Reconciling the FOPT and CIPT predictions for the tau hadronic spectral function moments

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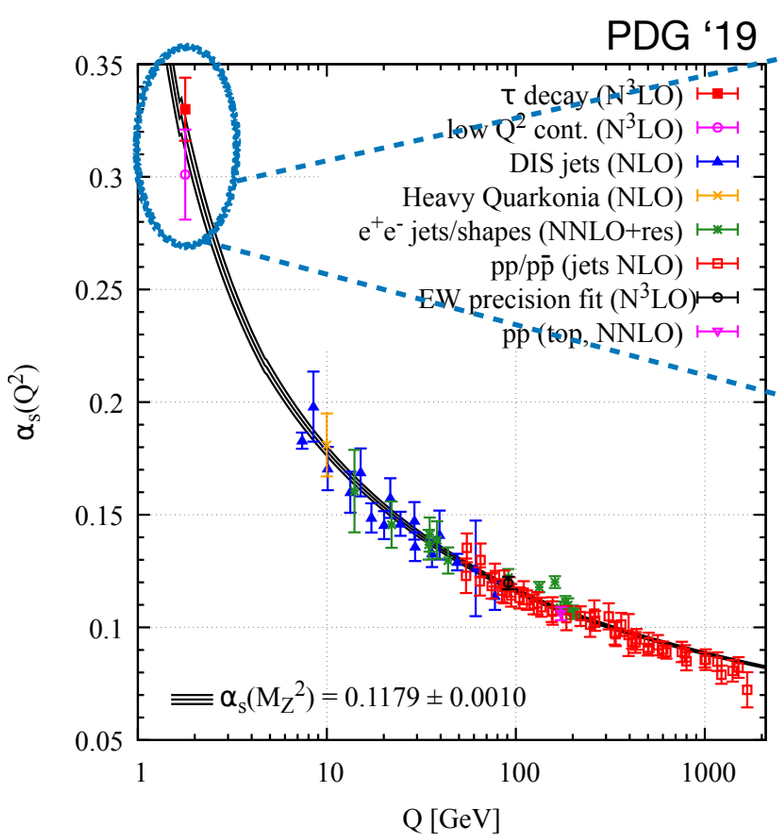
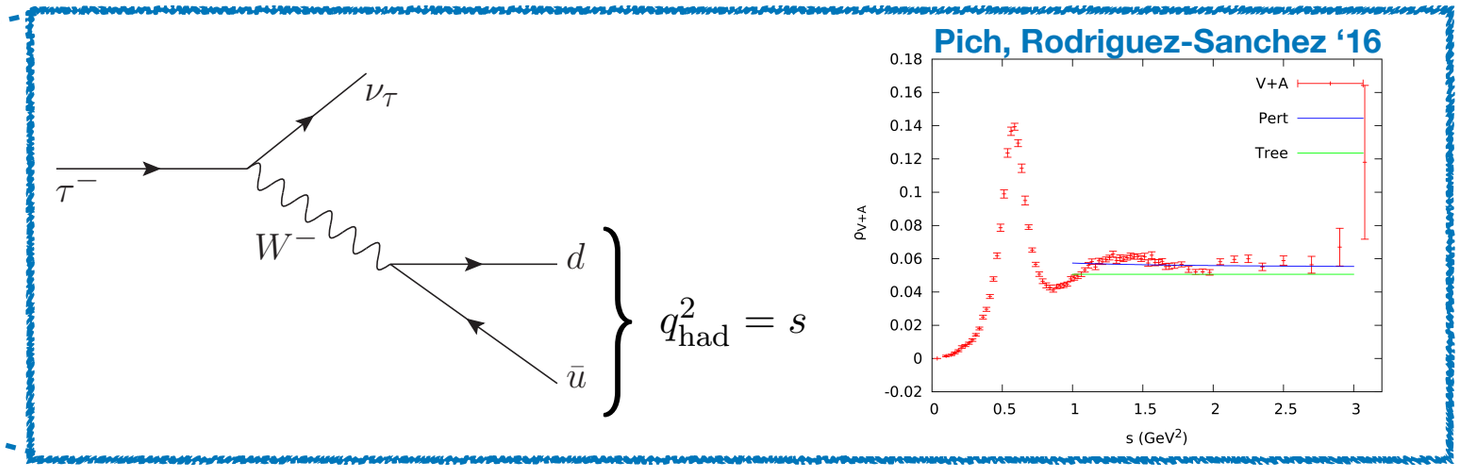
MBR, Boito, Hoang, Jamin, arXiv:2202.10957 (JHEP 07 (2022) 016)
MBR, Boito, Hoang, Jamin, arXiv:2207.01116

Strong coupling determinations





- Extraction of α_s from inclusive τ decays one of the most precise determinations of QCD coupling

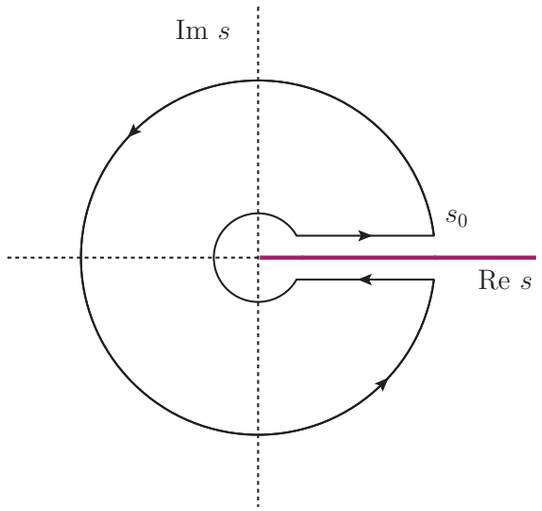
- Weighted integrals over exp. spectral functions tested against theory predictions

$$R^{(w)}_{V+A}(s_0) \sim 12\pi^2 \int_0^{s_0} \frac{ds}{s_0} w(s/s_0)\rho_{V+A}(s) \qquad \rho(s) = \frac{1}{\pi}\mathrm{Im}[\Pi(s+i\epsilon)]$$

- For $w(x) = (1-x)^2(1+2x)$

$$R^{(w)}_{V+A}(m_\tau^2) \simeq \Gamma(\tau \to \nu_\tau + \text{hadrons})$$

Hadronic tau decays: Theory



Massless correlation function: $\Pi_{\mu\nu}(p) \equiv i \int \mathrm{d}x\, e^{ipx} \langle\Omega| T\{j_\mu(x)\, j_\nu(0)^\dagger\}|\Omega\rangle$
(Only consider vector correlator)

Use Finite Energy Sum Rule (FESR) to relate experiment and theory
(Cauchy's theorem)

Experiment Theory

$$\frac{1}{s_0}\int_0^{s_0} ds\; w(s)\rho(s) = -\frac{1}{2\pi i}\oint_{|s|=s_0} ds\; w(s)\Pi(s) \;\sim\; \delta_W^{\text{tree}} + \delta_W^{(0)}(s_0) + \sum_{d\geq 4}\delta_{W,V/A}^{(d)}(s_0) + \delta_{W,V/A}^{(\text{DV})}(s_0)$$

Perturbative contribution:
(Coefficients known up to $\mathcal{O}(\alpha_s^4)$)

$$\Pi^{(0)}(s) \sim \sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \sum_{k=0}^{n+1} c_{n,k}\ln^k\left(\frac{-s}{\mu^2}\right)$$

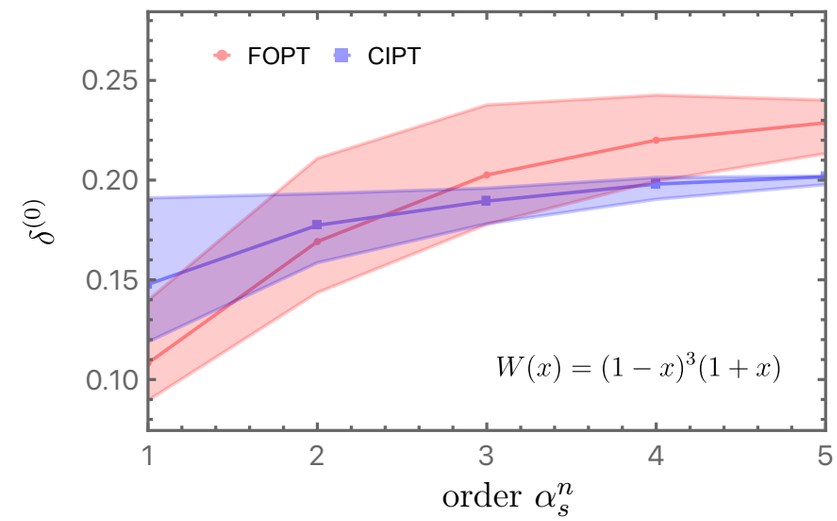
$c_{0,1} = c_{1,1} = 1, \quad c_{2,1} = 1.640$
$c_{3,1} = 6.371$ Gorishnii, Kataev, Larin, '91 Surguladze&Samuel, '91
$c_{4,1} = 49.076$ Baikov, Chetyrkin, Kühn, '08
$c_{5,1} = 280 \pm 140$ Beneke, Jamin, '08 Boito, Masjuan, Oliani, '18 Caprini, '19

OPE contribution: $\Pi^{\text{OPE}} \sim \sum_{d=4}^{\infty}\frac{\langle\mathcal{O}_d\rangle}{s^{d/2}}$ Shifman, Vainshtein, Zakharov, '78

FOPT and CIPT

Fixed-Order Pt. Th. (FOPT): Fixed ren. scale, expansion in powers of coupling ($x = s/s_0$)

$$\delta_W^{(0),\mathrm{FO}}(s_0) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n} k\, c_{n,k}\, \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W(x)\, \ln^k(-x)$$



Contour-Improved Pt. Th. (CIPT): Running ren. scale, no expansion parameter after contour integration

$$\delta_W^{(0),\mathrm{CI}}(s_0) = \sum_{n=1}^{\infty} c_{n,1}\, \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W(x) \left(\frac{\alpha_s(-x s_0)}{\pi}\right)^n$$

One of the dominant sources of theoretical uncertainty: prescription for setting ren. scale, FOPT/CIPT predictions not compatible

CIPT consistently higher than FOPT, independent of treatment of NP corrections

	Pich et al. '16	Boito et al. '21
FOPT $[\alpha_s(m_\tau^2)]$	0.317(14)	0.308(7)
CIPT $[\alpha_s(m_\tau^2)]$	0.336(17)	0.324(9)

Motivation for a new scheme

Hoang, Regner '20 '21

Conclusions from Hoang, Regner:

1. CIPT/FOPT have different IR sensitivity ⇒ NP corrections differ

2. CIPT inconsistent with standard OPE in presence of IR renormalons in Π (or D)

   ↳ Asymptotic separation ≡ Discrepancy of CIPT to FOPT at asymptotically large orders

   $$\sim \left(\frac{\Lambda_{\rm QCD}^d}{s_0^{d/2}} \right) \text{ for an } \mathcal{O}\left(\Lambda_{\rm QCD}^d\right) \text{ IR renormalon}$$

3. Effect dominated by GC renormalon

➡ Starting Point of this work:

CIPT can be (largely) cured when the GC renormalon is subtracted

Analogy: quark pole mass → short-distance mass (MSR, PS, RS,1S)

Renormalon-Free Gluon Condensate Scheme

MBR, Boito, Hoang, Jamin, JHEP 07 (2022) 016

Original OPE approach ('$\overline{\text{MS}}$' OPE):

Barred quantities ≡ C-scheme coupling (Boito, Jamin, Miravtllas, '16)

$$\frac{1}{4\pi^2}\left[1+D(s)\right] \equiv -s\,\frac{\mathrm{d}}{\mathrm{d}s}\,\Pi(s)$$

$$\langle\bar{G}^2\rangle = \langle\frac{\alpha_s}{\pi}G^2\rangle$$

$$D(s) \sim \sum_\ell c_\ell\,\bar{a}^\ell(s) \quad + \#\frac{\langle\bar{G}^2\rangle}{s^2} \quad + \dots$$

$$\bar{a}_Q \equiv \frac{\beta_0\,\bar{\alpha}_s(Q^2)}{4\pi}$$

GC renormalon contribution

Cancellation of renormalon numerically by renormalon behavior of GC

GC renormalon in QCD

$$B[\hat{D}](u) \sim \frac{N_g}{(2-u)^{1+4\hat{b}_1}} \xrightarrow{\text{GC renormalon contr. to pert. series}} c_\ell \sim r_\ell^{(4,0)}\bar{a}_Q^\ell \sim N_g\frac{(\ell-1)!}{2^\ell}\bar{a}_Q^\ell$$

$$\hat{b}_1 \equiv \frac{\beta_1}{2\beta_0^2}$$

$$r_\ell^{(4,0)} = \left(\frac{1}{2}\right)^{\ell+4\hat{b}_1}\frac{\Gamma(\ell+4\hat{b}_1)}{\Gamma(1+4\hat{b}_1)}$$

Renormalon-Free Gluon Condensate Scheme

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Define IR-subtracted scheme for GC

In analogy to RS scheme for quark mass advocated in Pineda '01

$$\langle \bar{G}^2 \rangle \equiv \langle G^2 \rangle (R^2) - R^4 \sum_{\ell=1} N_g r_\ell^{(4,0)} \bar{a}_R^\ell \qquad \bar{a}_R = \frac{\alpha_s(R^2)\beta_0}{4\pi}$$

IR factorization scale

Leads to modified GC contribution $\longrightarrow$ $B[\hat{D}^{\rm RF}(-Q^2)](u) \sim N_g \int_0^\infty du \left[\frac{e^{-\frac{u}{\bar{a}_Q}}}{(2-u)^{1+4\hat{b}_1}} - \frac{R^4}{Q^4} \frac{e^{-\frac{u}{\bar{a}_R}}}{(2-u)^{1+4\hat{b}_1}} \right]$

Ambiguity free!

More convenient to work with scale invariant GC

$$\langle \bar{G}^2 \rangle \equiv \langle G^2 \rangle^{\rm RF} - R^4 \sum_{\ell=1}^{n} N_g \, r_\ell^{(4,0)} \, \bar{a}_R^\ell + N_g \, \bar{c}_0(R^2) \qquad \bar{c}_0(R^2) \equiv R^4 \ \mathrm{PV} \int_0^\infty \frac{\mathrm{d}u \ e^{-\frac{u}{\bar{a}_R}}}{(2-u)^{1+4\hat{b}_1}}$$

R-dependence cancels

Treated as 'tree level' contribution

Expand consistently in $\alpha_s(\mu^2)$ at a common renormalization scale μ

Borel model study

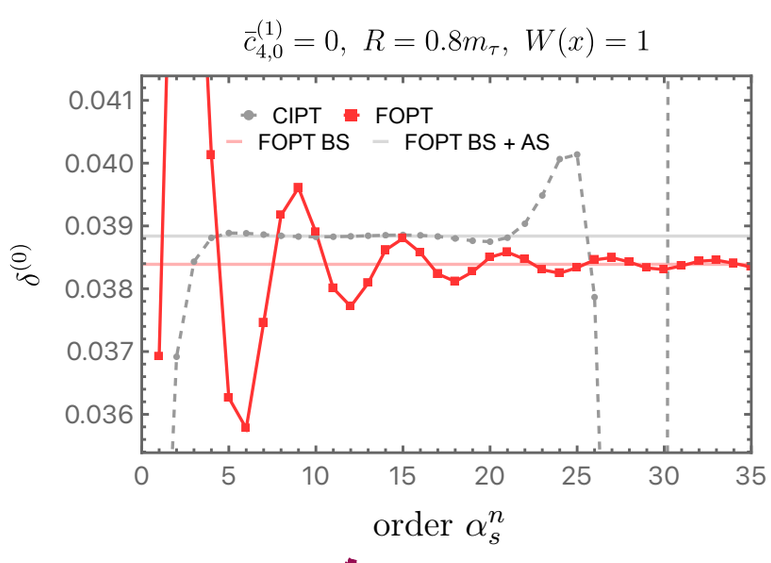
MBR, Boito, Hoang, Jamin, JHEP 07 (2022) 016

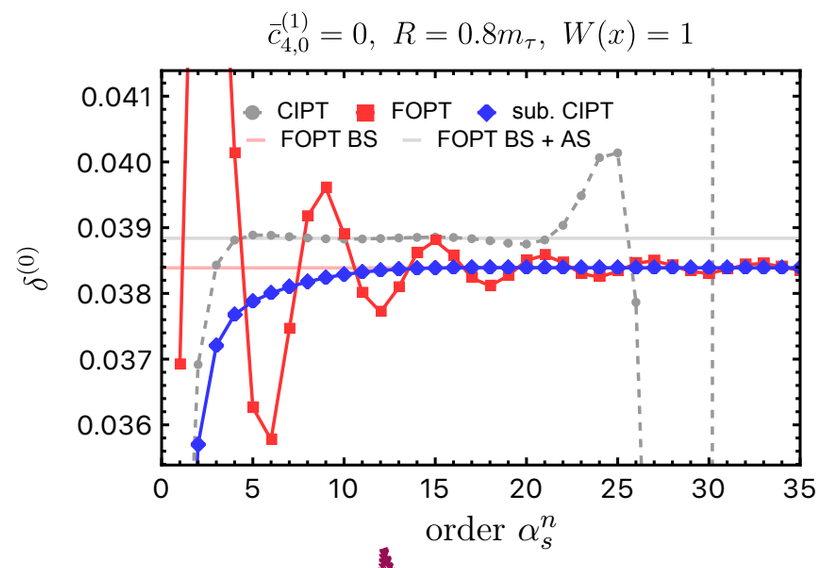
Take pure GC renormalon model

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}}$$

$$\longrightarrow \quad \hat{D}^{\text{model}}(s) = \sum_{\ell=1}^{\infty} r_\ell^{(4,0)} \bar{a}^\ell(-s) \qquad N_g = 1$$

Use GC suppressed moment





CIPT not consistent with standard OPE

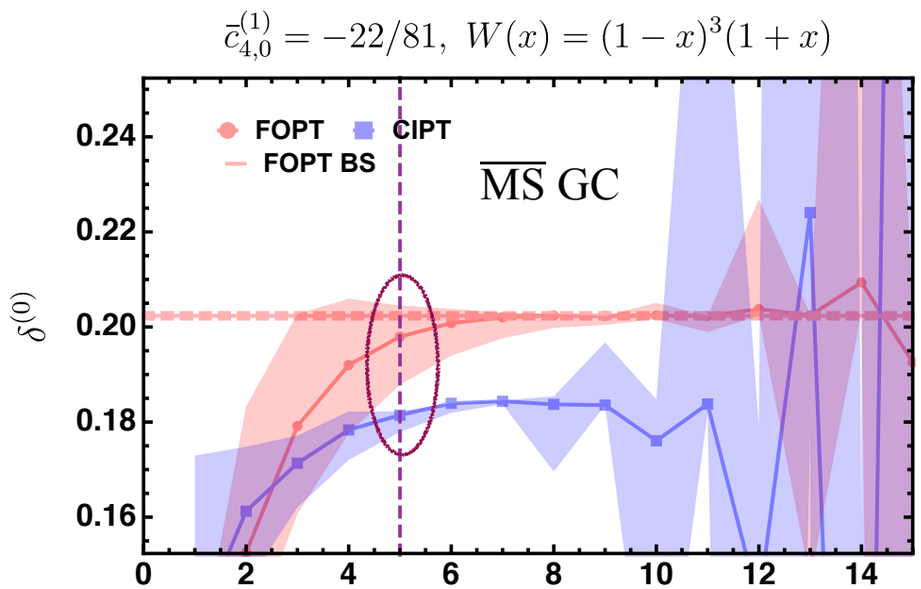
CIPT cured (+ better convergence than FOPT)

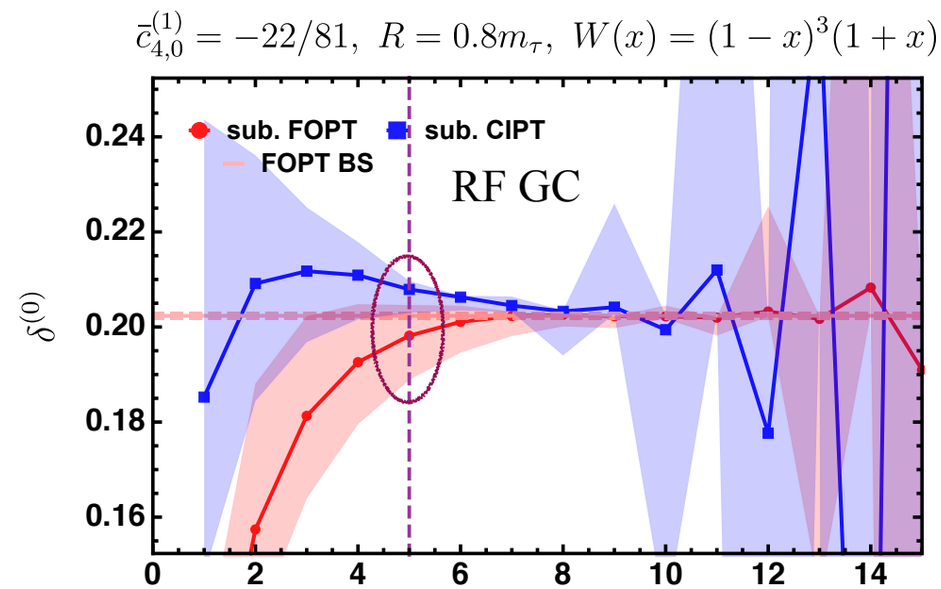
Renormalon-Free Gluon Condensate Scheme

MBR, Boito, Hoang, Jamin, JHEP 07 (2022) 016

Use Multi-Renormalon model (MRM) ($N_g = 0.64$)

Beneke, Jamin '08





<u>Observations:</u>

- Discrepancy between CIPT/FOPT removed in RF GC scheme
- CIPT/FOPT consistent within ren. scale variation errors already at $\mathcal{O}(\alpha_s^5)$

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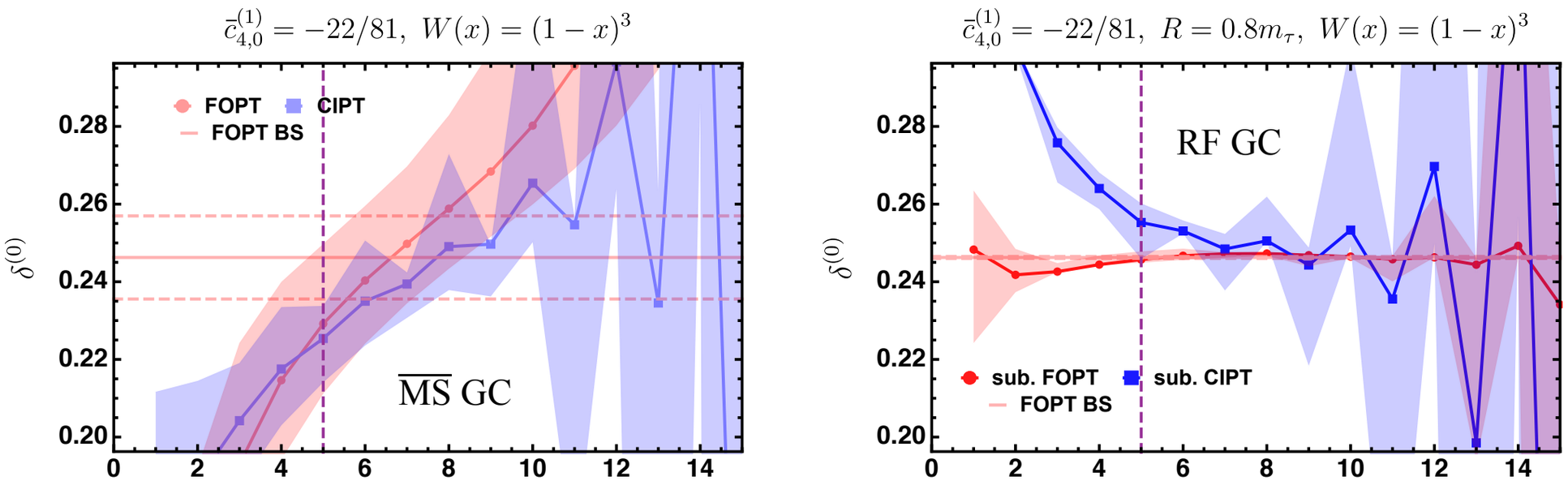

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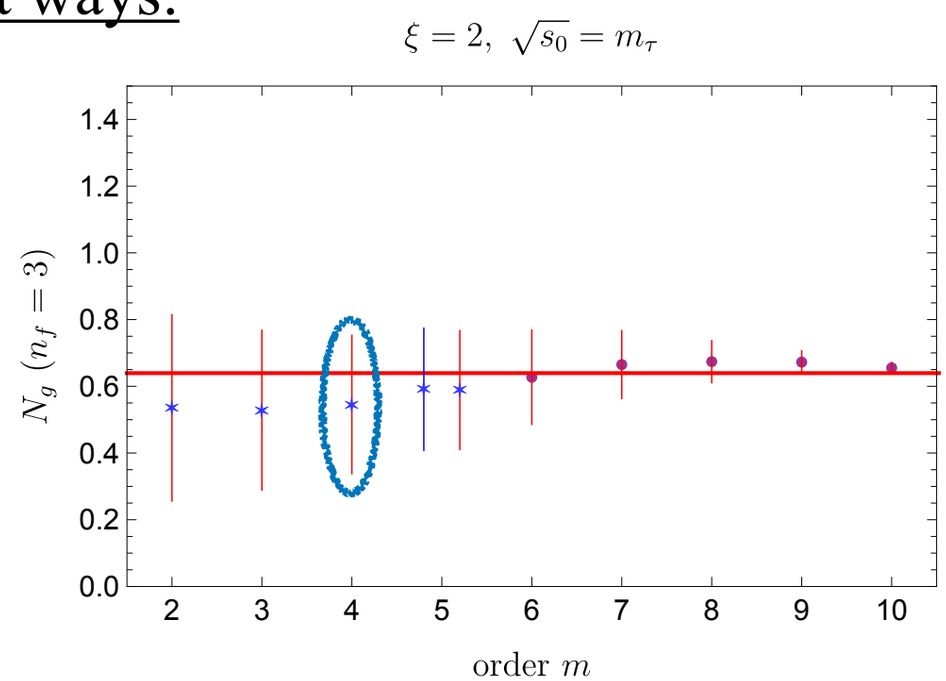
Determination of the norm of the GC

MBR, Boito, Hoang, Jamin, 2207.01116

Determine norm in three different ways:

- Varying the Borel Model
  Beneke, Jamin '08
- Use conformal mapping methods
  Lee '12 Caprini, Fischer '09
- Optimal subtraction approach



- Converges to correct N_g for MRM
- Use known $\mathcal{O}(\alpha_s^4)$ result
- Final result: $N_g = 0.57 \pm 0.23$

Optimal subtraction approach:

- Construct a suitable χ^2 - type function
- Improvements from RF GC scheme provide quantitative measure for χ^2 - function

$$\chi^2_m(N_g) = \chi^2_{m,\mathrm{GCS}}(N_g) + \chi^2_{m,\mathrm{GCE}}(N_g)$$

Small discrepancy for five GC supressed moments

Good convergence of five GC enhanced moments

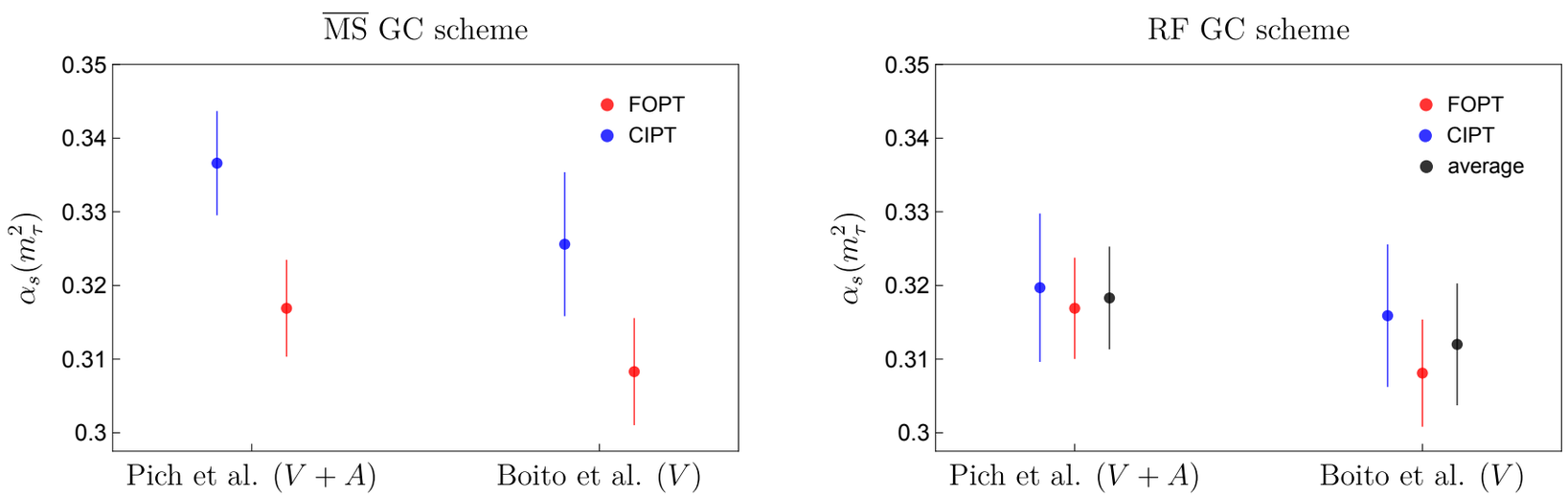
The RF GC scheme in strong coupling determinations

MBR, Boito, Hoang, Jamin, 2207.01116

Strategy:

Repeat in detail two state-of-the-art determination methods in the RF GC scheme:

- Truncated OPE approach Pich, Rodriguez-Sanchez '16
- Duality violation model approach Boito, Golterman, Maltman, Peris, Rodrigues, Schaaf '21



Observations:

- In contrast to original $\overline{\text{MS}}$ GC scheme determinations we obtain consistent results in the RG GC scheme
- CIPT becomes consistent with FOPT
- Additional errors from IR factorization scale variation as well as uncertainty related to norm of GC have minor impact on error of the strong coupling

Summary:

- Introduction of RF GC scheme
- In RF GC scheme, size of discrepancy between FOPT and CIPT strongly reduced
- In original '$\overline{\text{MS}}$' GC scheme, CIPT not compatible with standard OPE
- In RF GC scheme, inconsistency of CIPT w.r.t. standard OPE largely 'cured'
- Strong coupling extractions based on FOPT/CIPT in RF GC scheme lead to compatible results in contrast to using original $\overline{\text{MS}}$ GC scheme

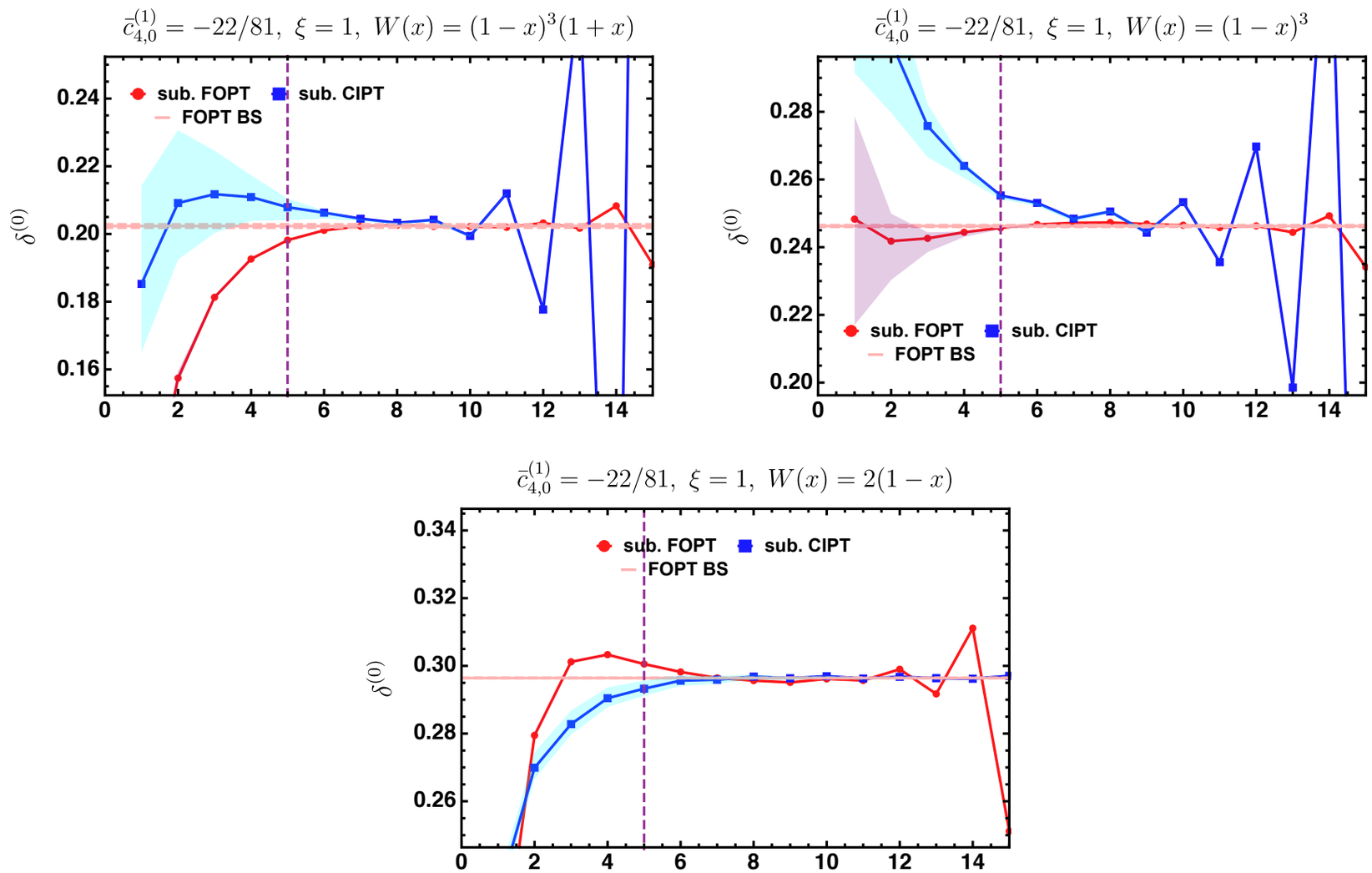
Back-up

IR factorization scale variation in the RF GC scheme

MBR, Boito, Hoang, Jamin, JHEP 07 (2022) 016

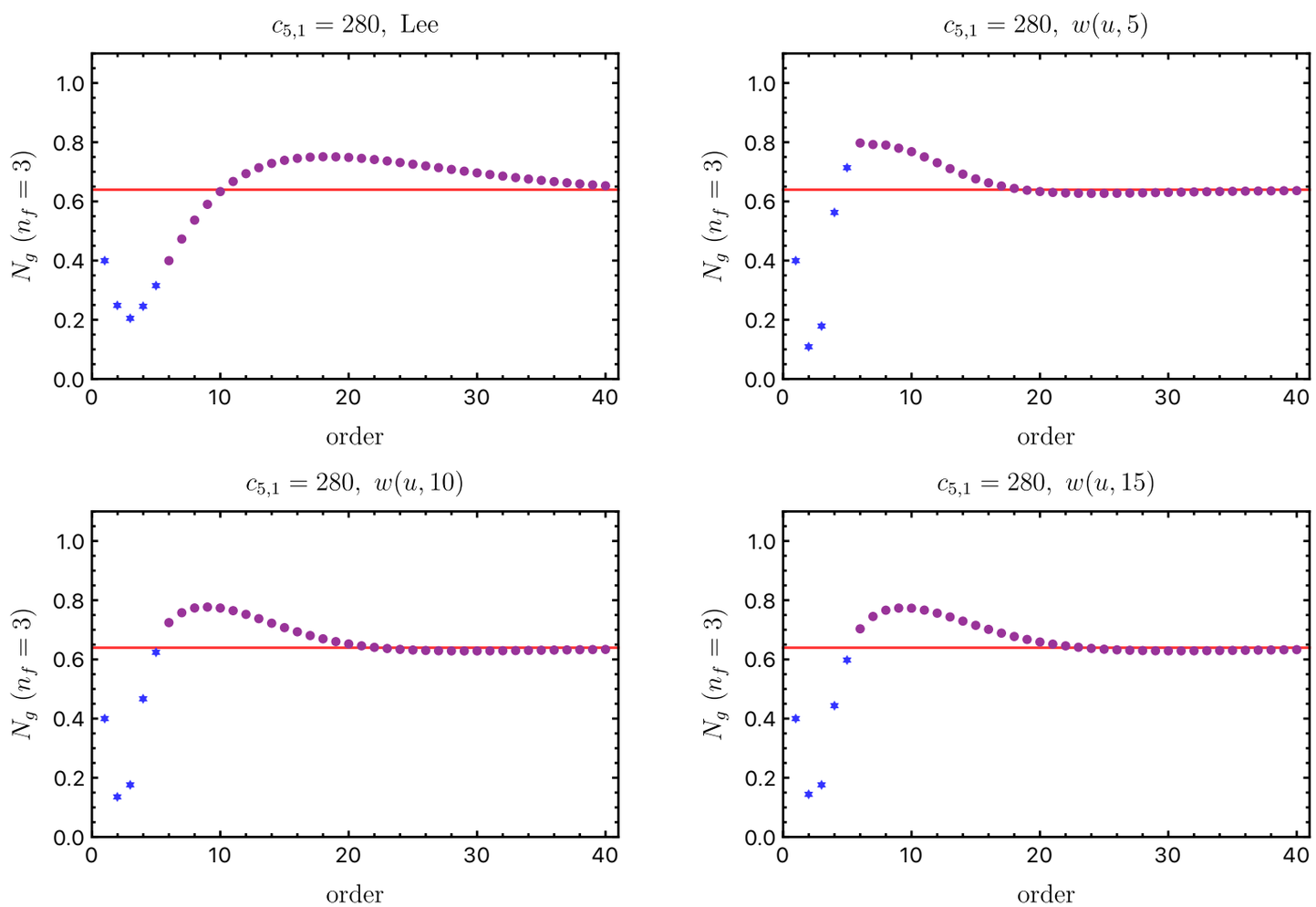
Use MRM ($N_g = 0.64$)

Beneke, Jamin '08



Results for GC norm from Conformal mapping approach

MBR, Boito, Hoang, Jamin, 2207.01116



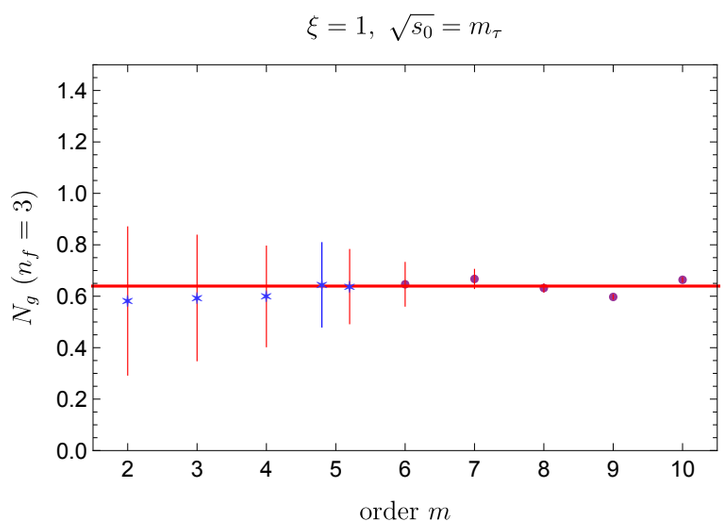
Conformal mapping approach:

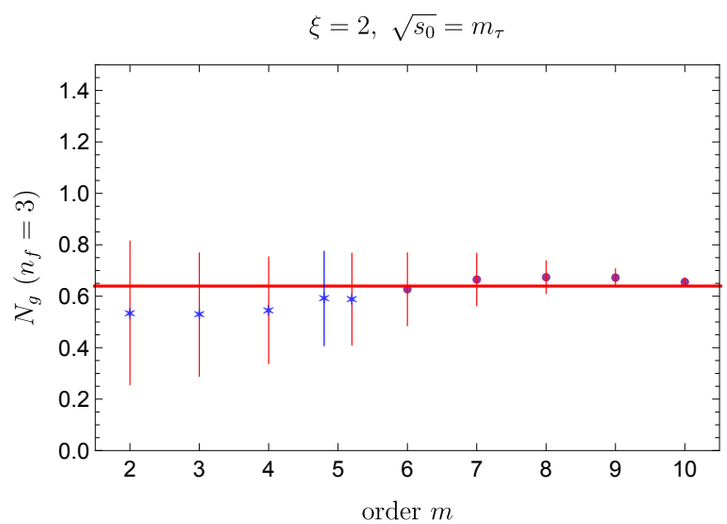
	$w(u,5)$	$w(u,10)$	$w(u,15)$
$\mathcal{O}(\alpha_s^4)$	0.57	0.47	0.45
$\mathcal{O}(\alpha_s^5)$	0.72 ± 0.24	0.63 ± 0.17	0.60 ± 0.15

$$w(u,p) = \frac{\sqrt{1+u} - \sqrt{1-\frac{u}{p}}}{\sqrt{1+u} + \sqrt{1-\frac{u}{p}}}$$

Results for GC norm from Optimal subtraction approach

MBR, Boito, Hoang, Jamin, 2207.01116





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Optimal subtraction approach:

	$\sqrt{s_0} = m_\tau$	$\sqrt{s_0} = 3$ GeV
$m = 4\ (\xi = 1)$	0.60 ± 0.20	0.51 ± 0.17
$m = 4\ (\xi = 2)$	0.54 ± 0.21	0.55 ± 0.11
$m = 5\ (\xi = 1)$	0.64 ± 0.16	0.50 ± 0.19
$m = 5\ (\xi = 2)$	0.59 ± 0.18	0.52 ± 0.15

➡ Envelope of $\mathcal{O}(\alpha_s^4)$ results our final result

$$N_g = 0.57 \pm 0.23$$

Construction of the Optimal subtraction approach

MBR, Boito, Hoang, Jamin, 2207.01116

Perturbative series in RF GC scheme

$$\delta^{(0),\mathrm{FO/CI}}_{w,m}(N_g, s_0; \alpha_s(s_0)) \;=\; \sum_{n=0}^{m} r^{\mathrm{FO/CI}}_{w,n}(N_g, R^2, \xi; s_0, \alpha_s(s_0))$$

Consider five GC suppressing (GCS) and enhancing (GCE) moments separately

$$\chi^2_{m,\mathrm{GCS}}(N_g) = \sum_i \left(\delta^{(0),\mathrm{CI}}_{w_i,m}(N_g) - \delta^{(0),\mathrm{FO}}_{w_i,m}(N_g) \right)^2$$

$$\chi^2_{m,\mathrm{GCE}}(N_g) = \sum_i \left(r^{\mathrm{FO}}_{w_i,m}(N_g) - r^{\mathrm{FO}}_{w_i,m-1}(N_g) \right)^2$$

→ Vary IR factorization scale between $0.7\sqrt{s_0} \leq R \leq \sqrt{s_0}$ to obtain error estimate for norm of GC

Moments:

$$w_1(x) = 1 - 3x^2 + 2x^3$$
$$w_2(x) = 1 - 4x^3 + 3x^4$$
$$w_3(x) = 1 - 5x^4 + 4x^5$$
$$w_4(x) = 1 - 6x^5 + 5x^6$$
$$w_5(x) = 1 - 7x^6 + 6x^7$$

GCS

$$w_6(x) = \frac{3}{2}(1-x)^2 = \frac{3}{2} - 3x + \frac{3x^2}{2}$$
$$w_7(x) = (1-x)^2\left(\frac{13}{12} + \frac{5x}{3}\right) = \frac{13}{12} - \frac{x}{2} - \frac{9x^2}{4} + \frac{5x^3}{3}$$
$$w_8(x) = \frac{1}{2}(1-x)^2(5-8x) = \frac{5}{2} - 9x + \frac{21x^2}{2} - 4x^3$$
$$w_9(x) = (1-x)^2\left(\frac{3}{2} + x - 3x^2 + x^3\right) = \frac{3}{2} - 2x - \frac{7x^2}{2} + 8x^3 - 5x^4 + x^5$$
$$w_{10}(x) = (1-x)\left(1 - \frac{x^3}{2} + \frac{3x^4}{4}\right) = 1 - x - \frac{x^3}{2} + \frac{5x^4}{4} - \frac{3x^5}{4}$$

GCE

Final χ^2- type function: $\chi^2_m(N_g) = \chi^2_{m,\mathrm{GCS}}(N_g) + \chi^2_{m,\mathrm{GCE}}(N_g)$

Sum GCS and GCE χ^2 - type functions and average over results → final value at each m