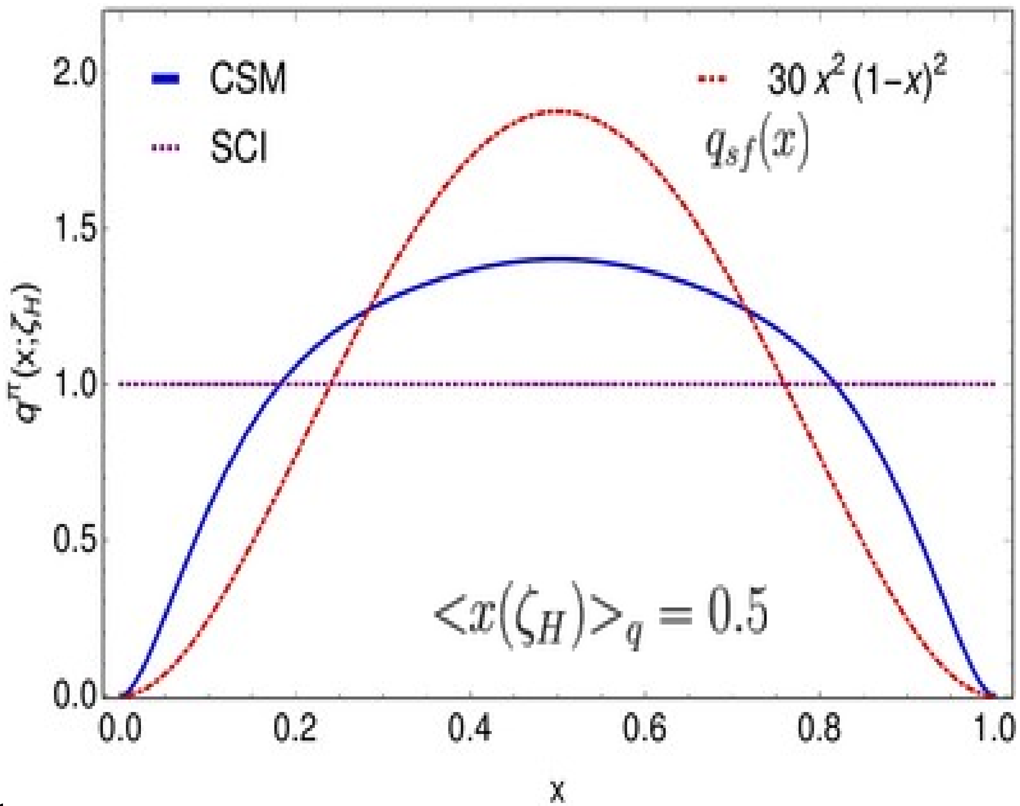
$(M_u = M_d)$

- **Fully-dressed valence quarks**
  - At this scale, **all properties** of the hadron are contained within their valence quarks.
  - **QCD constraints** are defined from here (e.g. large-x behavior of the PDF)

$$u^{\pi}(x;\zeta) \overset{x\simeq 1}{\approx} (1-x)^{\beta = 2+\gamma(\zeta)}$$

Parton distributions: energy scales





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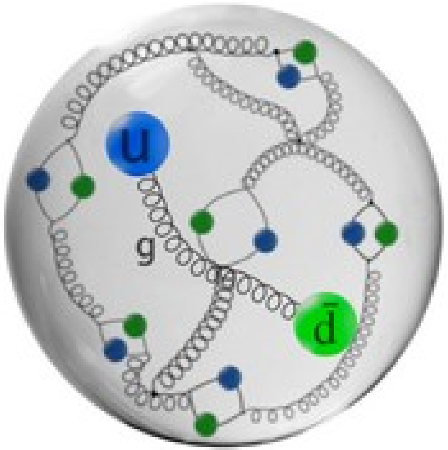
- **CSM** results produce:
  - **EHM-induced** dilated distributions
  - Soft end-point behavior

Cui:2020tdf

Parton distributions: energy scales

$$\zeta > \zeta_H$$

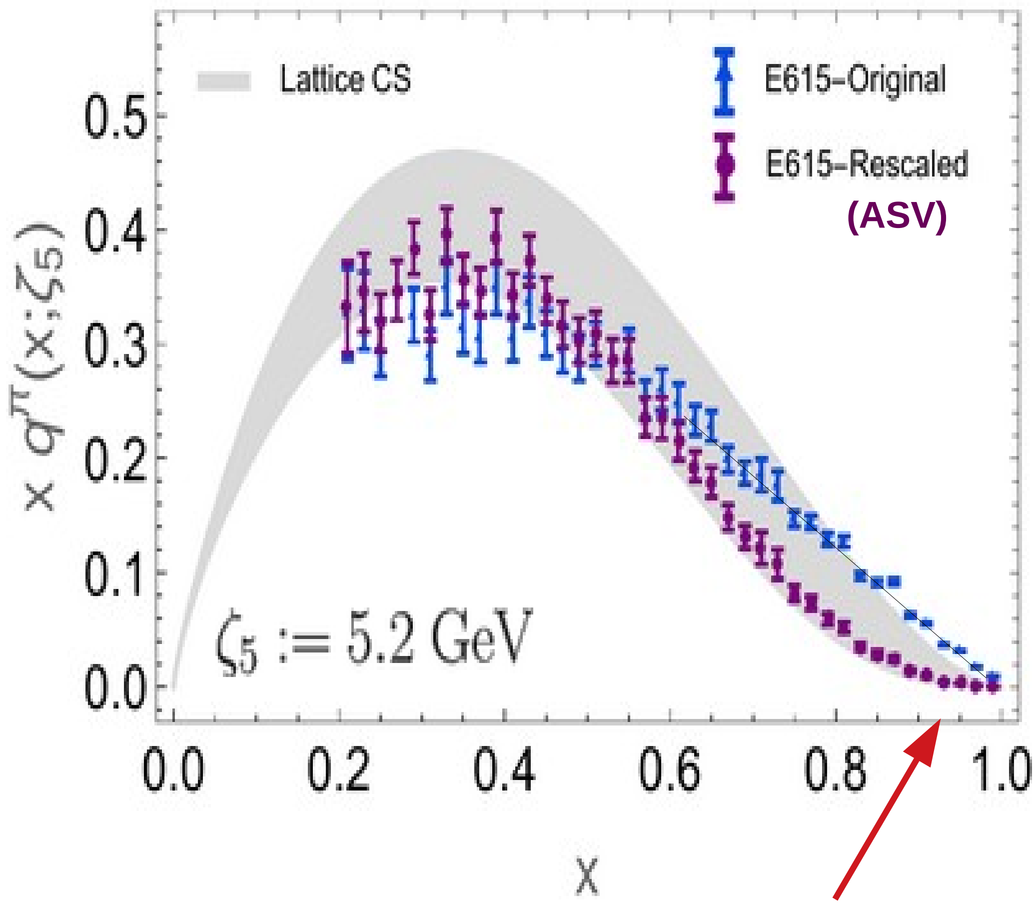


- **Unveiling of glue and sea d.o.f.**

- **Experimental** data is given **here**.
- The interpretation of parton distributions from cross sections demands **special care**.
- In addition, the synergy with **lattice QCD** and phenomenological approaches is welcome.

Parton distributions: energy scales

Conway:1989fs Aicher:2010cb Sufian:2020vzb



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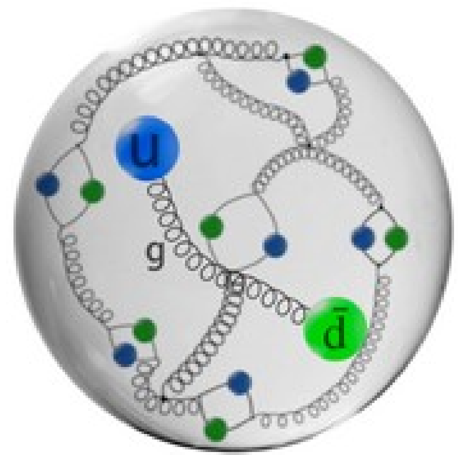

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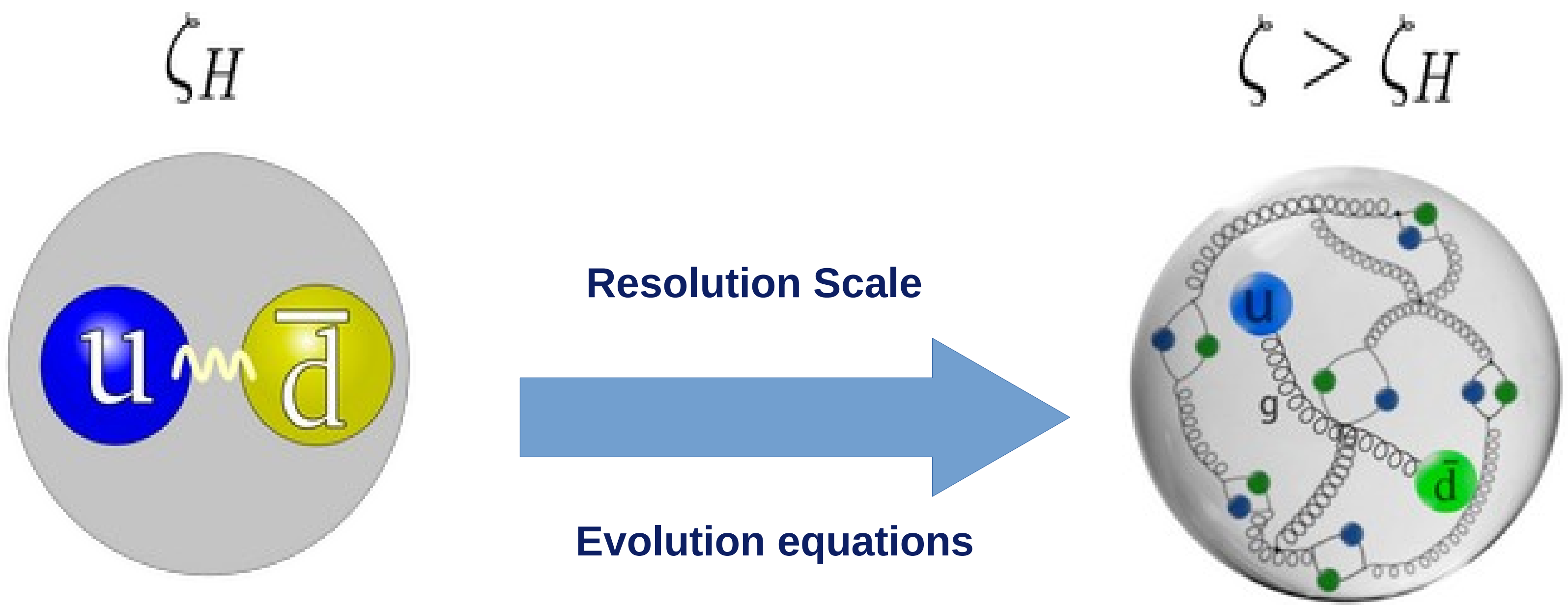

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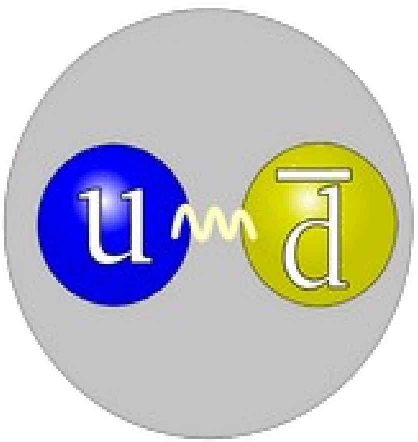
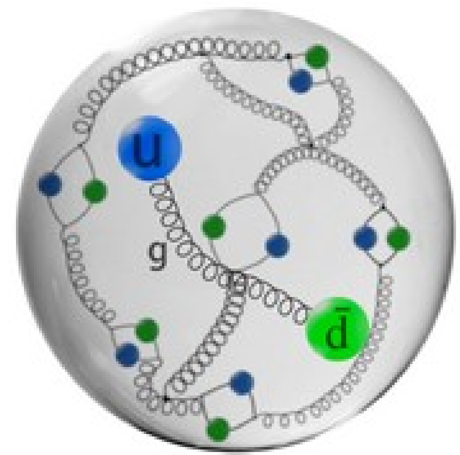


Resolution Scale

Evolution equations

$\zeta > \zeta_H$



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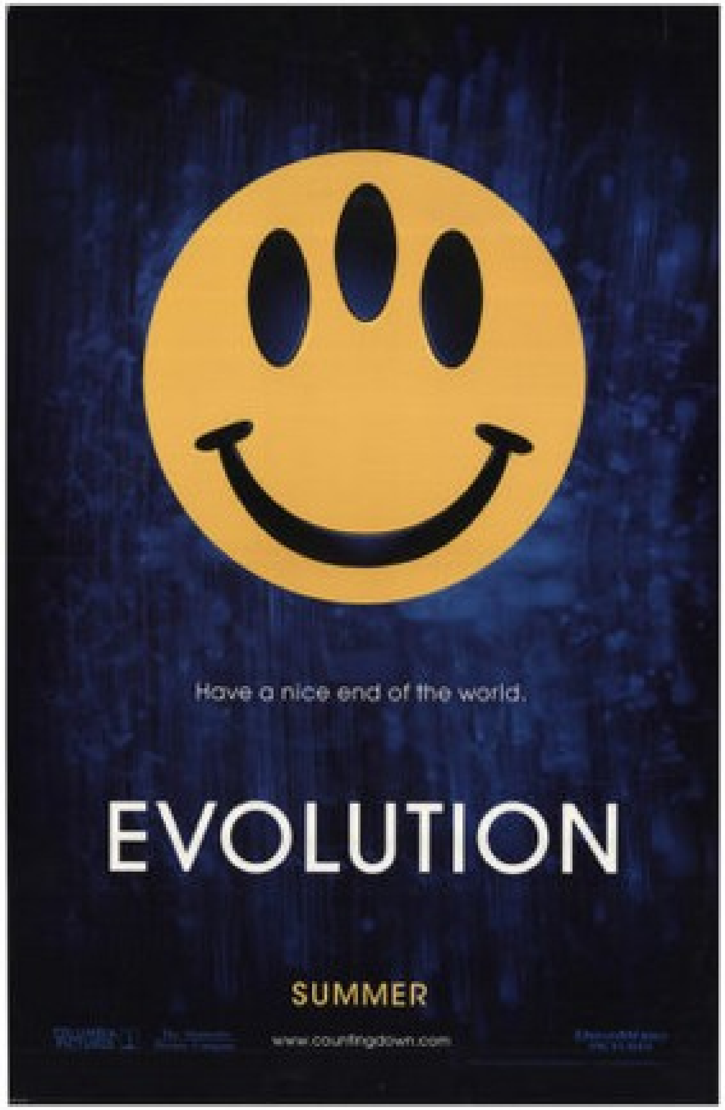
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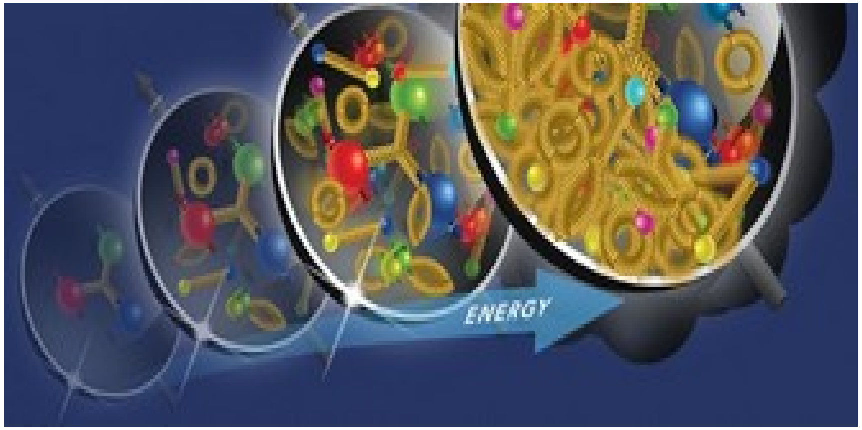
DGLAP: All orders evolution

Raya:2021zrz

Cui:2020tdf

$$\left\{\zeta^2 \frac{d}{d\zeta^2} \int_0^1 dy \delta(y-x) - \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \begin{pmatrix} P_{qq}^{NS}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^S\left(\frac{\mathbf{x}}{\mathbf{y}}\right) \end{pmatrix}\right\} \begin{pmatrix} H_\pi^{NS,+}(y,t;\zeta) \\ \mathbf{H}_\pi^S(y,t;\zeta) \end{pmatrix} = 0$$

DGLAP leading-order evolution equations



DGLAP: All orders evolution

Assumption: define an **effective** charge such that

Raya:2021zrz
Cui:2020tdf

Starting from fully-dressed **quasiparticles**, at ζ_H → **Sea** and **Gluon** content unveils, as prescribed by **QCD**

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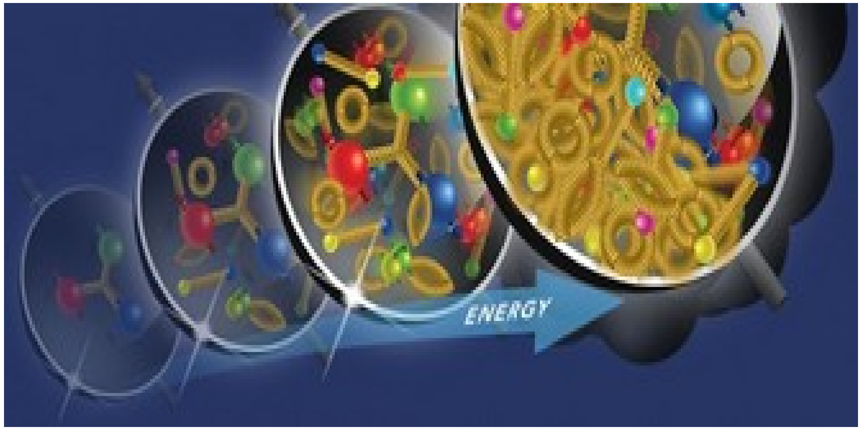

- **Not** the **LO** QCD coupling but an **effective** one.
- Making this equation **exact**.
- Connecting with the **hadron scale**, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)

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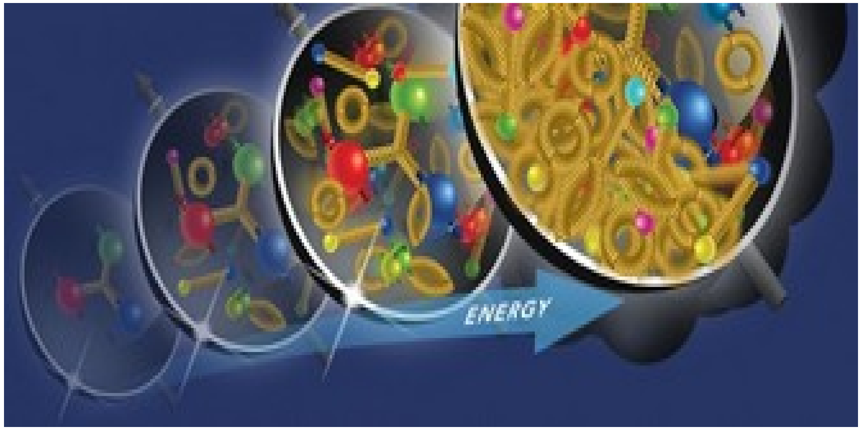
$$\left\{ \zeta^2 \frac{d}{d\zeta^2} \mathbb{1} + \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma_{qq}^{(n)} & 0 & 0 \\ 0 & \gamma_{qq}^{(n)} & 2n_f \gamma_{qg}^{(n)} \\ 0 & \gamma_{gq}^{(n)} & \gamma_{gg}^{(n)} \end{pmatrix} \right\} \begin{pmatrix} \langle x^n \rangle_{\rm NS}(\zeta) \\ \langle x^n \rangle_{\rm S}(\zeta) \\ \langle x^n \rangle_{g}(\zeta) \end{pmatrix} = 0$$

DGLAP ~~leading order~~ evolution equations

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$



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Cui:2020tdf

Implication 1: valence quark PDF

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi} I(\zeta_0,\zeta_f)\right)\langle x^n(\zeta_0)\rangle_q$$

$$I(\zeta_0,\zeta_f) = \int_{2\ln(\zeta_0/\Lambda_{\rm QCD})}^{2\ln(\zeta_f/\Lambda_{\rm QCD})} dt\,\alpha(t)$$

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This ratio encodes the information of the charge and use isospin symmetry

$$\langle x(\zeta_H)\rangle_u = \langle x(\zeta_H)\rangle_{\bar{d}} = 1/2$$

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve "all" the Mellin moments up and reconstruct the PDF.

DGLAP: All orders evolution

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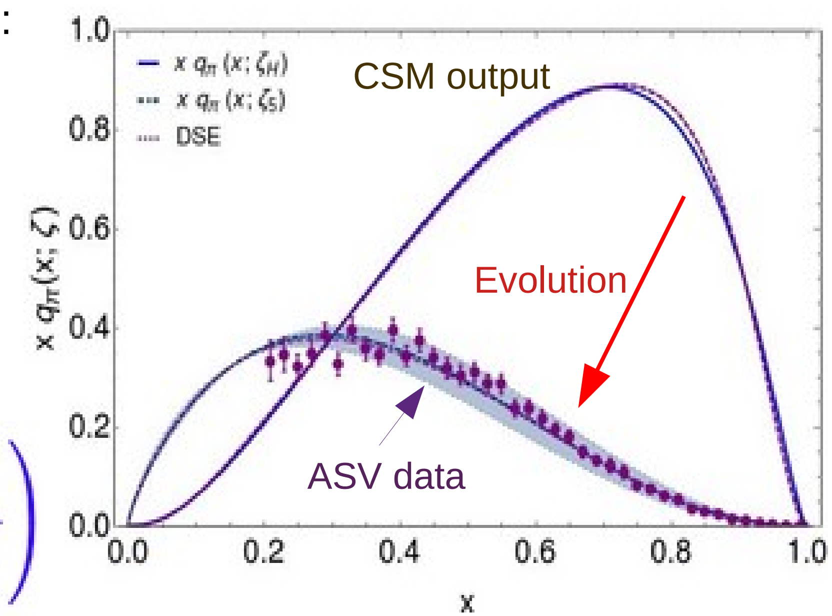
Reconstruction after evolving a CSM PDF



DGLAP: All orders evolution

Implication 2: glue and sea-quark distributions (n_f=4)

$$
\begin{aligned}
\langle 2x(\zeta_f)\rangle_q &= \exp\left(-\frac{8}{9\pi} I(\zeta_H,\zeta_f)\right), \qquad q = u, \bar{d}; \\
\langle x(\zeta_f)\rangle_{\rm sea} &= \langle x(\zeta_f)\rangle_{\sum_q q+\bar{q}} - (\langle x(\zeta_f)\rangle_u + \langle x(\zeta_f)\rangle_{\bar{d}}), \\
&= \frac{3}{7} + \frac{4}{7}\langle 2x(\zeta_f)\rangle_u^{7/4} - \langle 2x(\zeta_f)\rangle_u \\
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$$\langle 2x(\zeta_f)\rangle_q + \langle x(\zeta_f)\rangle_{\text{sea}} + \langle x(\zeta_f)\rangle_g = 1$$

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$\zeta_f/\zeta_H \to \infty$

Asymptotic (massless) limit is manifestly in agreement with textbook results: G. Altarelli, Phys. Rep. 81, 1 (1982)

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R.S. Sufian et al., arXiv:2001.04960

ζ_5	$\langle 2x\rangle_q^\pi$	$\langle x\rangle_g^\pi$	$\langle x\rangle_{\rm sea}^\pi$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

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Implication 2: glue and sea-quark distributions (n_f=4)

$$\langle 2x(\zeta_f)\rangle_q = \exp\left(-\frac{8}{9\pi}I(\zeta_H,\zeta_f)\right), \qquad q=u,\bar{d};$$
$$\langle x(\zeta_f)\rangle_{\rm sea} = \langle x(\zeta_f)\rangle_{\sum_q q+\bar{q}} - (\langle x(\zeta_f)\rangle_u + \langle x(\zeta_f)\rangle_{\bar{d}}),$$
$$= \frac{3}{7}+\frac{4}{7}\langle 2x(\zeta_f)\rangle_u^{7/4} - \langle 2x(\zeta_f)\rangle_u$$
$$\langle x(\zeta_f)\rangle_g = \frac{4}{7}\left(1-\langle 2x(\zeta_f)\rangle_u^{7/4}\right);$$

Obtained from valence-quark inputs

Momentum sum rule:

$$\langle 2x(\zeta_f)\rangle_q + \langle x(\zeta_f)\rangle_{\rm sea} + \langle x(\zeta_f)\rangle_g = 1$$

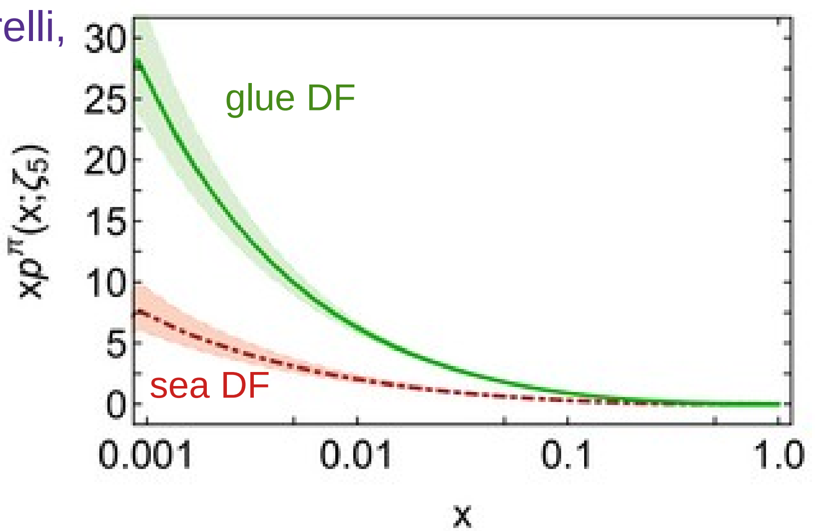
$\zeta_f/\zeta_H \to \infty$

Asymptotic (massless) limit is manifestly in agreement with textbook results: G. Altarelli, Phys. Rep. 81, 1 (1982)

R.S. Sufian et al., arXiv:2001.04960

ζ_5	$\langle 2x\rangle_q^\pi$	$\langle x\rangle_g^\pi$	$\langle x\rangle_{\rm sea}^\pi$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

Compute all the moments and reconstruct:



DGLAP: All orders evolution

Implication 3: recursion of Mellin moments

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{(\langle 2x\rangle^{\zeta}_{u_\pi})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)}$$
$$\times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} (\langle 2x\rangle^{\zeta}_{u_\pi})^{-\gamma_0^j/\gamma_0^1} .$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

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Reported lattice moments

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	
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5	0.014(4)(5)	
6	0.009(3)(3)	
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[99] C. Alexandrou et al., PRD104(2021)054504

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Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

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4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	0.0065(24)	0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 3: recursion of Mellin moments

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Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

Moments from global fits can be also compared to the estimated from recursion !

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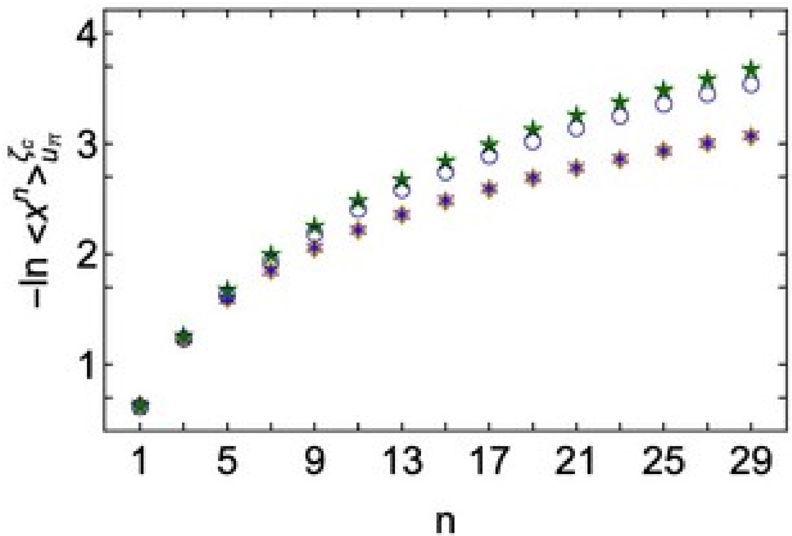
7	0.0065(24)	0.0078

Moments computed from: P. Barry et al., PRL127(2021)232001



[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 4: physical bounds

$$\langle x^n \rangle^{\zeta}_{u_\pi} (\langle 2x \rangle^{\zeta}_{u_\pi})^{-\gamma_0^n/\gamma_0^1}$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

DGLAP: All orders evolution

Implication 4: physical bounds

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1}$$

↑

$$q(x;\zeta_H) = \delta(x - 1/2)$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Lower bound** is imposed by considering the limit of a strongly interacting system of two (then maximally correlated) partons: both carry half of the momentum.

DGLAP: All orders evolution

Implication 4: physical bounds

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x - 1/2) \qquad q(x;\zeta_H) = 1$$

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- **Lower bound** is imposed by considering the limit of a strongly interacting system of two (then maximally correlated) partons: both carry half of the momentum.
- **Upper bound** comes out from considering the opposite limit of a weekly interacting system of two (then fully decorrelated) partons: all the momentum fractions are equally probable.

DGLAP: All orders evolution

Implication 4: physical bounds

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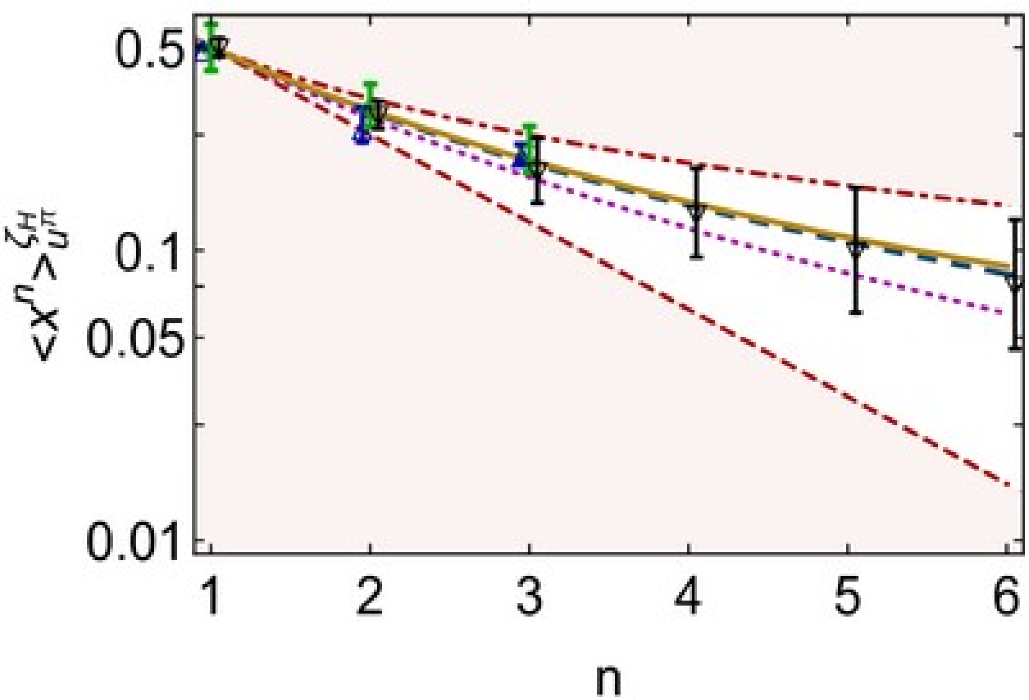

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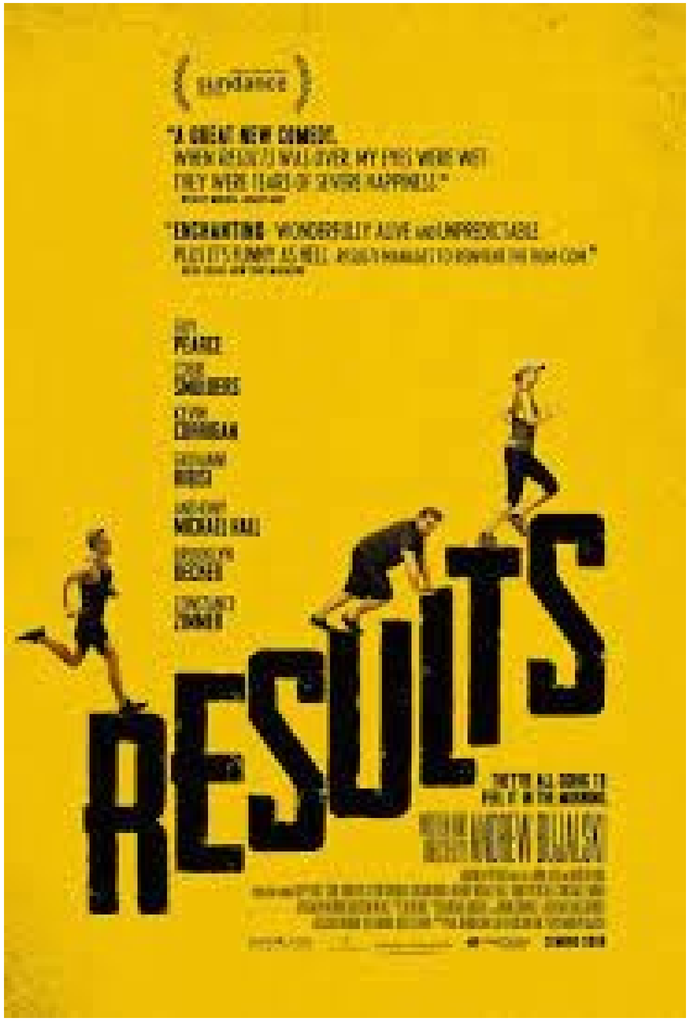
n	[61] Joo:2019bzr	[62] Sufian:2019bol	[63] Alexandrou:2021mmi
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
4			0.023(05)(06)
5			0.014(04)(05)
6			0.009(03)(03)

Lattice moments verifying the **recurrence relation** too.


sundance
"A GREAT NEW COMEDY.
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PEARCE
SMULDERS
CORRIGAN
RESULTS
ANDREW BUJALSKI

Pion PDF: from CSM (DSEs) to the experiment

Symmetry-preserving DSE computation of the valence-quark PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]
[M. Ding et al., Phys.Rev.D101(2020)054014

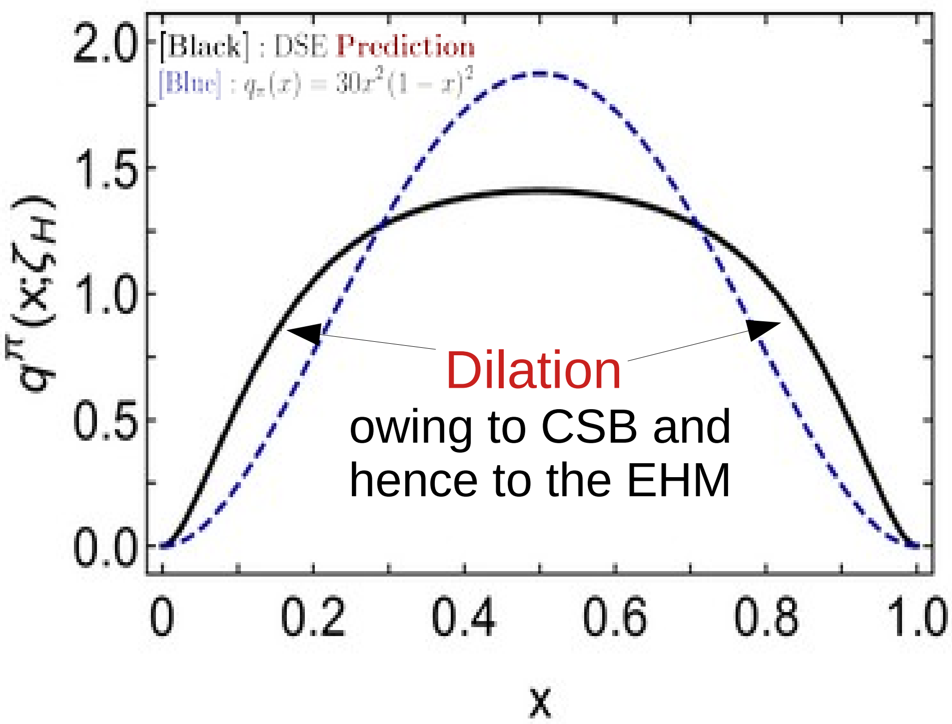
$$q^{\pi}(x;\zeta) = N_c \mathrm{tr} \int_{dk} \delta_n^x(k_\eta) \Gamma_\pi^P(k_{\bar{\eta}\eta};\zeta)\, S(k_{\bar{\eta}};\zeta) \times \{n \cdot \frac{\partial}{\partial k_\eta} [\Gamma_\pi^{-P}(k_{\eta\bar{\eta}};\zeta) S(k_\eta;\zeta)]\}\,.$$

$$q_{\mathrm{O}}^{\pi}(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times [1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)]$$

$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)} (1 + \mathcal{O}(1-x))$$

$$\beta(\zeta_H) = 2$$

Farrar, Jackson, Phys.Rev.Lett 35(1975)1416
Berger, Brodsky, Phys.Rev.Lett 42(1979)940



- The EHM-triggered broadening shortens the extent of the domain of convexity lying on the neighborhood of the endpoints, induced too by the QCD dynamics
- It cannot however spoil the asymptotic QCD behaviour at large-x (and, owing to isospin symmetry, at low-x)

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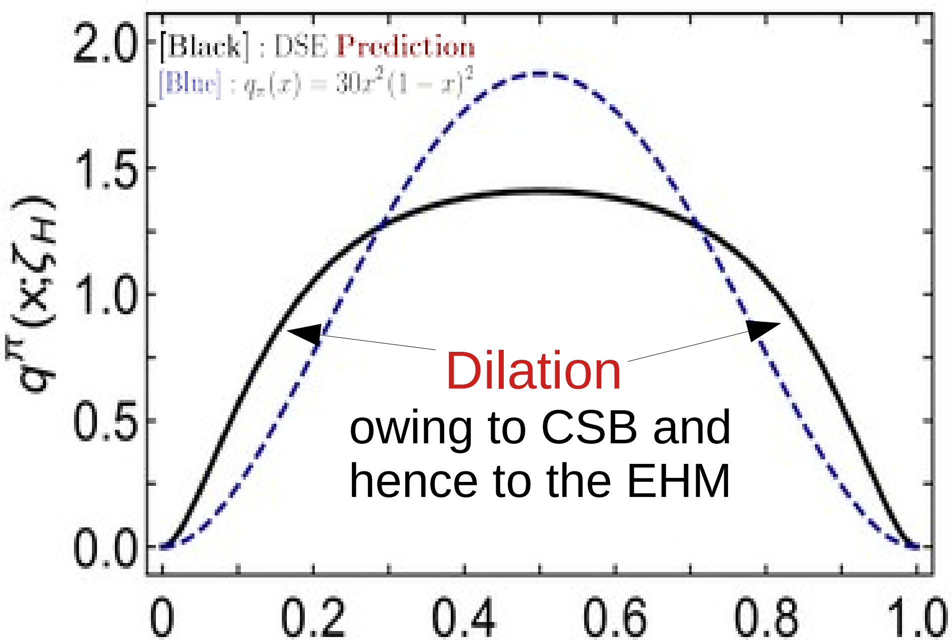

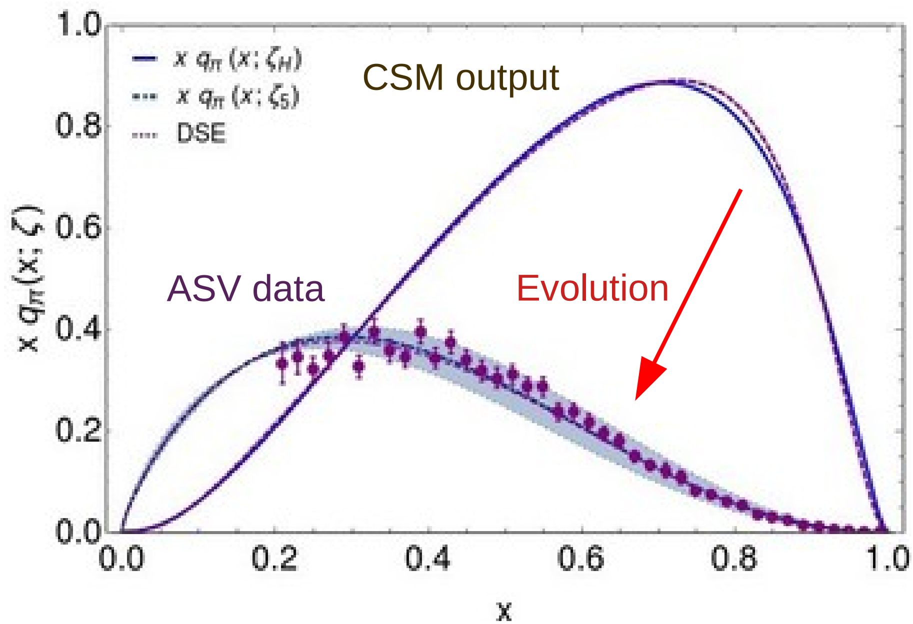

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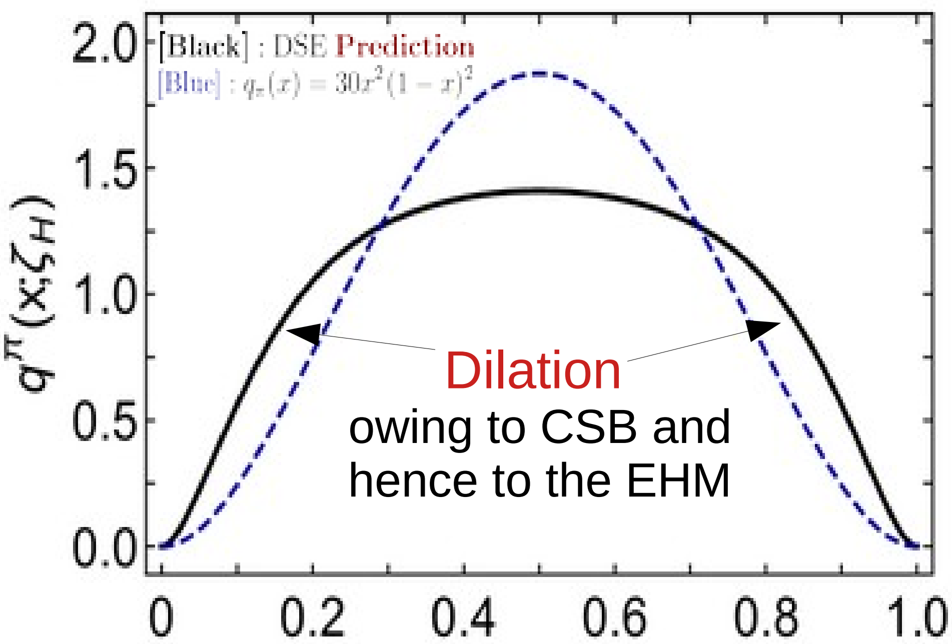

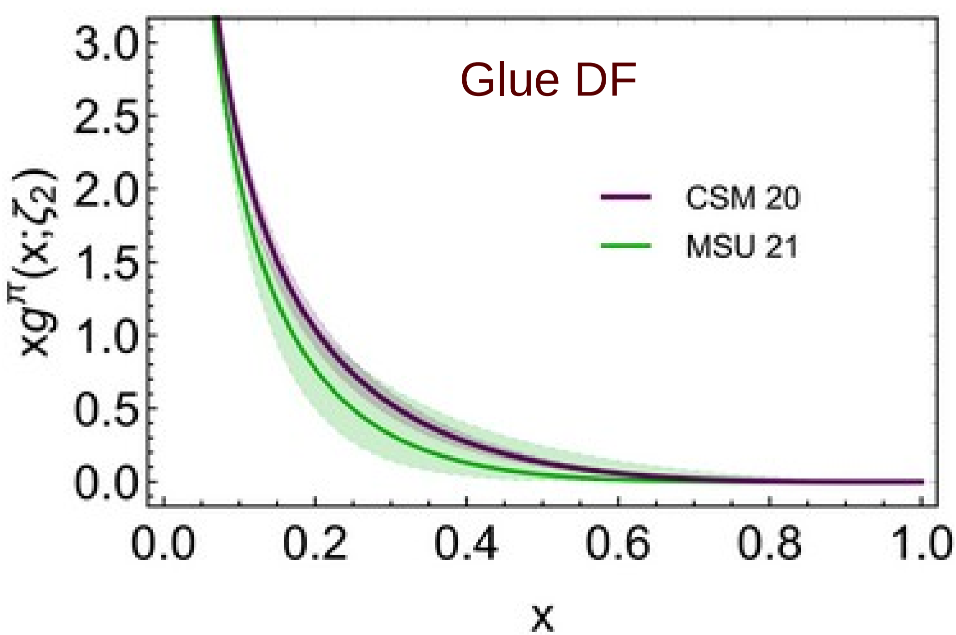

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An analogous symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach:
[L. Chang et al., Phys.Lett.B829 (2022) 137078]
[Y. Lu et al., Phys.Lett.B830 (2022) 137130]

And analogous evolution approach:

$$\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{q_H} = -\frac{\alpha_{1\ell}(\zeta^2)}{4\pi}\gamma^n_{qq}\langle x^n\rangle^{\zeta}_{q_H}$$

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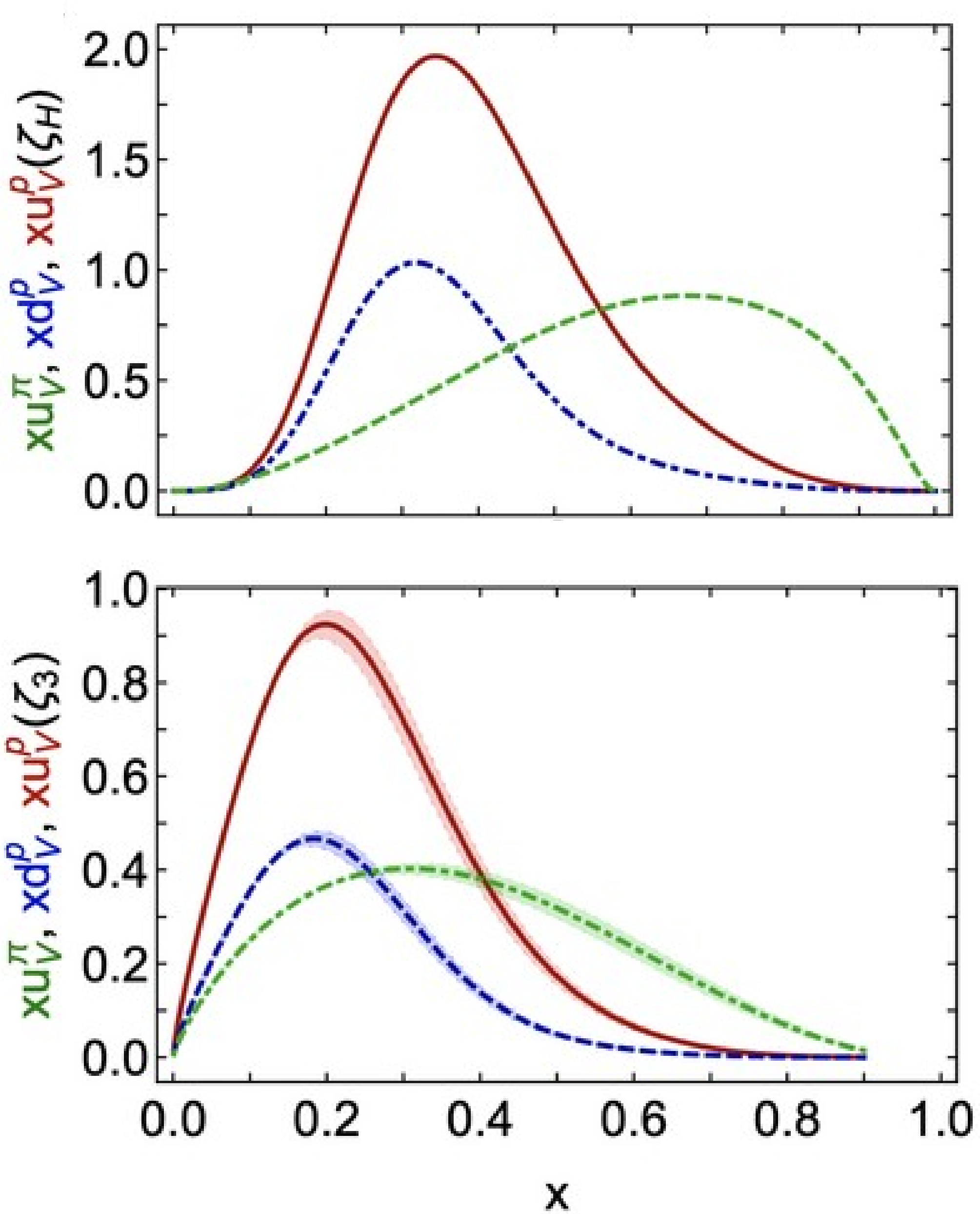

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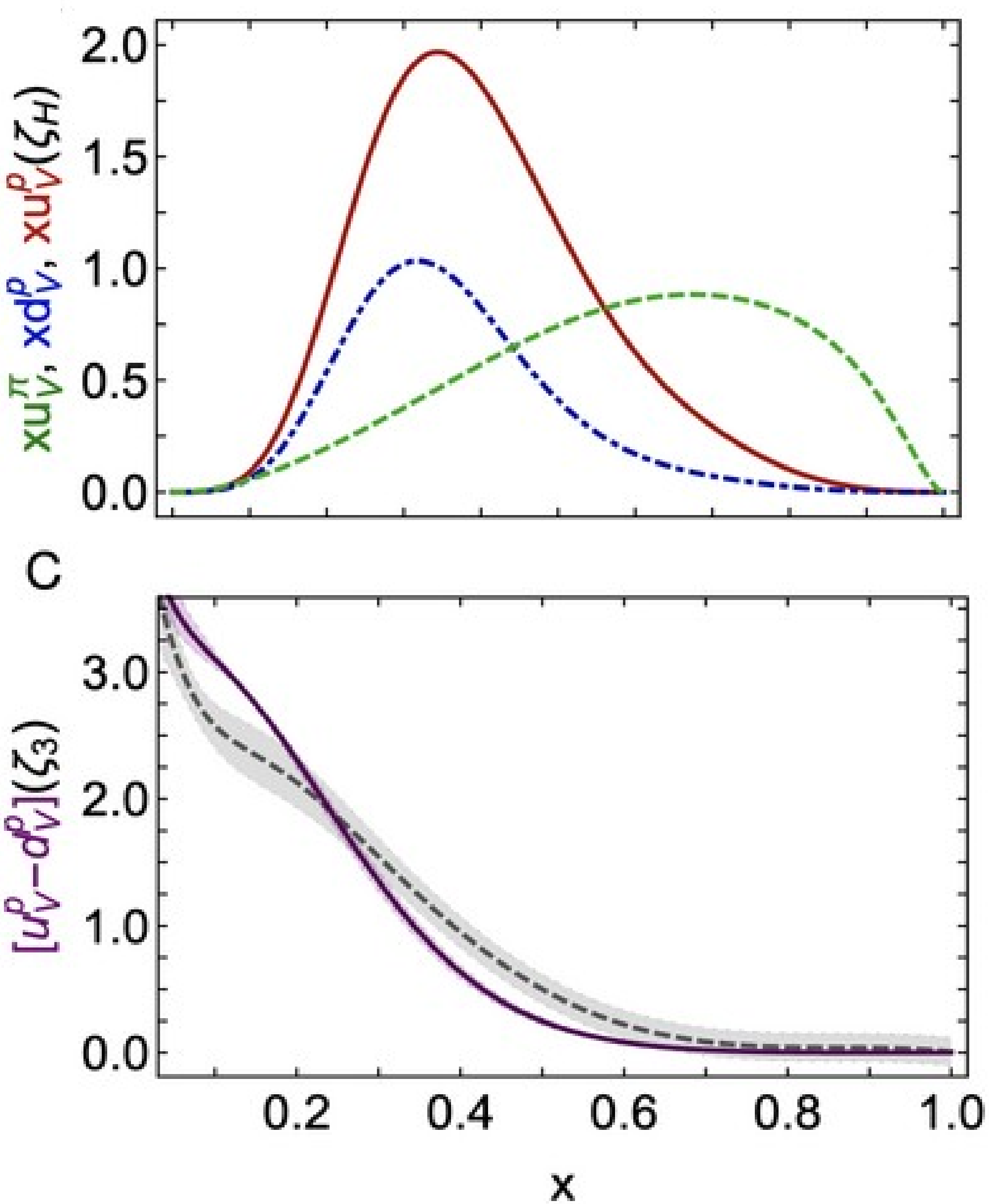
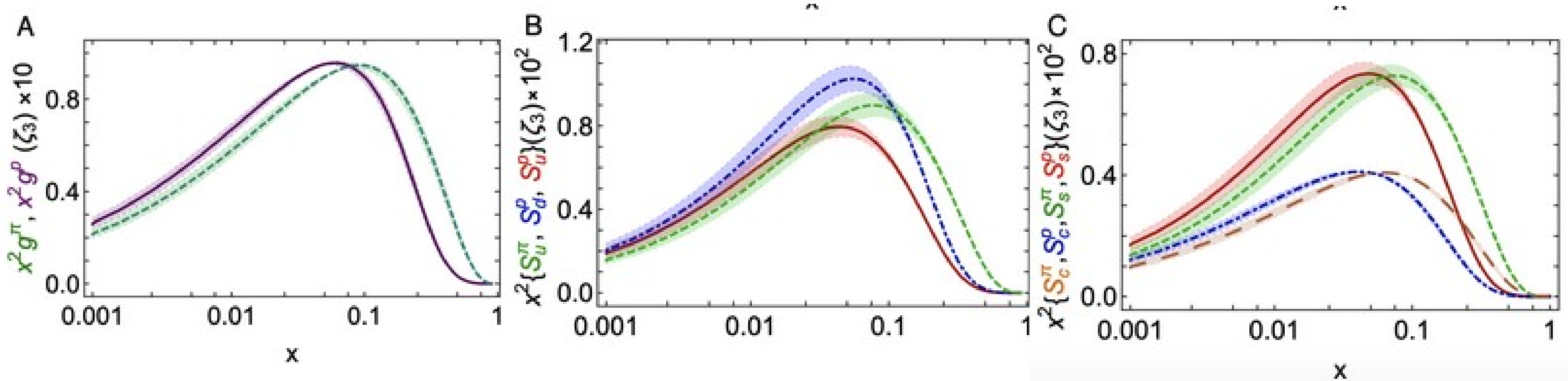
Producing an isovector distribution in fair agreement with lattice results
[H-W. Lin et al., arXiv:2011.14791]



Proton PDF: pion and proton in counterpoint



pion	u^π	$\bar{d}^\pi$	g^π	S_π^u	$S_\pi^{\bar{d}}$	S_π^s	S_π^c
$\langle x \rangle^{\zeta_2}$	24.0(1.1)	24.0(1.1)	41.0(1.2)	3.3(3)	3.3(3)	2.65(22)	1.33(5)
$\langle x^2 \rangle^{\zeta_2}$	9.5(7)	9.5(7)	3.7(1)	0.27(1)	0.27(1)	0.21(1)	0.092(2)
$\langle x^3 \rangle^{\zeta_2}$	4.7(4)	4.7(4)	0.92(6)	0.057(1)	0.057(1)	0.044(0)	0.018(1)
$\langle x \rangle^{\zeta_3}$	22.1(1.0)	22.1(1.0)	42.9(1.0)	3.7(3)	3.7(3)	3.0(2)	1.83(6)
$\langle x^2 \rangle^{\zeta_3}$	8.4(6)	8.4(6)	3.5(1)	0.27(1)	0.27(1)	0.22(1)	0.120(3)
$\langle x^3 \rangle^{\zeta_3}$	4.0(3)	4.0(3)	0.82(5)	0.056(0)	0.056(0)	0.044(0)	0.022(1)

proton	u^p	d^p	g^p	S_p^u	S_p^d	S_p^s	S_p^c
$\langle x \rangle^{\zeta_2}$	32.9(1.4)	15.0(0.7)	40.9(1.1)	2.9(2)	3.7(3)	2.64(22)	1.32(5)
$\langle x^2 \rangle^{\zeta_2}$	8.7(6)	3.6(2)	2.4(1)	0.14(1)	0.21(1)	0.13(0)	0.059(2)
$\langle x^3 \rangle^{\zeta_2}$	2.9(3)	1.1(1)	0.39(2)	0.019(0)	0.030(1)	0.019(0)	0.008(0)
$\langle x \rangle^{\zeta_3}$	30.4(1.3)	13.8(0.6)	42.8(1.0)	3.3(3)	4.1(3)	3.0(2)	1.82(6)
$\langle x^2 \rangle^{\zeta_3}$	7.7(5)	3.2(2)	2.2(1)	0.15(1)	0.21(1)	0.14(0)	0.075(2)
$\langle x^3 \rangle^{\zeta_3}$	2.5(2)	0.9(1)	0.35(2)	0.019(0)	0.028(0)	0.019(0)	0.010(1)

Hadron PDF: hard-threshold model

Let us focus on the evolution equations

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} = -\frac{\alpha_{1\ell}(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_H}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} = -\frac{\alpha_{1\ell}(\zeta^2)}{4\pi} \left[\gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + 2\mathcal{P}^{\zeta}_{qg} \gamma^n_{qg} \langle x^n \rangle^{\zeta}_{g_H} \right]$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha_{1\ell}(\zeta^2)}{4\pi} \left[\sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right]$$

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Let us focus on the evolution equations and consider a simpler but insightful model: a system of **massless partons** with **hard thresholds** for each flavor activation, that can be analytically solved!

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$\mathcal{P}^\zeta_{qg}$: $\theta(\zeta - M_q)$

$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s, c$$

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Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$

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Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors.

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In pion's (proton's) case

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Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors. Specializing for the averaged momentum fraction.

In pion's (proton's) case $\langle x \rangle^{\zeta_H}_{s_\pi} = 0$

$$\langle x \rangle^{\zeta}_{\Sigma^s_\pi} \equiv 0$$

$$\begin{pmatrix} \langle x \rangle^{\zeta}_{\Sigma^{u+d}_\pi} \\ \langle x \rangle^{\zeta}_{g_\pi} \end{pmatrix} = \begin{pmatrix} \frac{3}{11} + \frac{8}{11} [S(\zeta_H, \zeta]^{11/8} \\ \frac{8}{11} \left(1 - [S(\zeta_H, \zeta]^{11/8}\right) \end{pmatrix}$$

$$S(\zeta_H, \zeta) = \exp\left(-\frac{\gamma_{uu}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

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In kaon's case (after some algebra) $\langle x\rangle^{\zeta_H}_{s_K} = s_0$

$$\langle x\rangle^\zeta_{\Sigma^s_K} = s_0\, S(\zeta_H,\zeta)$$

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Same for all hadrons!

$$S(\zeta_H,\zeta) = \exp\left(-\frac{\gamma_{uu}}{2\pi}\int_{\zeta_H}^{\zeta}\frac{dz}{z}\alpha(z^2)\right)$$

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$$\langle x \rangle^{\zeta}_{q_H} = \langle x \rangle^{\zeta_H}_{q_H} S(\zeta_H, \zeta)$$

$$\langle x \rangle^{\zeta}_{g_H} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle^{\zeta}_{u_\pi} \right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-7/4} - \frac{24}{275} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-3/16} \left[\langle 2x \rangle^{M_s}_{u_\pi} \right]^{-25/16} + \frac{8}{11} \left[\langle 2x \rangle^{M_c}_{u_\pi} \langle 2x \rangle^{M_s}_{u_\pi} \right]^{-3/16}$$

Capitalizing on the universality of the effective charge, **all hadrons'** momentum fraction averages can be expressed in terms of **pion's** ones.

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3 (always) active flavors →

$$\tau(\zeta_H, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u} \right]^{-7/4} + \frac{16}{25} \left[\langle 2x \rangle^{M_c}_{u} \right]^{-3/16}$$

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$$\langle x \rangle^\zeta_{g_H} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle^\zeta_{u_\pi} \right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-7/4} - \frac{24}{275} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-3/16} \left[\langle 2x \rangle^{M_s}_{u_\pi} \right]^{-25/16} + \frac{8}{11} \left[\langle 2x \rangle^{M_c}_{u_\pi} \langle 2x \rangle^{M_s}_{u_\pi} \right]^{-3/16}$$

3 (always) active flavors →

$$\tau(\zeta_H, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u} \right]^{-7/4} + \frac{16}{25} \left[\langle 2x \rangle^{M_c}_{u} \right]^{-3/16}$$

4 (always) active flavors →

$$\tau(\zeta_H, \zeta_H) = \frac{4}{7}$$

Previous result then recovered!

Hadron PDF: hard-threshold model

Let us focus on the evolution equations and consider a simpler but insightful model: a system of **massless partons** with **hard thresholds** for each flavor activation, that can be analytically solved!

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_H}$$

$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s, c$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + \theta(\zeta - M_q) 2\gamma^n_{qg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

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$$\langle x \rangle^{\zeta}_{S^q_H} = \langle x \rangle^{\zeta}_{\Sigma^q_H} - \langle x \rangle^{\zeta}_{q_H} = \theta(\zeta - M_q) \frac{1}{3\pi} \int_{M_q}^{\zeta} \frac{dz}{z} \alpha(z^2) \langle x \rangle^{z}_{g_H} S(z, \zeta)$$

Any flavor **sea-quark** momentum fraction can be evaluated and seen to depend explicitly on the **mass threshold**, The same for **all hadrons** in this approximated scheme!

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$$\sum_q \langle x\rangle^{\zeta}_{S^q_H} = \frac{3}{7} + \tau(M_s, M_c)\left[\langle 2x\rangle^{\zeta}_{u_\pi}\right]^{7/4} - \sum_q \langle x\rangle^{\zeta}_{q_H}$$

Hadron PDF: hard-threshold model

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$$\sum_q \langle x \rangle^\zeta_{S^q_H} = \frac{3}{7} + \tau(M_s, M_c) \left[\langle 2x \rangle^\zeta_{u_\pi} \right]^{7/4} - \sum_q \langle x \rangle^\zeta_{q_H}$$

Momentum conservation!

Summary

Summary

- The **EHM** is argued to be intimately connected to a **PI effective** charge which enters a conformal regime, below a given momentum scale, where gluons acquiring a dynamical mass decouple from interaction.

- Capitalizing on the latter, two main ideas emerge: (I) the identification of that decoupling with a **hadronic scale** at which the structure of hadrons can be expressed only in terms of valence dressed partons; and (ii) the reliability of an **all-orders** evolution scheme to describe the splitting of valence into more partons, generating thus the glue and sea, when the resolution scale decreases.

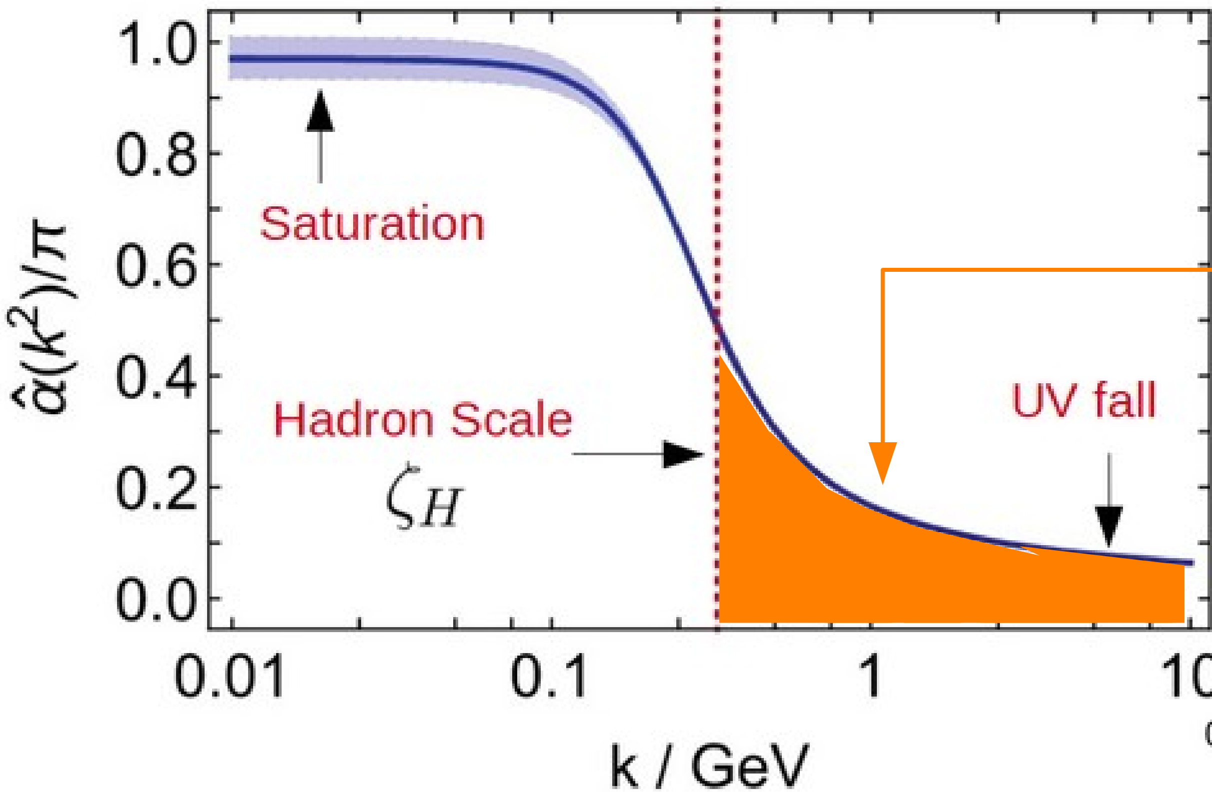
- Key implications stemming from both ideas have been derived and tested for the pion PDFs. Grounding on them, **Lattice QCD** and data from **ASV** or **JAM MF** analyses have been shown to confirm **CSM** results.

- The robustness of the approach based on **all-orders** evolution from **hadronic** to experimental scale has been proved with its application to the proton case. A model featuring massless evolution for quark flavors activated after a hard momentum threshold has been solved analytically, and seen to expose some of the main results implied by the approach.

To be continued...

Backslides

QCD effective charge

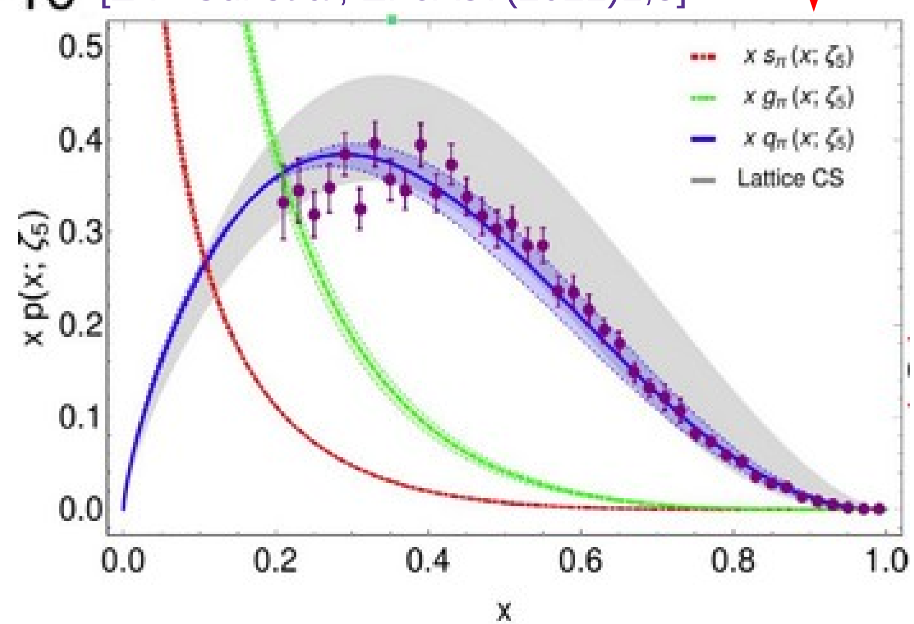


The strength of the charge defines de input for the evolution

$$S(\zeta_H, \zeta_f) = \int_{2\ln(\zeta_H/\Lambda_{\rm QCD})}^{2\ln(\zeta_f/\Lambda_{\rm QCD})} dt\, \hat{\alpha}(t)$$

$$\langle x(\zeta_5)\rangle_q^\pi = \frac{1}{2}\exp\left(-\frac{8}{9\pi} S(\zeta_H, \zeta_5)\right) = \boxed{0.20(2)}$$

[Z-F. Cui et al, EPJC80(2020)11,1064]
[Z-F. Cui et al, EPJA57(2021)1,5]



Then, the glue, valence- and sea-quark DFs can be predicted, with no tuned parameter, on the ground of the effective charge definition, from the LFWF (or, equivalentely, from a symmetry-preserving DSE/BSE computation of the valence-quarks Mellin moments

[M. Ding et al, CPC44(2020)3,031002]