Information entropy and fragmentation functions

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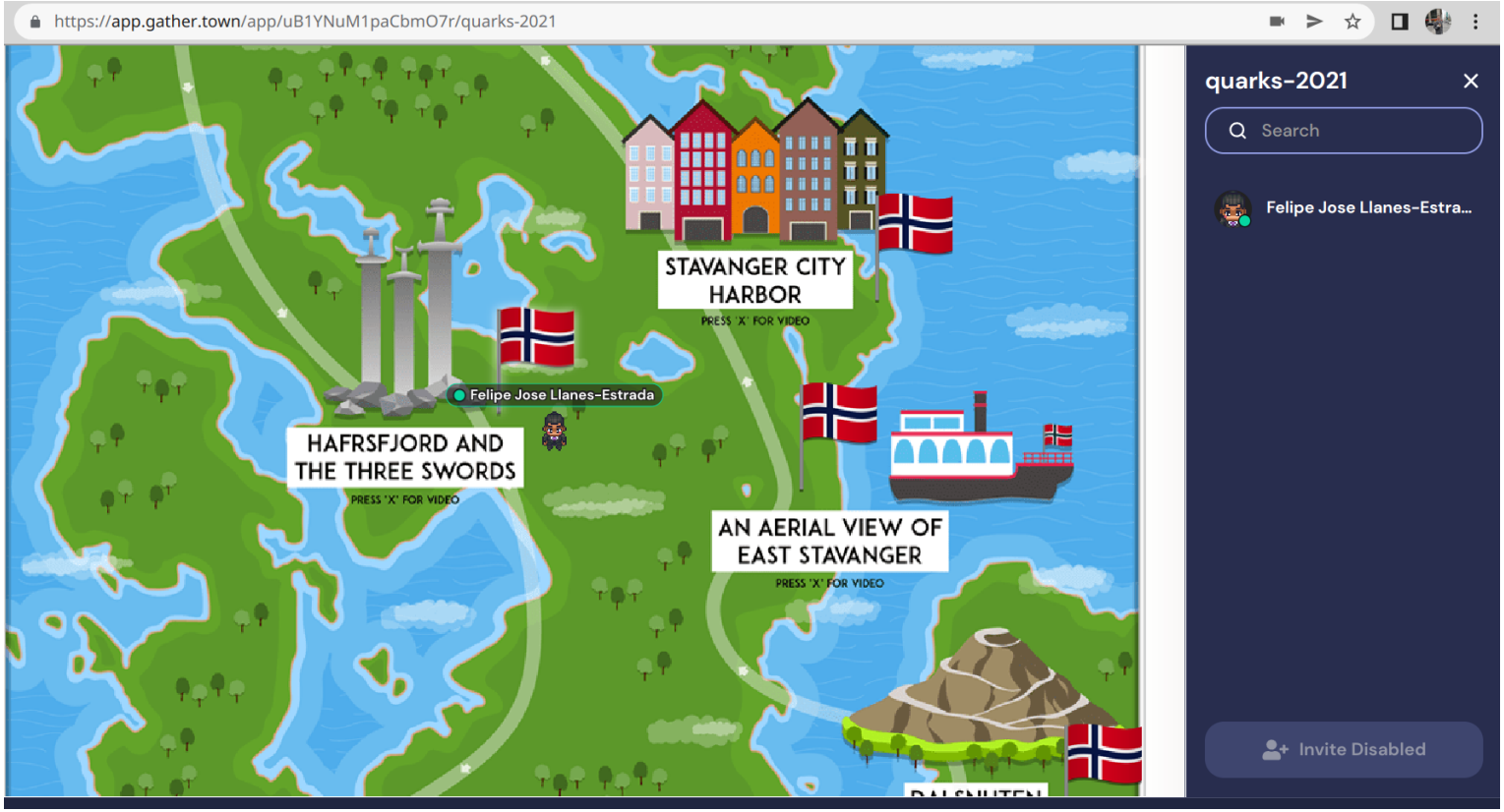
1/Aug/2022 XV Quark Confinement & Hadron Spectrum

Remember the meeting in Gathertown?



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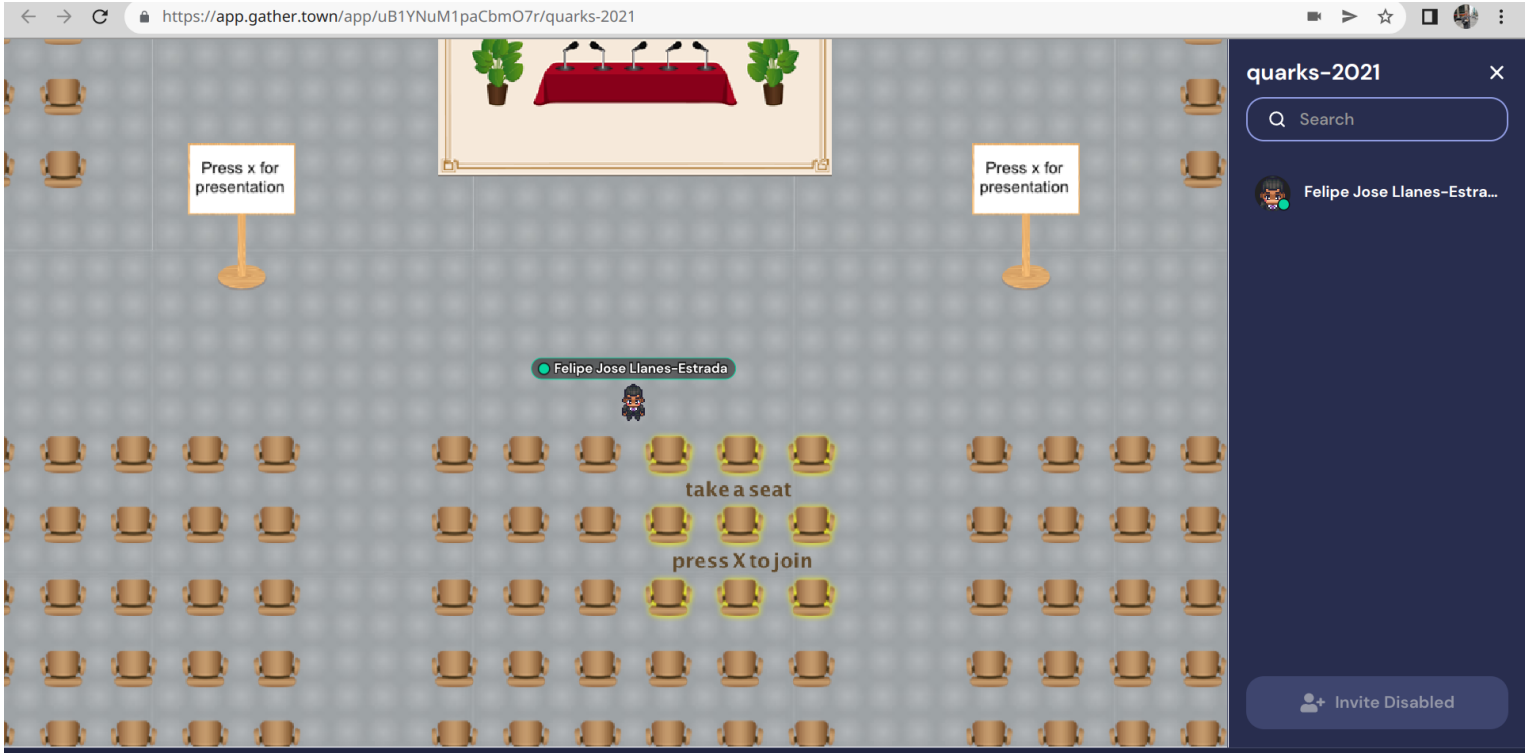
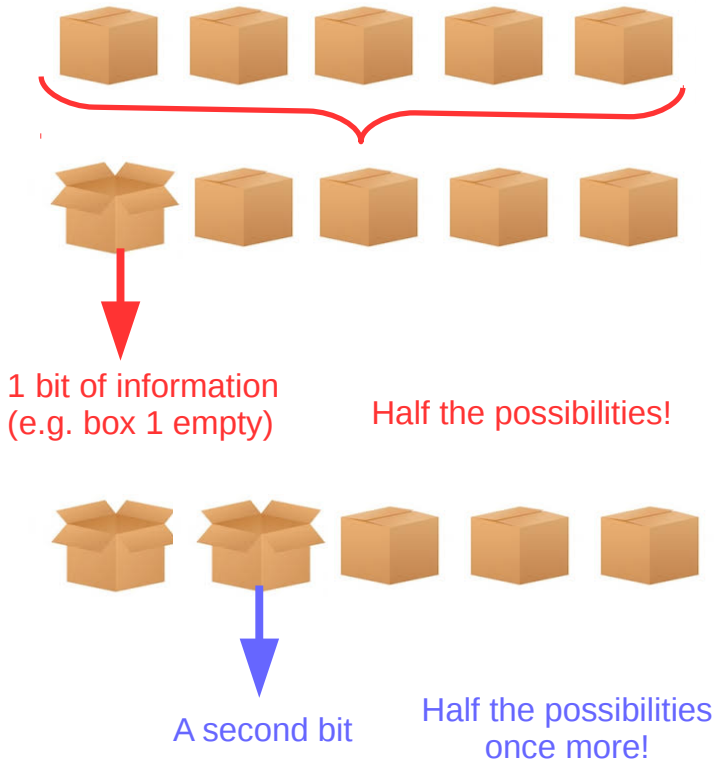


Information entropy



Probability of a configuration in terms of the number of bits of information obtained:

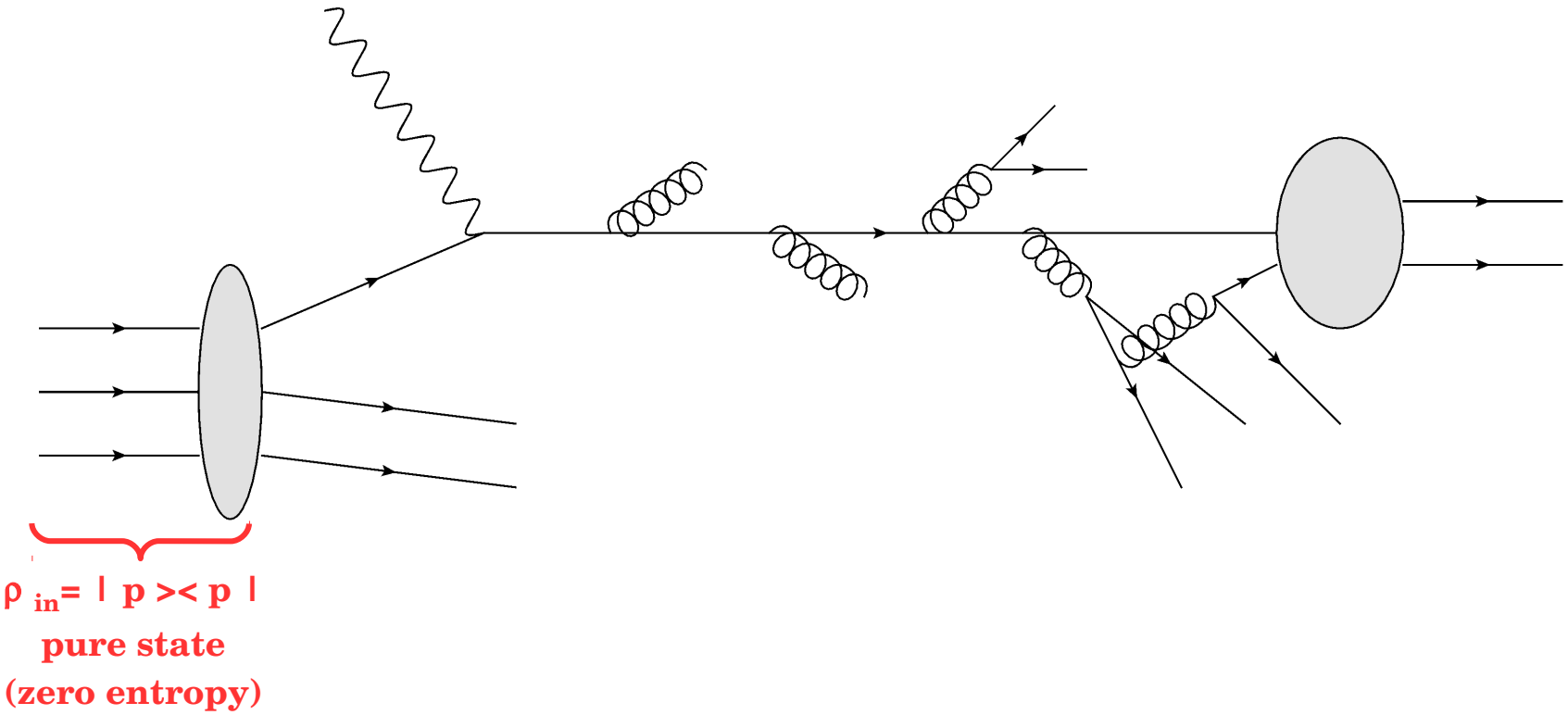
$$p = \left(\frac{1}{2}\right)^I \implies I = -\log_2(p)$$

Shannon entropy: expectation value of the information in a probability distribution

$$\boxed{S = -\sum_i p_i \log p_i}$$

Quantum Von Neumann entropy $S = -\mathrm{Tr}(\rho \log \rho)$

Entropy in high-energy physics



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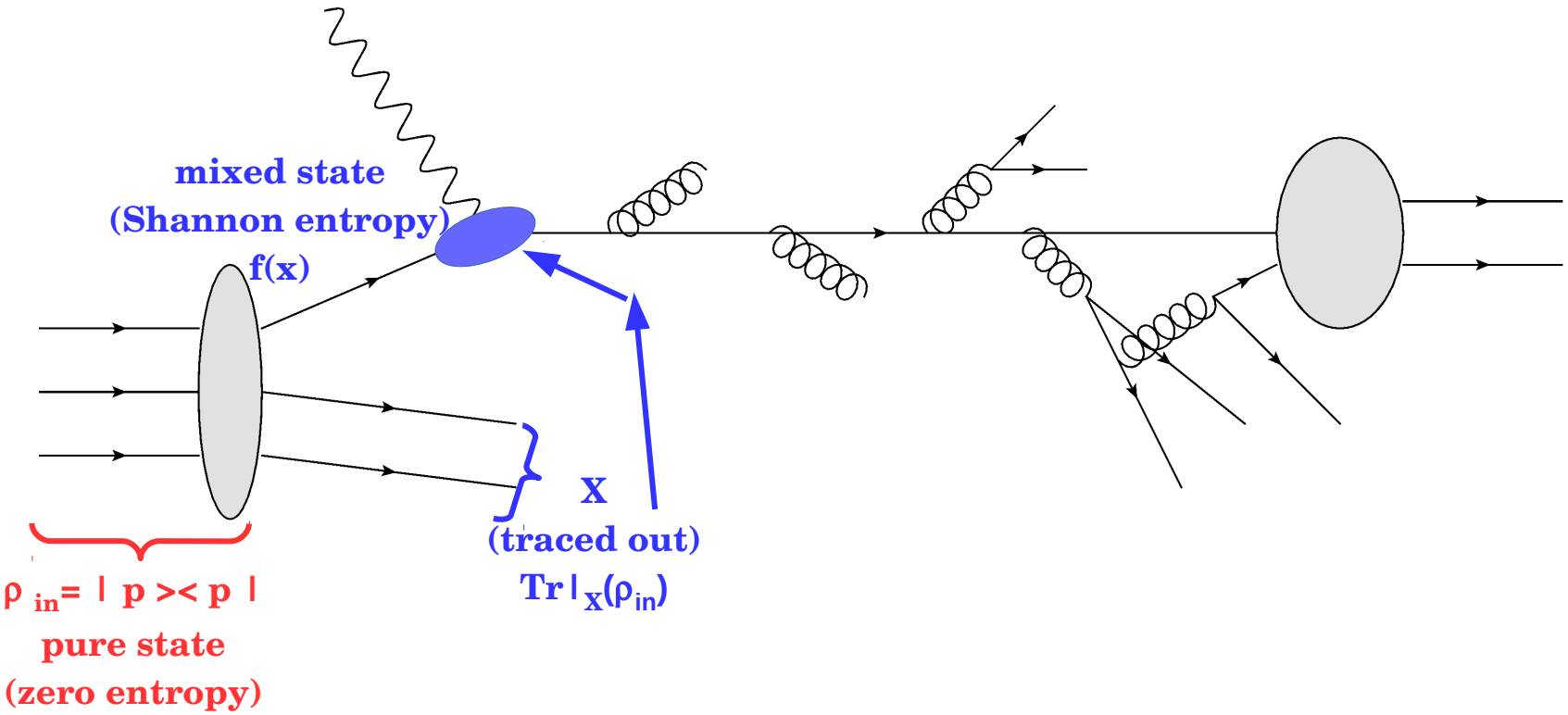

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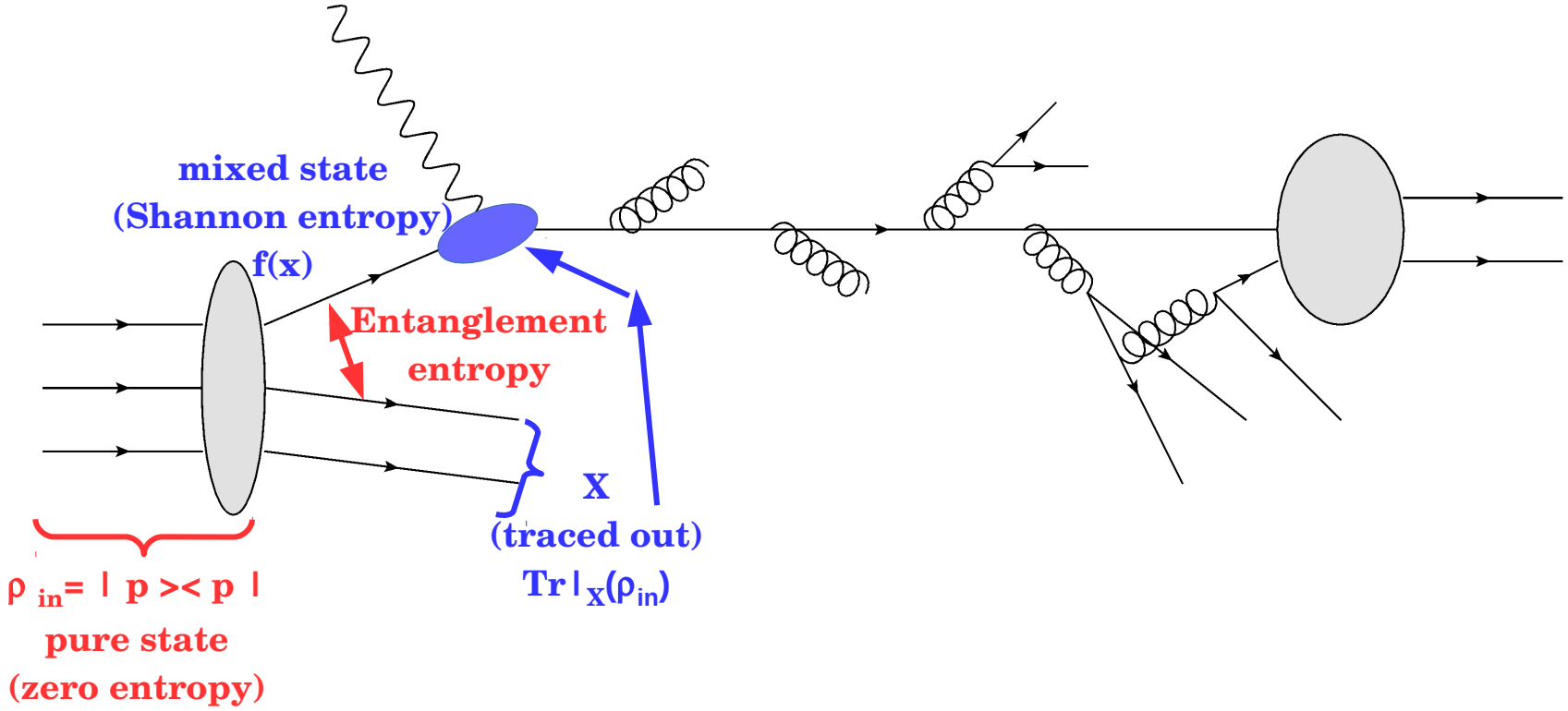
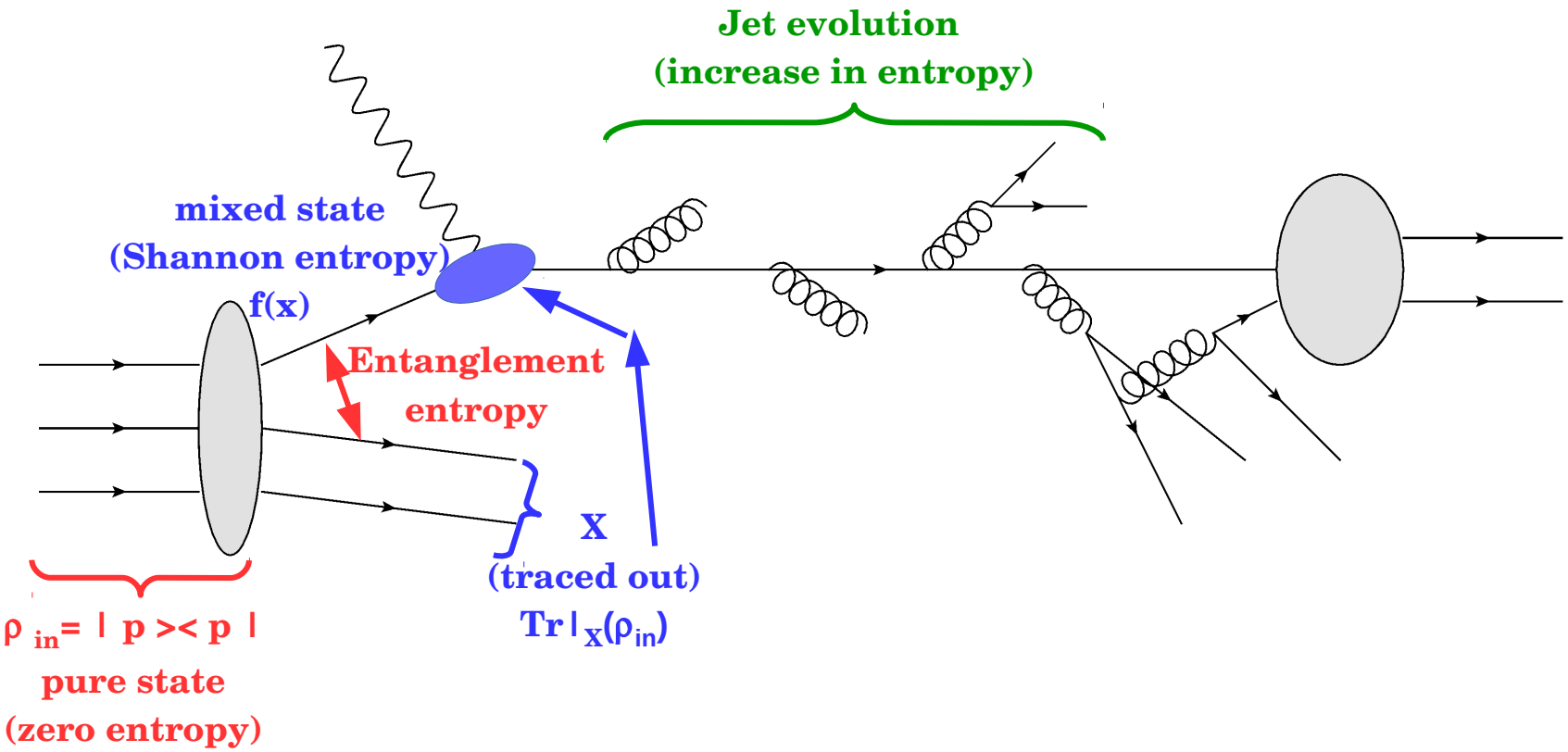


Kharzeev and Levine, Phys.Rev.D 104 (2021) 3, L031503

Entropy in high-energy physics



Neill and Waalewijn, Phys.Rev.Lett. 123 (2019) 14, 142001

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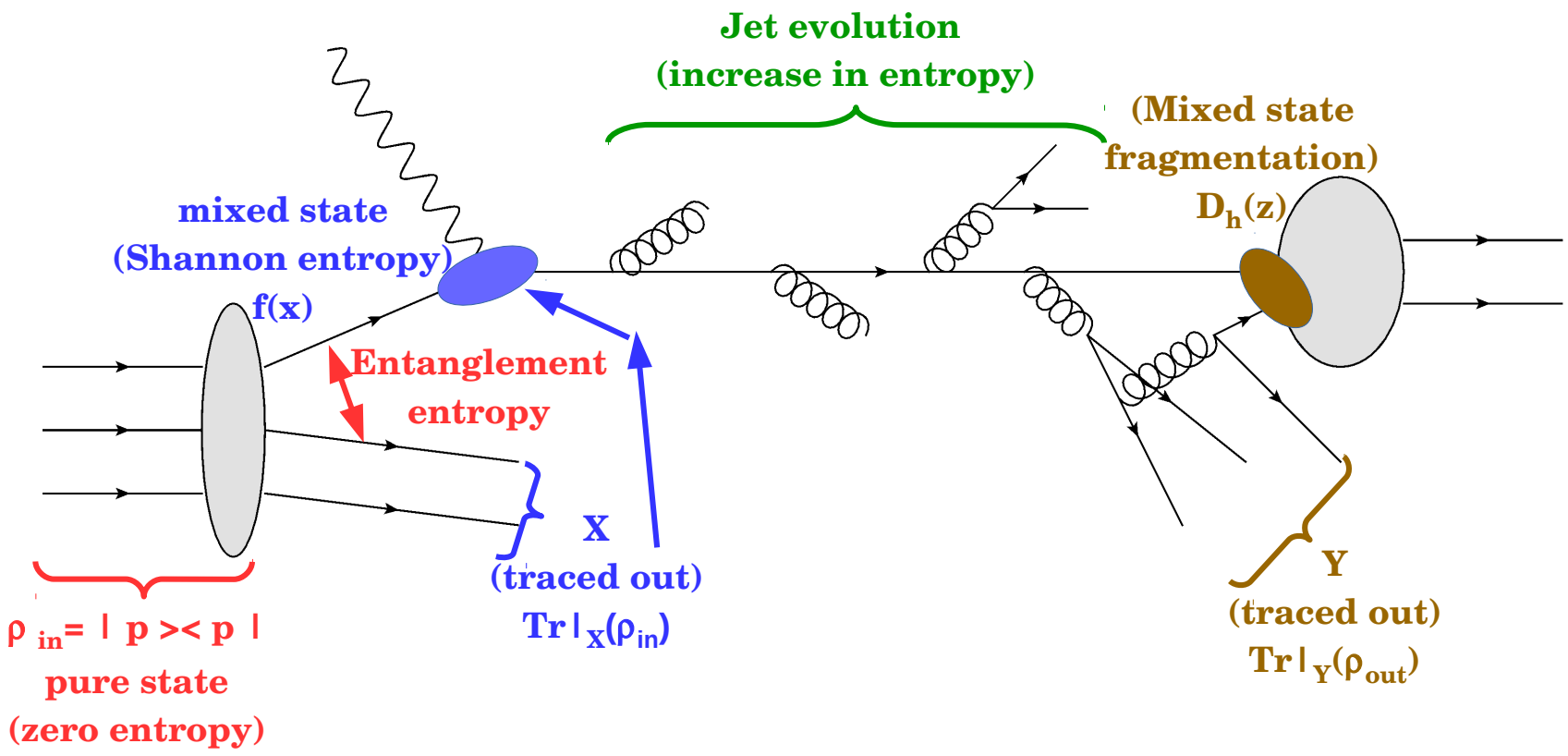

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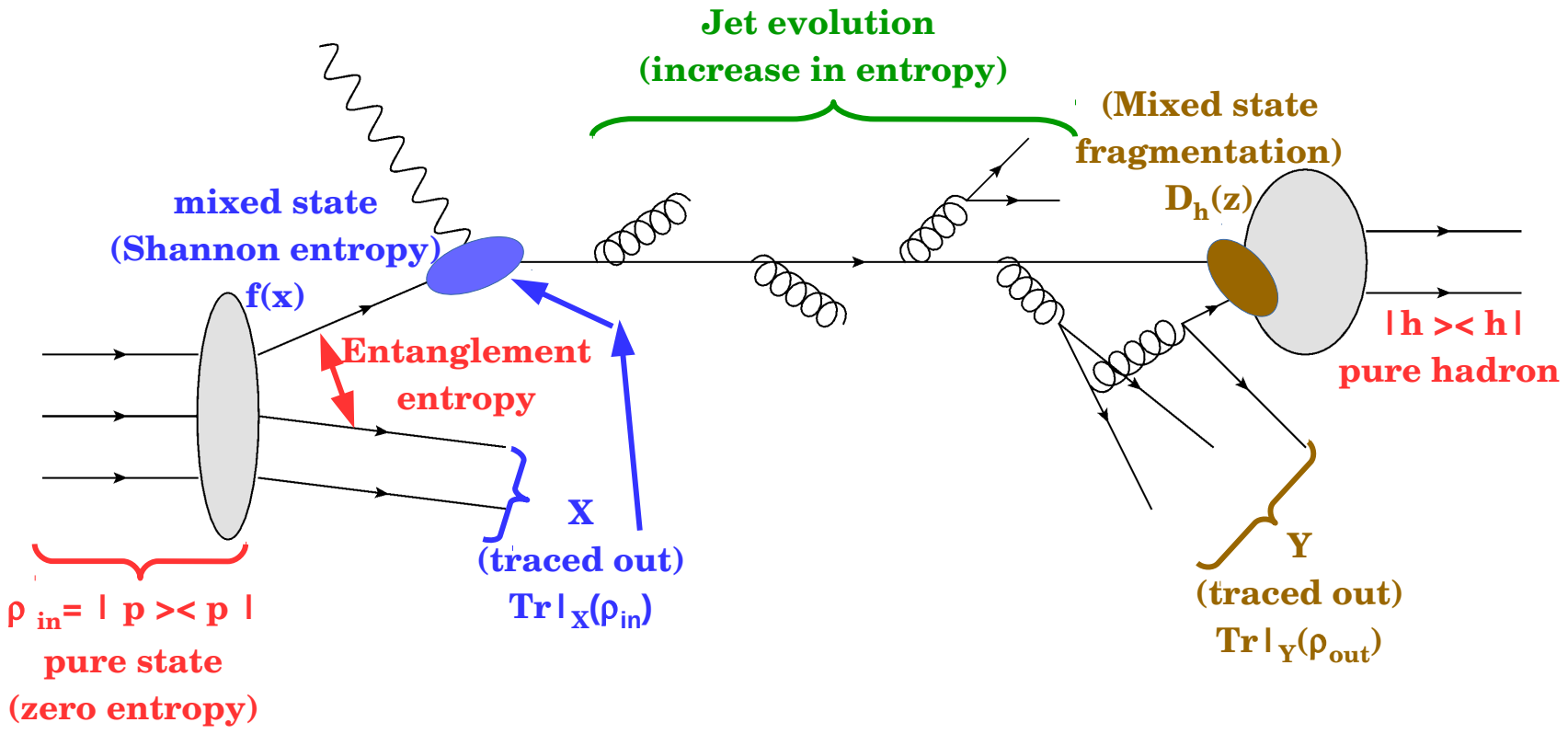
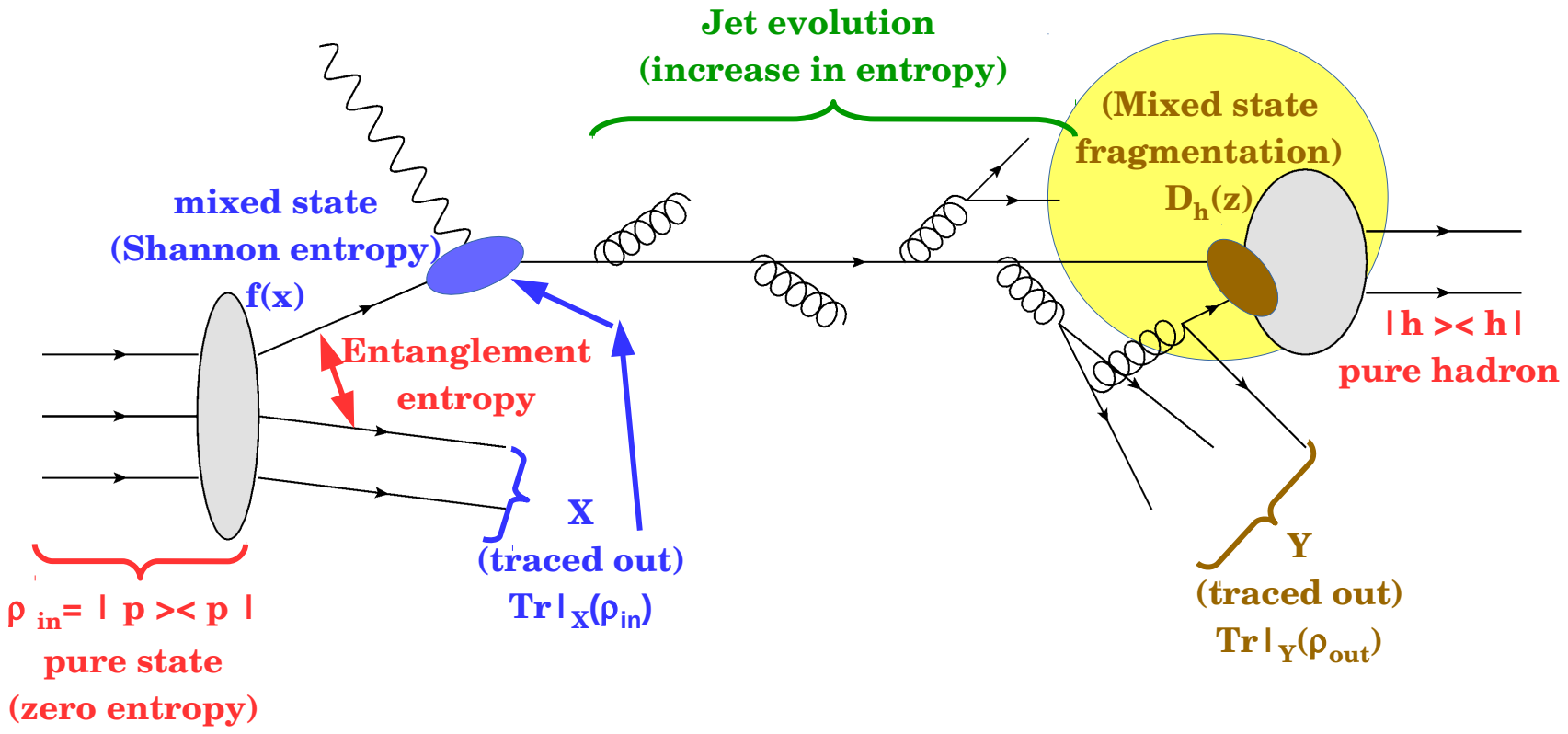


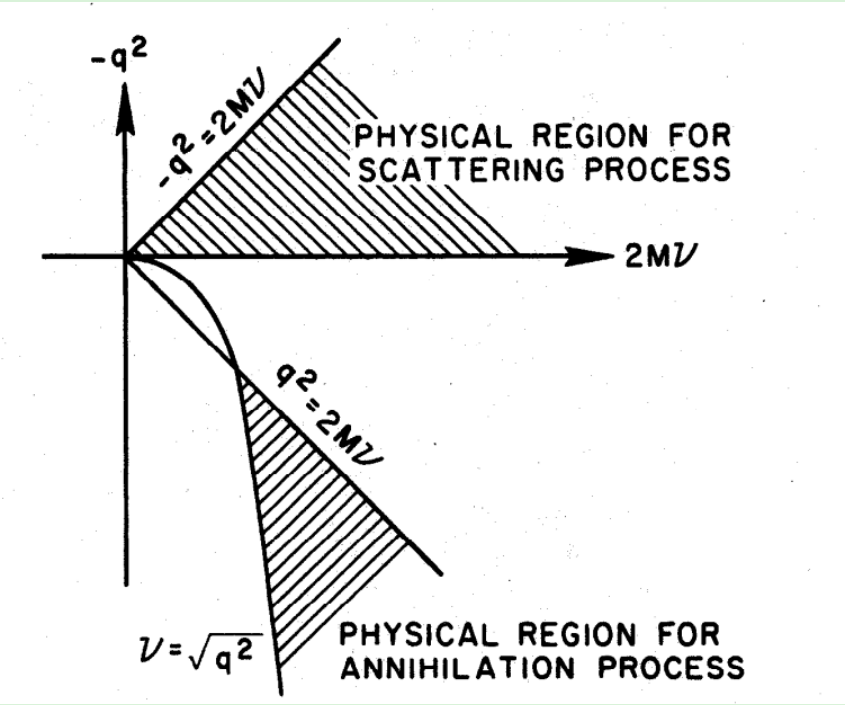
This work: entropy in fragmentation functions



Pdfs vs FFs

Parton distributions	Fragmentation functions
$\frac{d\sigma(e^- h \to e^- X)}{dx dy_1} = \frac{2\pi\alpha^2}{Q^4} s[1 + (1 + y_1)^2 F_2(x)]$	$\frac{d\sigma(e^- e^+ \to hX)}{dz} = \frac{4\pi\alpha^2}{3Q^2} z^2 \hat{F}_2(z)$
$F_2(x) = \sum_i e_i^2 x[f_i(x) + \bar{f}_i(x)]$	$\hat{F}_2(z) = -3 \sum_q e_q^2 \frac{D^q(z) + D^{\bar{q}}(z)}{z^2}$
$\sum_i \int x f_i(x) dx = 1$	$\sum_h \int_0^1 z D_h^q(z) dz = 1$

DIS and fragmentation in e^-e^+ related by continuation



Drell, Levy, Yan, Phys. Rev. D 6, 1617 (1970)

Proposed relations between pdfs and FFs

- Drell-Levy-Yan relation $D(z) = z\, f\left(\frac{1}{z}\right)$

  Problem: if $z \in [0,1]$, $\frac{1}{z} \in [1,\infty)$
  (unphysical except at $z \to 1$)

- Gribov-Lipatov "reciprocity" relation $3D(z) = f(z)$

  theoretical status: rather a model

- For $z > 0.6$ Barone-Drago-Ma eq. $D(z) \simeq zf\left(2 - \frac{1}{z}\right)$

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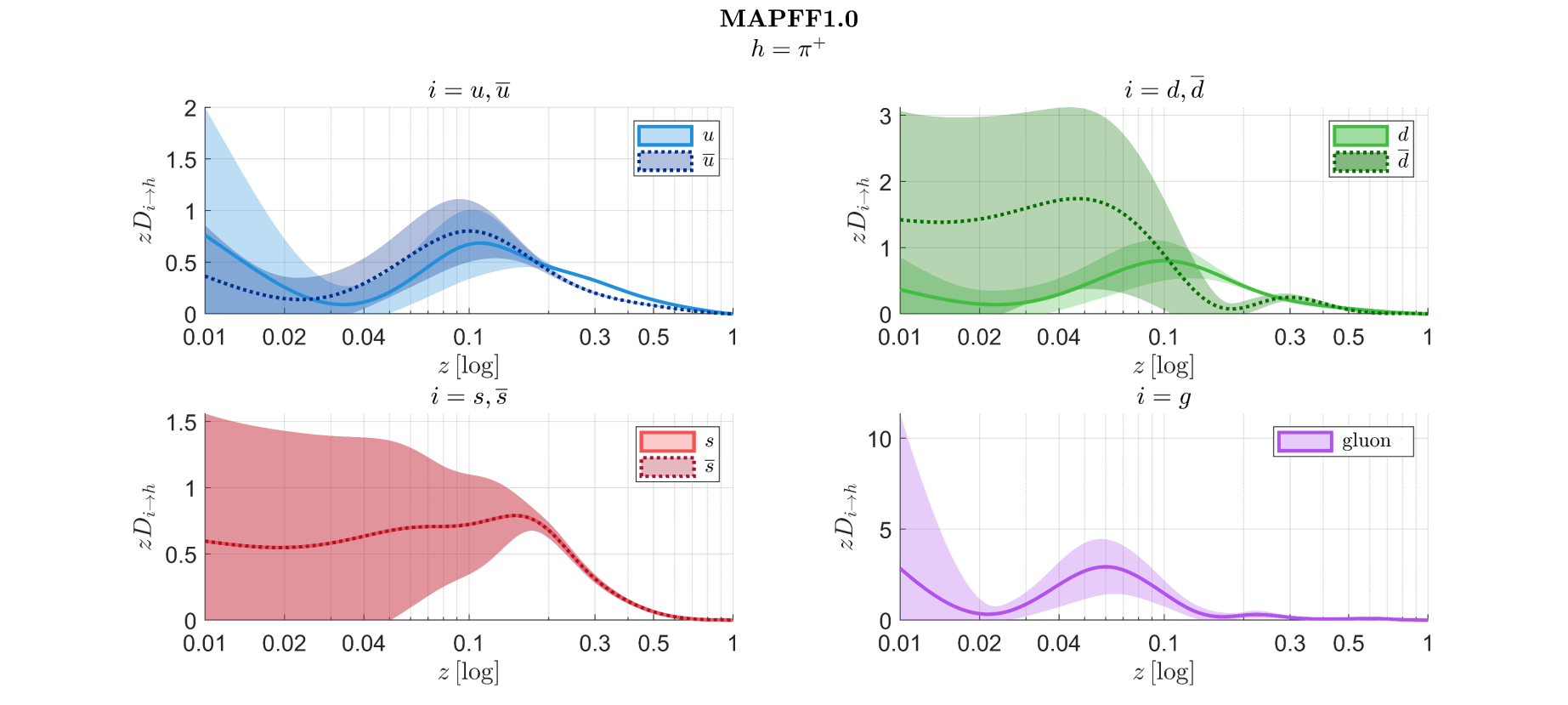
FFs from neural network fits in the literature



NNPDF coll. EPJC 77, 516 (2017)

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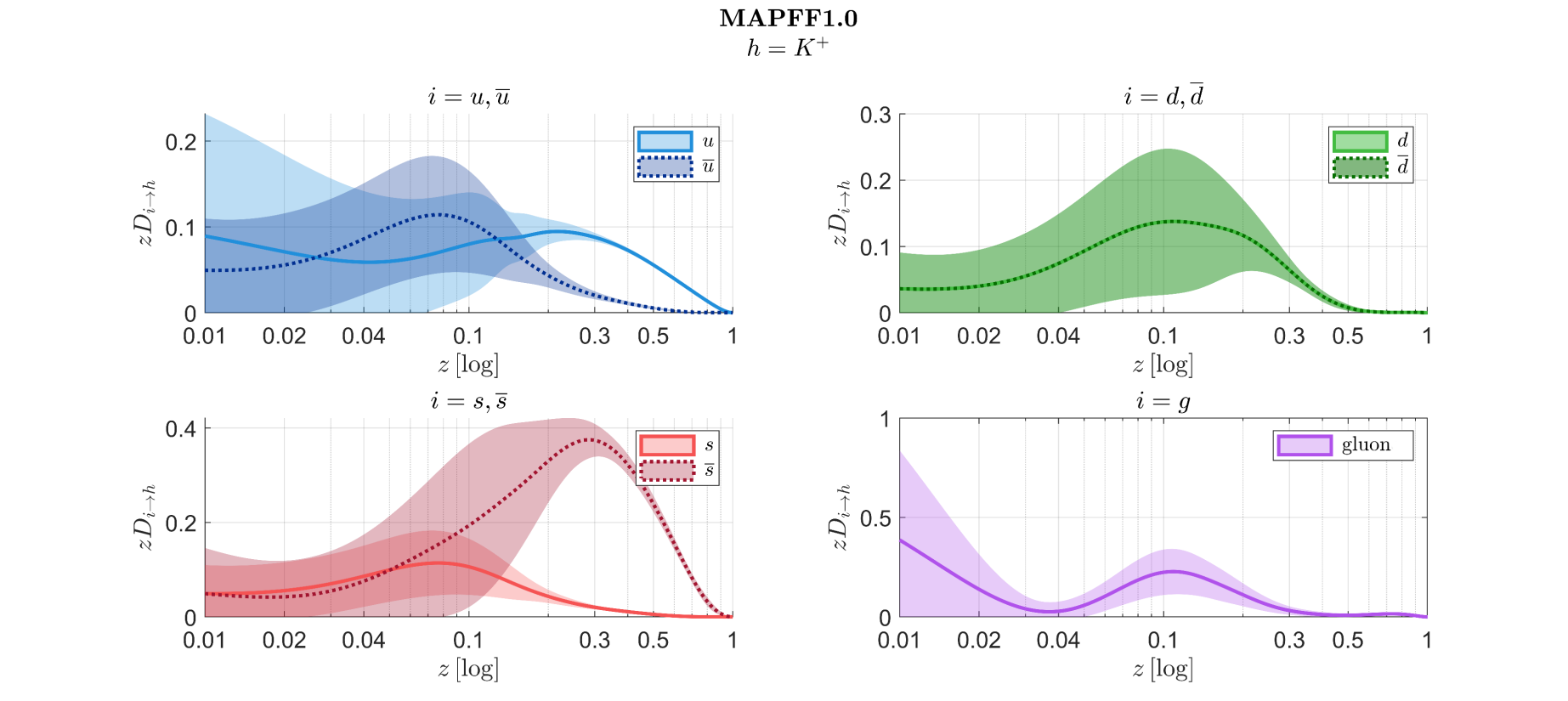

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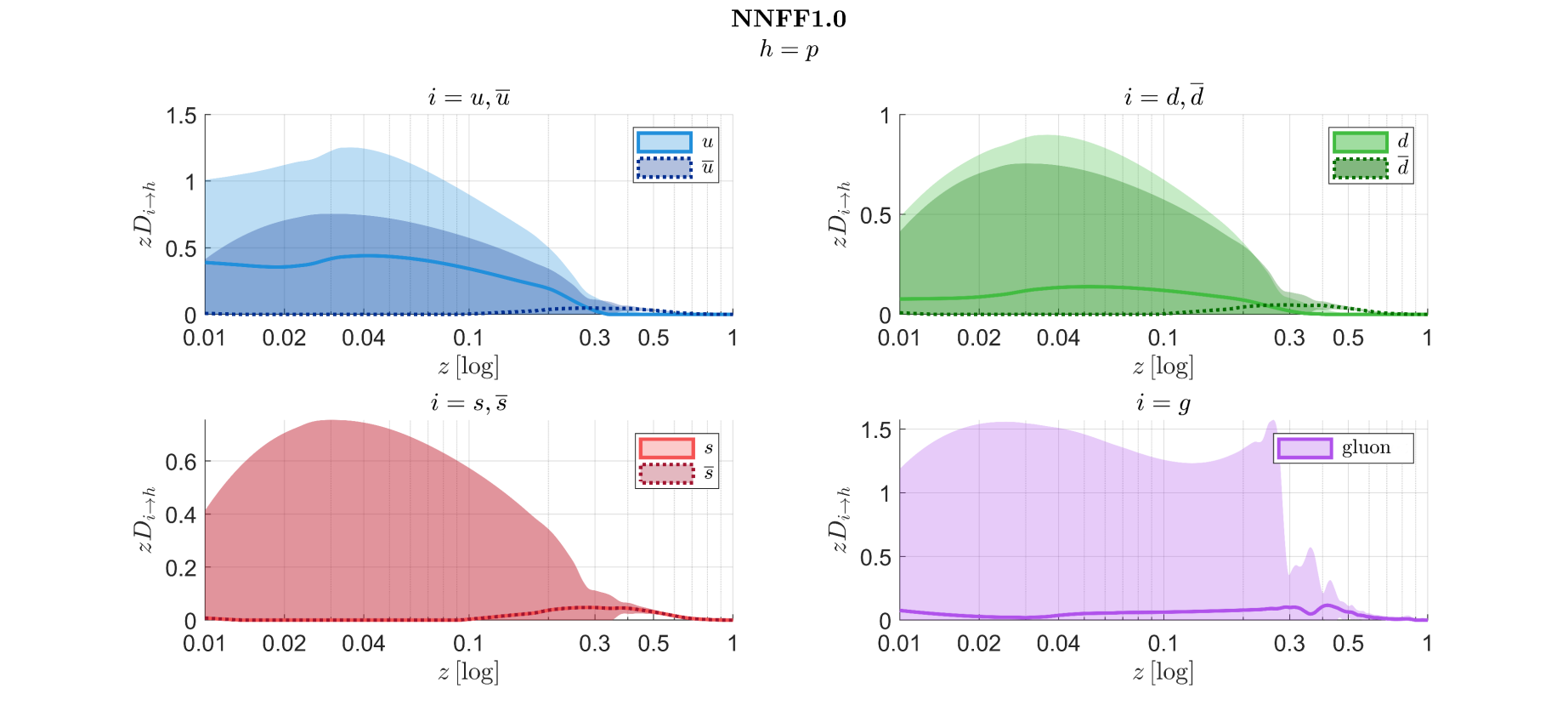

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Entropy for a continuous probability distribution

Simple discretization fails:

$zD(z) \to \{p_1, p_2 \dots p_i \dots p_N\}; \qquad \sum_i p_i = 1$

$S = -\sum_{i=1}^{N} (p_i \log p_i) \xrightarrow{N \to \infty} \log N \to \infty$

Generalization to a continuum distribution:

$S(F) := -\int f(x) \ln f(x) \mathrm{d}x$

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Kullback-Leibler divergence

$$D_{\mathrm{KL}}(P\|Q) := \sum_x p(x)\log\frac{p(x)}{q(x)} \to$$

How different are the distributions P and Q

Continuum generalization:

$$D_{\mathrm{KL}}(F\|G) = \int f(x)\log\frac{f(x)}{g(x)}\mathrm{d}x$$

$S(F) := -\int f(x)\ln f(x)\mathrm{d}x$ is then the KL divergence to the uniform distribution (that with least information)

What to do with insufficient data?

$$\underbrace{\sum_h}_{\text{Need them all!}} \int_0^1 z D_h^q(z) dz = 1$$

In practice, have $\pi^\pm$, $K^\pm$, p and little more...

Split: $D_q(z) = D_q^{\text{measured}}(z) + D_q^{\text{unknown}}(z)$

$$\underbrace{\sum_h}_{\text{measured}} \int_0^1 z D_h^q(z) dz = p < 1 \qquad \underbrace{\sum_h}_{\text{unknown}} z D_h^q(z) := 1 - p$$

(maximize entropy: uniform distribution remainder)

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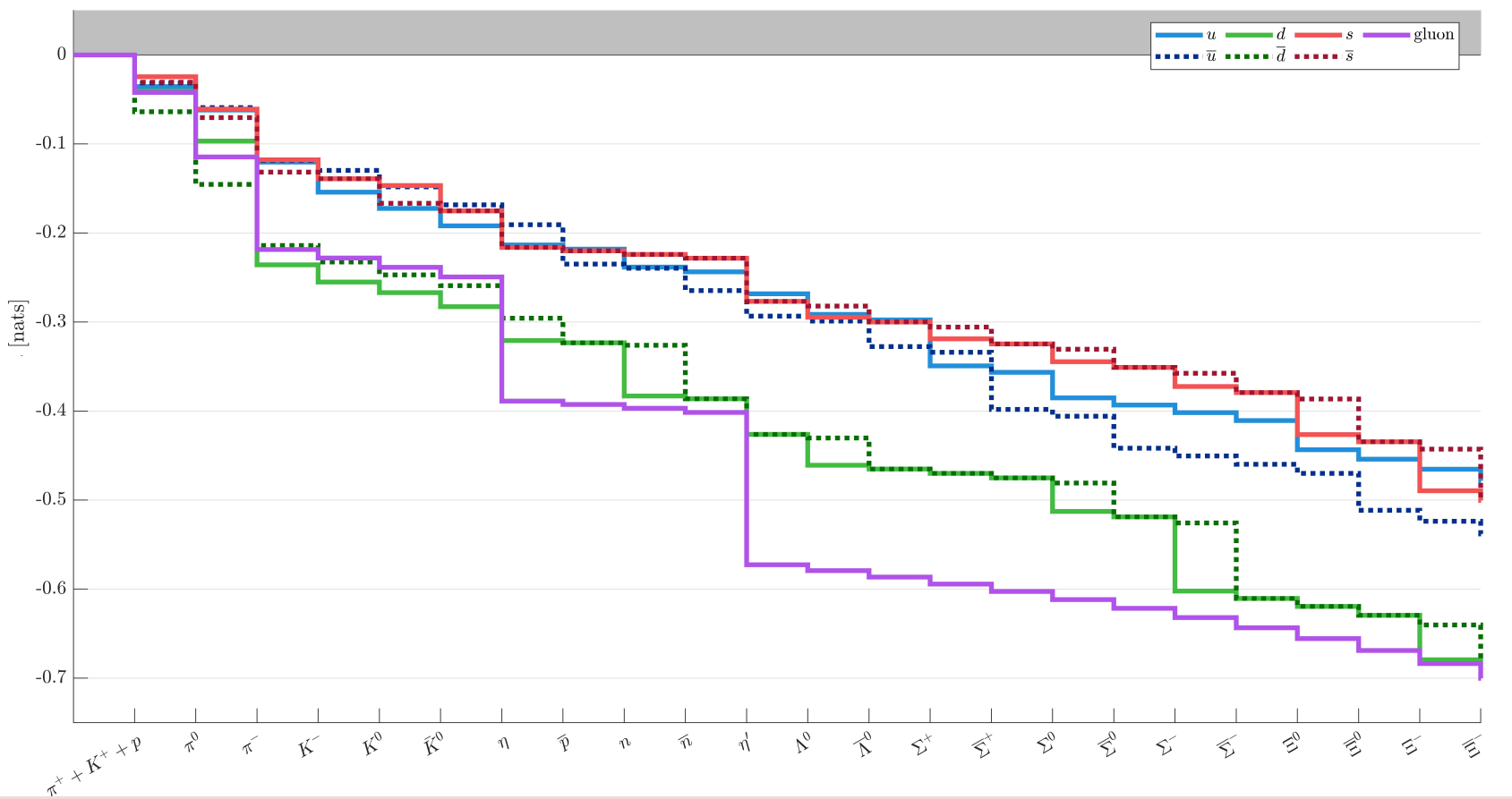
What would happen with future data? Upper bound on S



What would happened with future data?

- From $SU(3)$ relations + valence/sea separation as customary *e.g.* $D_{d\to K^0}$ (unknown) set $=$ $D_{u\to K^+}$ (already extracted)
- $\theta(\eta,\eta') = -15.5^{\circ}$
- S: one number that quantifies progress on Fragmentation Functions

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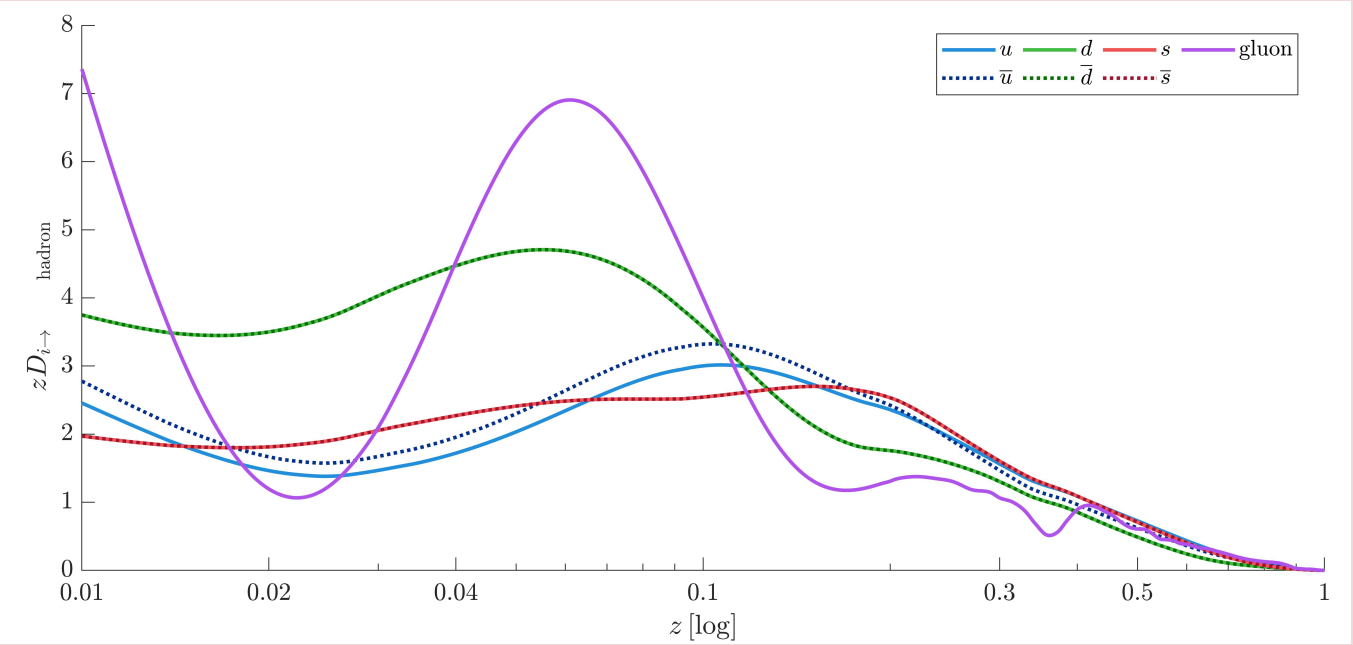
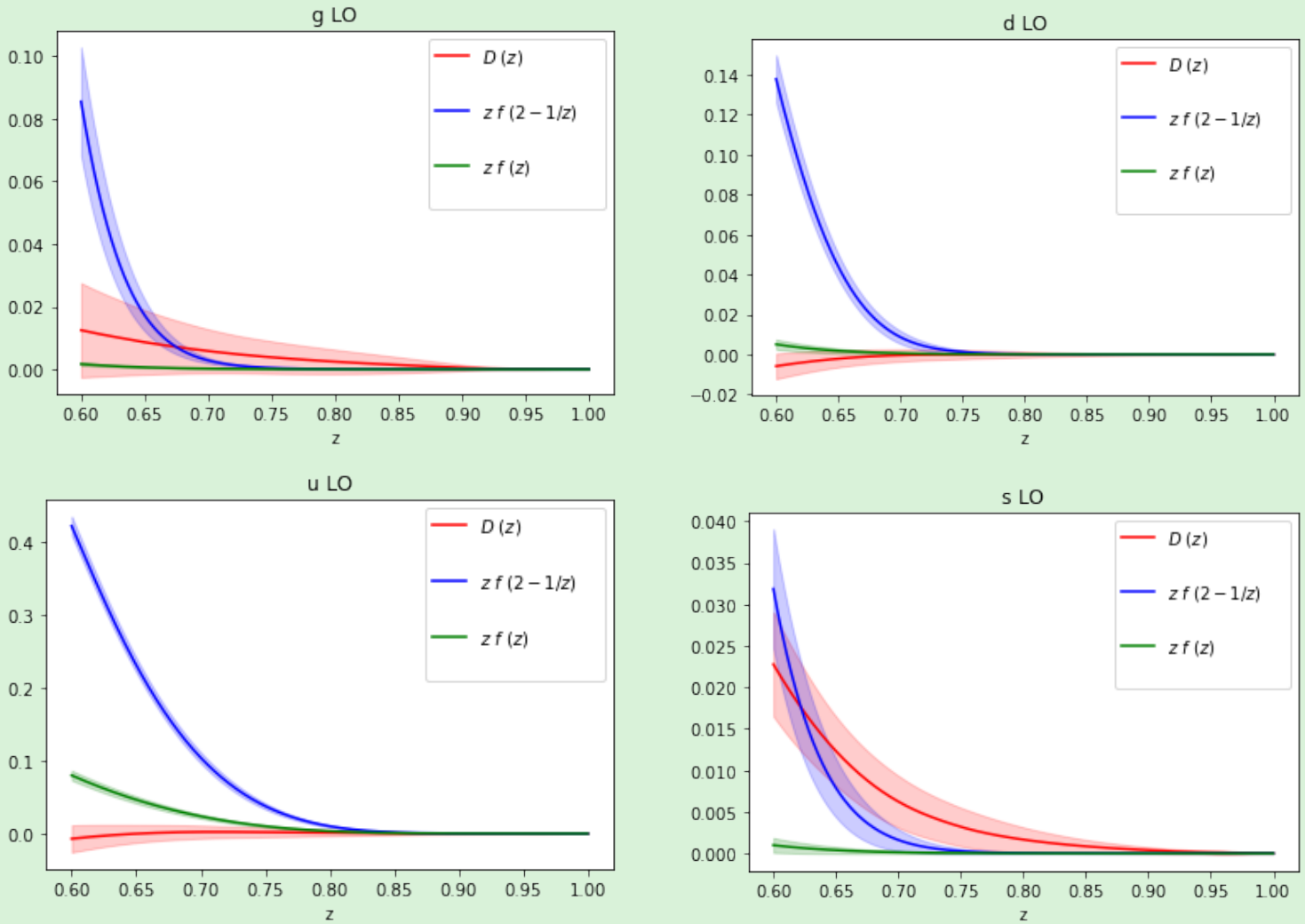
For completeness, the looks of the "reconstructed" FFs:



Click here for animation

Comparison between *D*s and *FF*s



(Using the proton's NNPDF parametrizations, $Q^2 = 100\mathrm{GeV}^2$)

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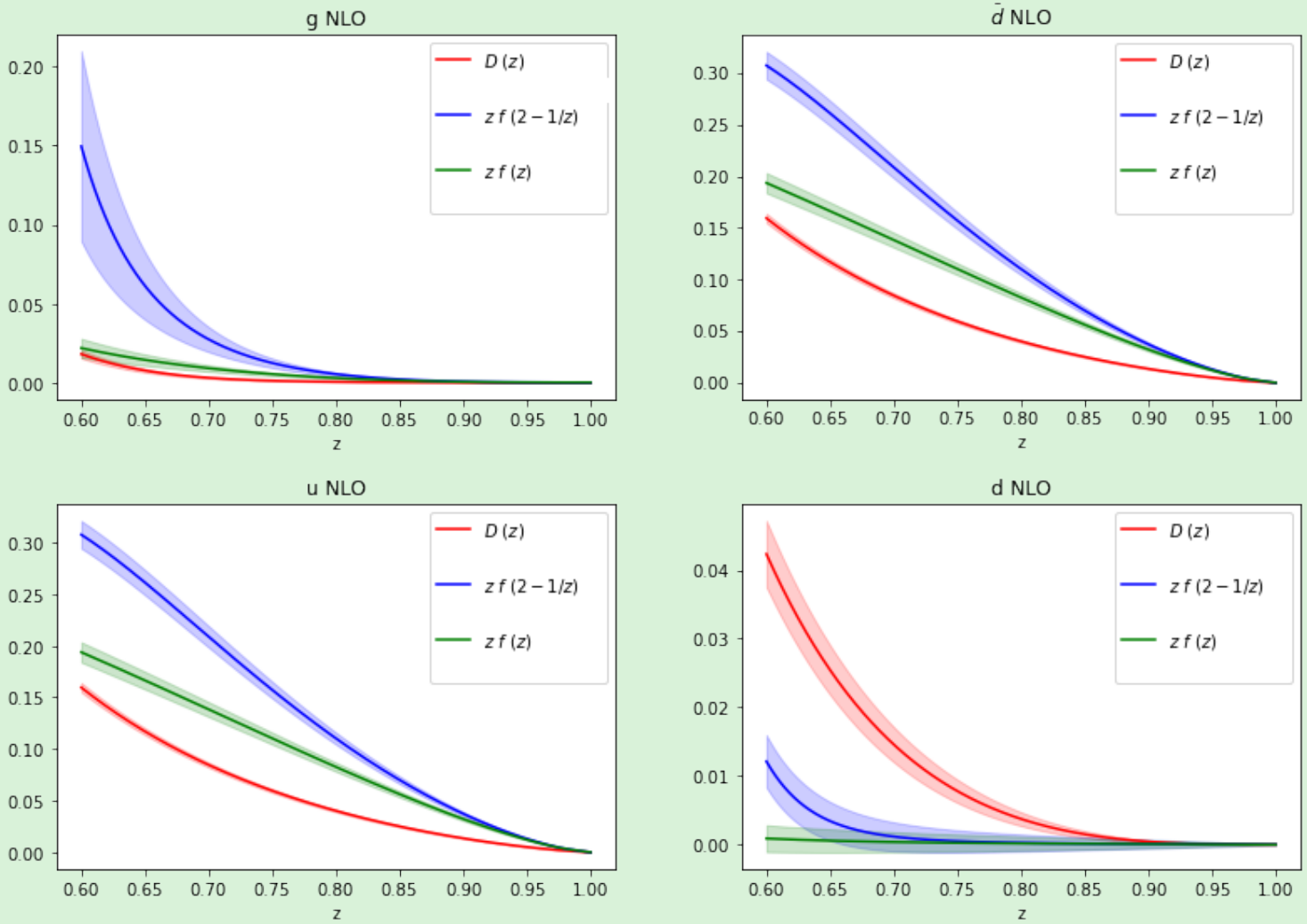
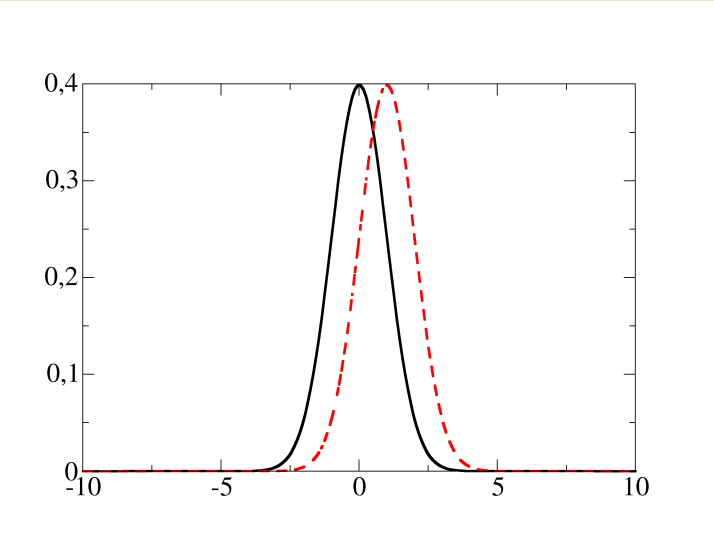


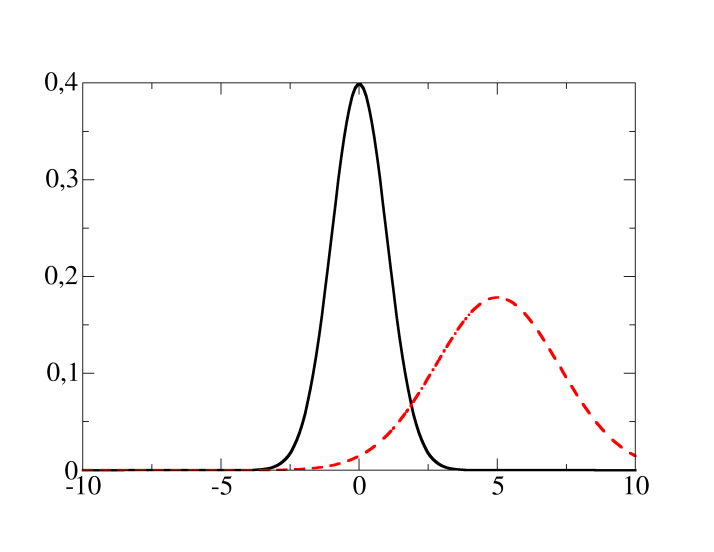
(Traditional template fit to π^+ data, $Q^2 = 100\mathrm{GeV}^2$)

Jefferson Lab Angular Momentum Collaboration, Phys.Rev.D 104 (2021) 1, 016015

Kullback-Leibler divergence: typical cases



$$D_{\mathrm{KL}} = 0.12$$



$$D_{\mathrm{KL}} = 1.63$$

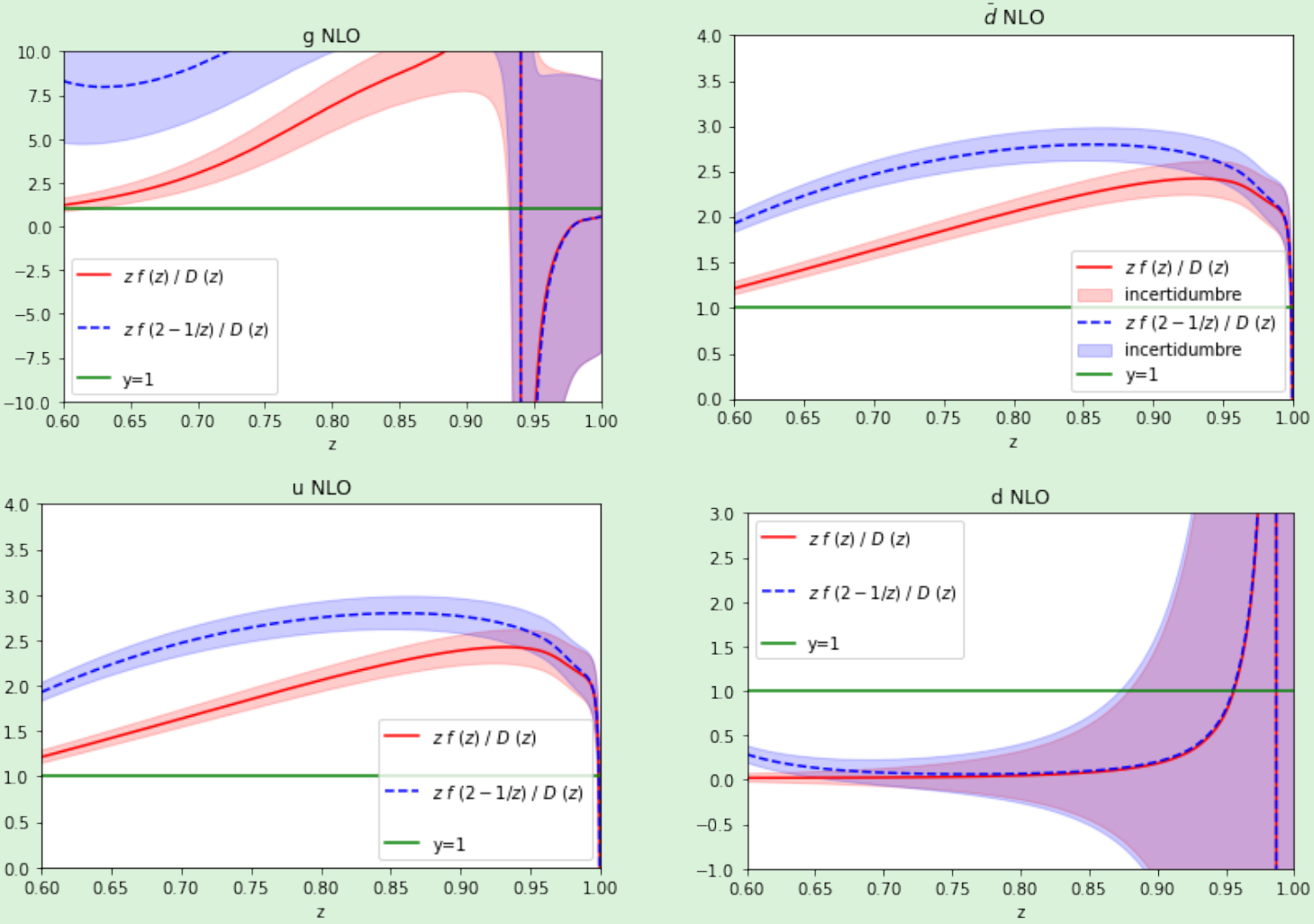
Kullback-Leibler divergence for two relations

Parton	$D_{KL}(zf_\pi(z)\|\|D^\pi(z))$	$D_{KL}(zf_\pi(2-1/z)\|\|D^\pi(z))$
gluon	2,28	7,70
up	1,19	2,36
anti-down	1,19	2,36
down	6,64	5,28

(contribution from $z \in (0.7, 1)$)

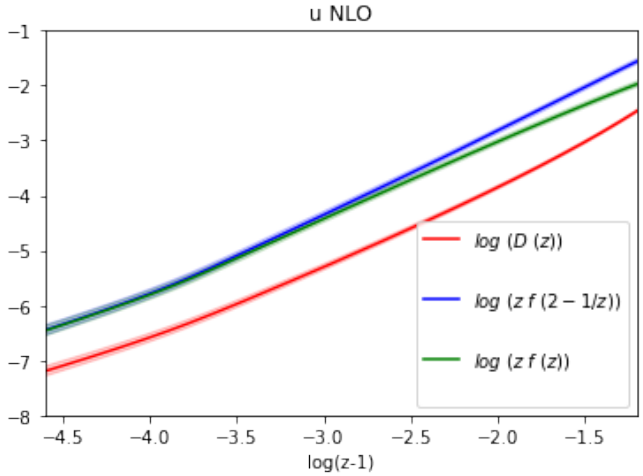
Not well satisfied!

Ratios between *D*s and *FF*s: not 1!

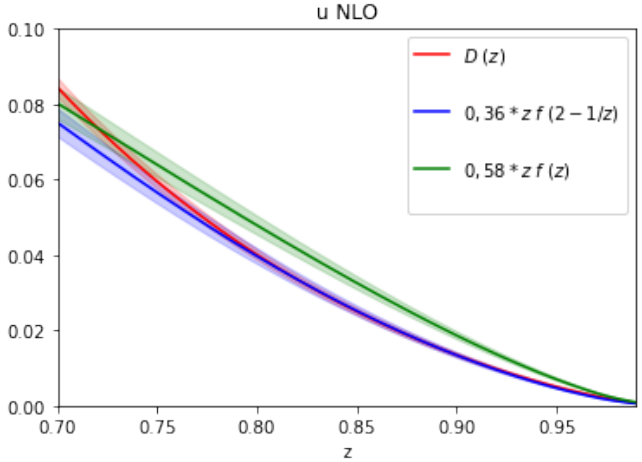


(NLO parametrizations of JAM coll. π^+ data, $Q^2 = 100\mathrm{GeV}^2$)

Relations between $f_\pi^u(x)$ and $D_u^\pi(z)$



Parallel lines
in a log-log plot



Barone-Drago-Ma relation off by a multiplicative constant only!

Further approximating $f(2-1/z) \to f(z)$ not so bad either

Slopes (power-law): $D(z)$, 1.48; $zf(2-\frac{1}{z})$, 1.52; $zf(z)$, 1.37

Conclusions

- Information entropy and KL divergence deployed for fragmentation functions
- S is one number to quantify progress on FF knowledge
- Relations $pdfs \leftrightarrow FFs$ not well satisfied with current sets
- Guess at what's happening? $f(x)$ measured on the proton; but $N^*, \Delta^* \to p \ldots$ contribute to $D(z)$, not only p
- Or Barone-Drago-Ma relation trivial and good only at $z = 1$ exactly?

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