# Nucleon scalar and tensor couplings from lattice QCD at the physical points

(For detail, see arXiv:2207.11914)

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In collaboration with: Y. Aoki, K.-I. Ishikawa, Y. Kuramashi, S. Sasaki, E. Shintani and T. Yamazaki for PACS Collaboration

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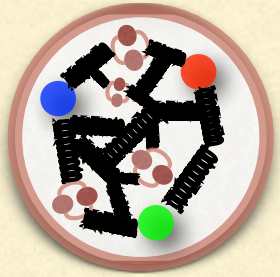
# Introduction

- Many body problem with QCD
- Nucleon structure study
- Parton Distributions
- The conventional studies and our works

# Nucleon has STRUCTURE

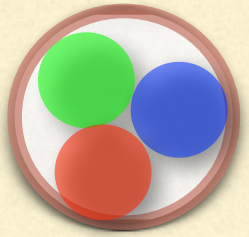
High Energy Nucleon

QUARK & GLUON pic.



? Spin crisis
? Single spin asymmetry
? Origin of mass

$\Lambda_{\mathrm{QCD}} \sim O(10^2)$ (MeV)



O Magnetic moment
O Mass gap
O Chiral SSB

CONSTITUENT QUARK pic.

Low Energy Nucleon

Is the properties of Nucleon interpretable in terms of the dynamics of quark & gluon?

Perturbation dose NOT work

Non perturbative analysis(*ab initio*) = lattice QCD (LQCD)

* "Energy" corresponds to the resolution when we see nucleon($q \sim 100$(GeV) $\leftrightarrow$ resolution $\sim 0.002$ (fm)).

# Parton Distributions

GTMD $W(x, \overrightarrow{k}_T, \Delta) \xrightarrow[\Delta^+ \to 0]{} \text{Wigner } W(x, \overrightarrow{k}_T, \vec{r}_T)$

$\Delta = P_{\text{ini.}} - P_{\text{fin.}}$

Quantum phase-space distributions

# Parton Distributions

Transverse Momentum Dependent Parton Distribution

Generalized Parton Distribution

3D structure

$\Delta \to 0$

$\int d\vec{r}_T$

$\int d^2k_T$

GTMD $W(x, \vec{k}_T, \Delta) \xrightarrow[\Delta^+ \to 0]{} \text{Wigner } W(x, \vec{k}_T, \vec{r}_T)$

$\int dx \int d^2k_T$

$\Delta = P_{\text{ini.}} - P_{\text{fin.}}$

ongoing projects

$\int d^2k_T, \Delta \to 0$

1D structure

## Form Factor

- Elastic scattering

→ Nucleon's SPATIAL dis.

Proton radius puzzle
Nucleon transversity
Quark EDM e.t.c.

## Parton Distribution Function

- Deep inelastic scattering

→ Partons' MOMENTUM/HEL-CITY dis. inside nucleon

Proton spin crisis, SSA,
Gluon saturation e.t.c.

# Parton Distributions

Transverse Momentum Dependent Parton Distribution

Generalized Parton Distribution

3D structure

$\Delta \to 0$

$\int d\vec{r}_T$

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GTMD $W(x, \vec{k}_T, \Delta) \xrightarrow[\Delta^+ \to 0]{} W(x, \vec{k}_T, \vec{r}_T)$ Wigner

$\Delta = P_{\text{ini.}} - P_{\text{fin.}}$

$\int dx \int d^2k_T$

1D structure

Form Factor

- Elastic scattering

→ Nucleon's SPATIAL dis.

Proton radius puzzle
Nucleon transversity
Quark EDM e.t.c.

Our works

**Paper**

- K.-I. Ishikawa et al., Phys. Rev. D**98** (2018) 074510 [1807.03974].
- E.Shintani et al., Phys. Rev. D**99** (2019) 014510 [1811.07292]; (Erratum; Phys. Rev. D**102** (2020) 019902.)
- N.Tsukamoto et al., PoS Lattice2019 (2020) 132 [1912.00654].
- K.-I. Ishikawa et al., arXiv:2107.07085 (2021). e.t.c.

**Talk**

- R.T. et al., "Nucleon tensor charges from lattice QCD", The 24th International Spin Symposium

# Parton Distributions

Transverse Momentum Dependent Parton Distribution

Generalized Parton Distribution

3D structure

$\Delta \to 0$

$\int d\vec{r}_T$

$\int d^2k_T$

GTMD $W(x, \vec{k}_T, \Delta)$ $\xrightarrow[\Delta^+ \to 0]{}$ Wigner $W(x, \vec{k}_T, \vec{r}_T)$

$\int dx \int d^2k_T$

$\Delta = P_{\text{ini.}} - P_{\text{fin.}}$

$\int d^2k_T, \Delta \to 0$

1D structure

Form Factor

Parton Distribution Function

this talk

- Elastic scattering

→ Nucleon's SPATIAL dis.

Proton radius puzzle

Nucleon transversity

Quark EDM e.t.c.

Isovector nucleon matrix element

$$g_i \equiv \langle p | \psi \Gamma_i \bar{\psi} | n \rangle \quad \text{(NME)}$$

| $\Gamma_i$ | $\gamma_\mu\gamma_5$ | $\sigma_{\mu\nu}$ | $1$ |
|---|---|---|---|
| $g_i$ | Axial | Tensor | Scalar |

# Neutron and Beyond S.M.

- Neutron EDM

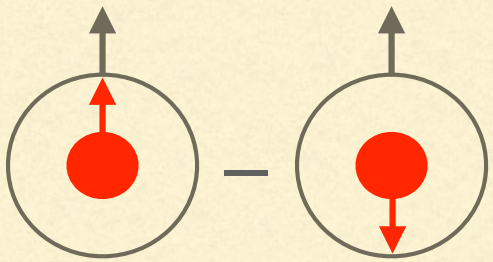
→ estimate CP-violation induced from quark chromo-EDM $d_q$

exp. bounds

Lat. calculation

$$d_n = \delta d d_u + \delta u d_d + \delta s d_s$$

$$\delta q(Q^2) = \int_0^1 dx[h_1^q(x,Q^2) - h_1^{\bar{q}}(x,Q^2)] : \; h_1^q =$$



Constraints on CP-violation quark chromo-EDM

- Neutron decay

→ $\beta$ decay NOT with V-A channel = Beyond S.M.

e.g.) Scalar, Tensor channel coupling

# Nucleon structure with lattice QCD

After 2011, lattice can approach Parton Distributions directly.
However, can lattice overcome experimental precision/accuracy?

→ ✓BENCHMARK calculations, indirect one, are also needed

| | Matrix elements | Feature | Experiments/Remark |
|---|---|---|---|
| ✓ | $g_A$ | Nucleon axial charge | $g_A^{\text{exp.}} = 1.2756(13)$ |
| | $g_S$ | Direct Dark Matter detection $\langle N \| \psi 1 \bar{\psi} \| N \rangle$ | Both isoscalar and isovector are needed for practical use |
| | $g_T$ | 0th moment of Colins func. $\langle N \| \psi \sigma_{\mu\nu} \bar{\psi} \| N \rangle$ | |
| ✓ | $\langle x \rangle_{u-d}$ | 1st moment of unp. PDF | $\langle x \rangle_{u-d}^{\text{PDF4LHC}} = 0.155(5)$ |
| ✓ | $\langle x \rangle_{\Delta u - \Delta d}$ | 1st moment of pol. PDF | $\langle x \rangle_{\Delta u - \Delta d}^{\text{BENCHMARK}} = 0.199(16)$ |
| | $\langle x \rangle_{\delta u - \delta d}$ | 1st moment of tra. PDF | ——— |

# Nucleon structure with lattice QCD

After 2011, Parton Distributions can be directly computed in LQCD
Whether can LQCD overcome experimental precision/accuracy?

→ ✓BENCHMARK calculations, indirect one, are also needed

| | Matrix elements | Feature | Experiments/Remark |
|---|---|---|---|
| ✓ | $g_A$ | Nucleon axial charge | $g_A^{\text{exp.}} = 1.2756(13)$ |
| this talk | $g_S$ | Direct Dark Matter detection $\langle N \| \psi 1 \bar{\psi} \| N \rangle$ | Both isoscalar and isovector are needed for practical use |
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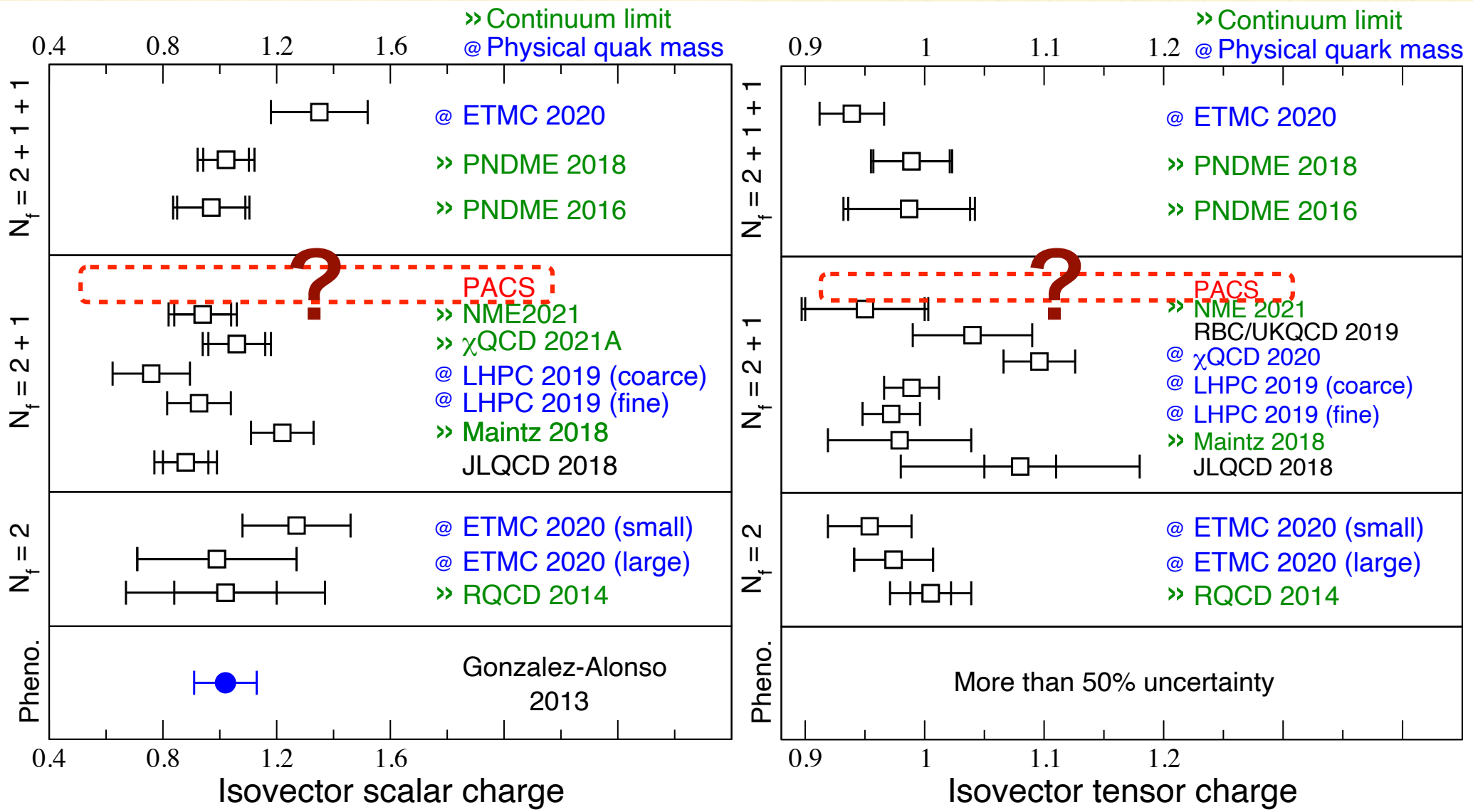
# Conventional studies -isovector




High-precision & High-accuracy = Purpose of PACS(this work)

# Lattice QCD & Assessment of errors

- Monte Carlo simulation (statistical error)
- Major systematic uncertainties by lattice set-up
- Systematic uncertainties induced by calculation methods

# Calculation strategy

Our targets :

- Non-perturbative information of nucleon

  → Calculate NMEs in Lattice QCD

- Observables need to be renormalized

  → The renormalization constants are additionally required

Therefore:

(Renormalized value)

= (Bare matrix element) × (Renormalization constant)

→ Evaluate both the bare matrix elements and the renorm -alization constants with high accuracy in Lattice QCD

High accuracy in Lattice QCD(*ab initio* cal.)?

$\overline{\mathrm{MS}}$ (Modified Minimal subtraction) scheme at 2 GeV for the subtraction point is widely used now.

# Lattice QCD and its accuracy

Path integration of QCD = High-dimensional integrals

$$\langle O \rangle = \frac{1}{Z} \int \mathscr{D}[U] \mathscr{D}[\overline{\psi}] \mathscr{D}[\psi] \, O[U, \overline{\psi}, \psi] \, \mathrm{e}^{-J_{\mathrm{QCD}}[U, \overline{\psi}, \psi]}$$

→ Estimate stochastically = Monte Carlo integration
(Importance sampling)

High accuracy in Lattice QCD means

1. Higher statistics for statistical improvement
   → All-mode-averaging[1]
2. Fewer systematic uncertainties
   → Patly[2] eliminated by lattice set-ups, but NOT enough

Needs assessment of the residual systematic uncertainties

[1] T. Blum et al., Phys.Rev. D **88,** 094503(2013)、[2] One stem from finite spatial volume and heavy pion mass

# Residual systematic uncertainties

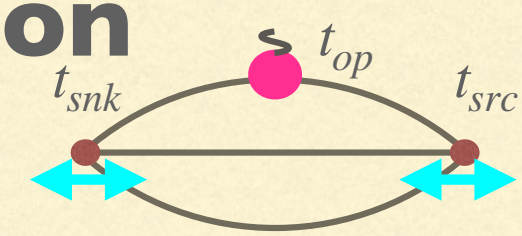
① : (Bare matrix element) × ② : (Renormalization constant)

Both have systematic uncertainties, and we mainly focus on

| Parts | Source of systematic uncertainty | Origin |
|---|---|---|
| ① | Excited state contamination | • Nucleon's excited states<br>e.g. $\langle N(t)N(0)^\dagger\rangle = \sum_i a_i \, \mathrm{e}^{-E_i t}$ |
| ② | Perturbative truncation<br>Non-perturbative effects ←<br>Fitting functions/range | • Chiral S.S.B<br>• Gluon condensation<br>e.g. $Z_O^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) \supset \frac{m_{val}^2}{p^2}, \frac{\langle q\bar{q}\rangle^2}{p^6}, \frac{\langle A_\mu\rangle^2}{p^2}$ |

Problem : How can we reduce systematic uncertainties?

# ① Excited state contamination



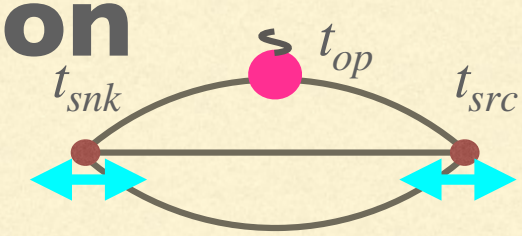

$t_{sep} = |t_{snk} - t_{src}|$

Nucleon matrix elements obtained from the ratio of 3pt. function to 2pt. function

$$\frac{\langle N(t_{snk})O(t_{op})N(t_{src})^\dagger\rangle}{\langle N(t_{snk})N(t_{src})^\dagger\rangle} = \frac{\sum_{ij}\langle 0|N(0)|i\rangle\langle i|O(0)|j\rangle\langle j|N^\dagger|0\rangle e^{-E_i t_{sep}}}{\sum_i |\langle 0|N(0)|i\rangle|^2 e^{-E_i t_{sep}}}$$

$$\xrightarrow[t_{sep} \gg t_{op} \gg 0]{} \langle N|O(0)|N\rangle$$

# ① Excited state contamination

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$$\rightarrow \quad \langle N|O(0)|N\rangle + Ae^{-(E_1 - M_N)t_{sep}} + \cdots$$

Actually, ~~$t_{sep} \gg t_{op} \gg 0$~~

All excited states appearing in the ratio depend on $t_{sep}$

→ Calculate the ratio for several choice of $t_{sep}$ and examine $t_{sep}$ independence = confirm no excited states contamination

→ Average over $t_{sep}$ where the ground state saturation is achieved

Preliminary

# ② Non-perturbative effect

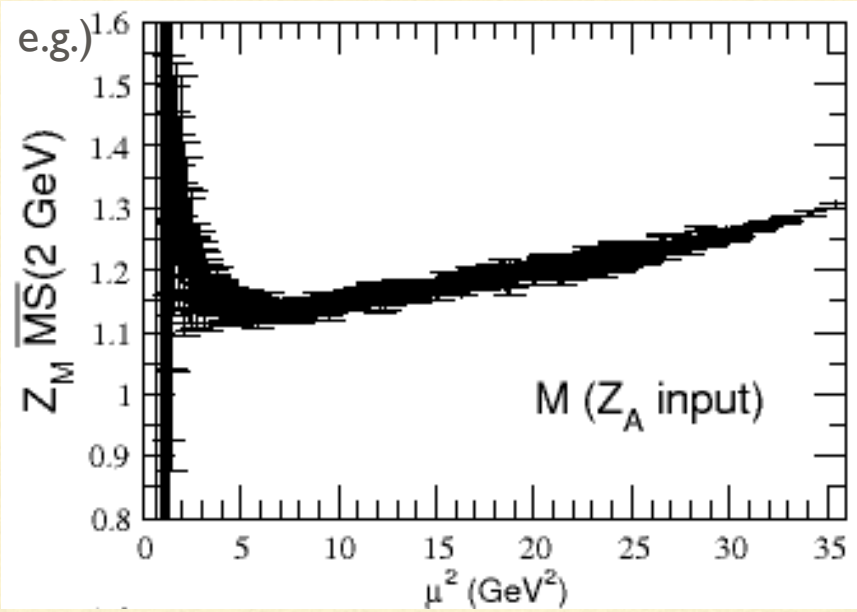
$$Z^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = \underbrace{\frac{Z^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV})}{Z^{\overline{\mathrm{MS}}}(\mu)} \cdot \frac{Z^{\overline{\mathrm{MS}}}(\mu)}{Z^{\mathrm{RI}}(\mu)}}_{\text{Perturbative matching}} \times \boxed{\underset{\text{Lattice}}{Z^{\mathrm{RI}}(\mu)}}$$

→ Ideally, $Z^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV})$ is independent of matching scale: $\mu$

However,
the residual dependence appears

↓

Remove such scale dependence by FIT ansatz and then extract $\mu$-independent renormalization constant




Skip how we can calculate the renormalization constants on lattice for saving time here.

# Numerical results

- Nucleon matrix elements
- Renormalization constants
- Renormalized axial, scalar and tensor charges

# Simulation details -PACS configuration

| | $128^4$ lattice | $64^4$ lattice |
|---|---|---|
| Lattice size | $128^4$ [1] | $64^4$ [2] |
| Lattice spacing | $\sim 0.086$ fm | |
| Pion mass | 135 MeV | 139 MeV[3] |
| Spatial vol. | $\sim (10.9\ \mathrm{fm})^3$ | $\sim (5.5\ \mathrm{fm})^3$ |

Eliminate two major systematic errors

~~Finite size effect~~

~~Chiral extrapolation~~

→ Highest precision of $g_A$[1]

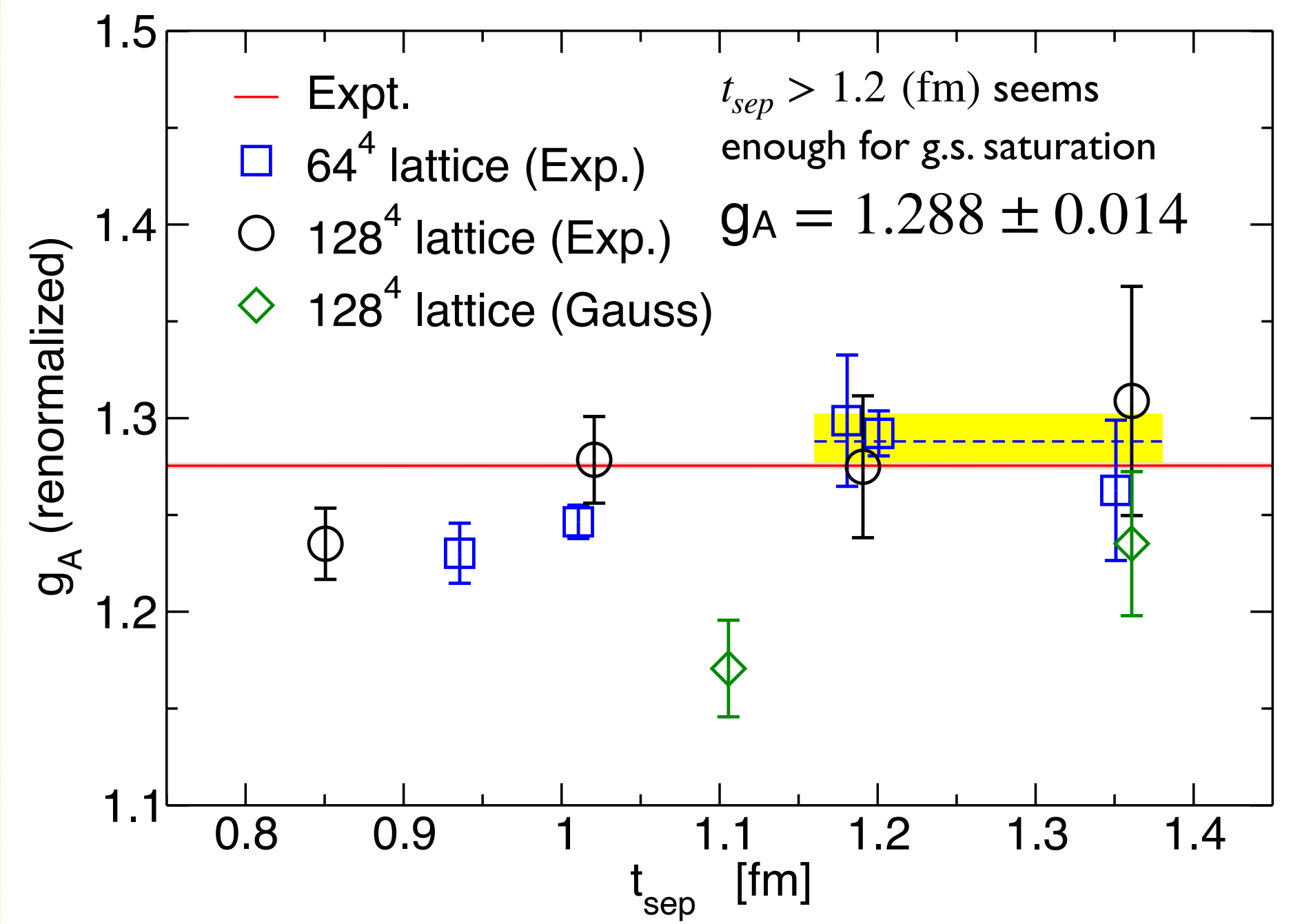
$$g_A^{128^4} = 1.273(24)_{\mathrm{sta.}}(5)_{\mathrm{sys.}}(9)_{\mathrm{ren.}}$$

$$g_A^{\mathrm{exp.}} = 1.2756(13)$$

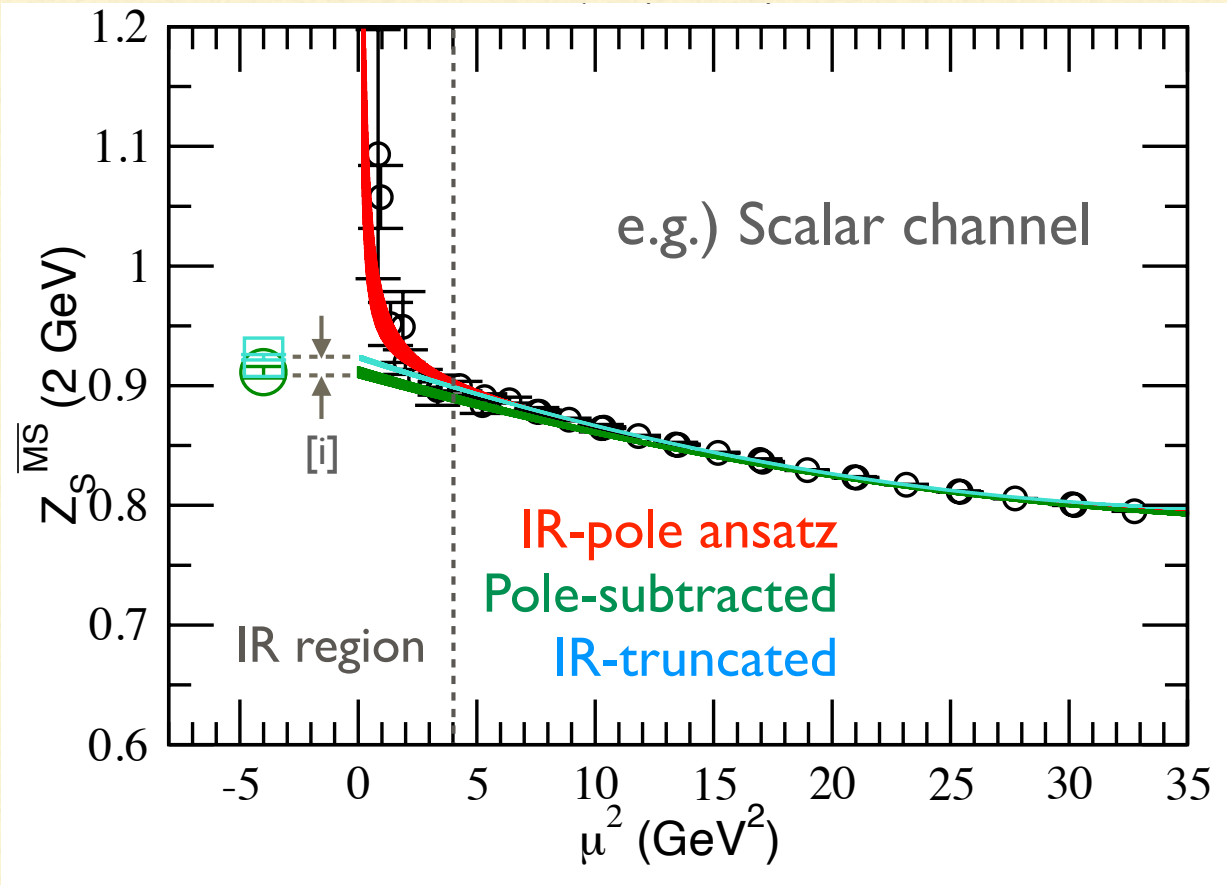
[1] E. Shintani et al., Phys. Rev. D **99**, 014510(2019) [2] K.-I. Ishikawa et al., Phys. Rev. D **99**, 014504(2019)
The stout-smeared $O(a)$ improved Wilson fermions and Iwasaki gauge action.
[3] Finite volume-size effect

Preliminary

# Nucleon axial charge $g_A$



# Renormalization and systematic error




Systematic uncertainties of

[i] Perturbative truncation

[ii] Non-perturbative effects

[iii] Fitting function/range

→ Three major systematic uncertainties are examined

High-precision & High-accuracy

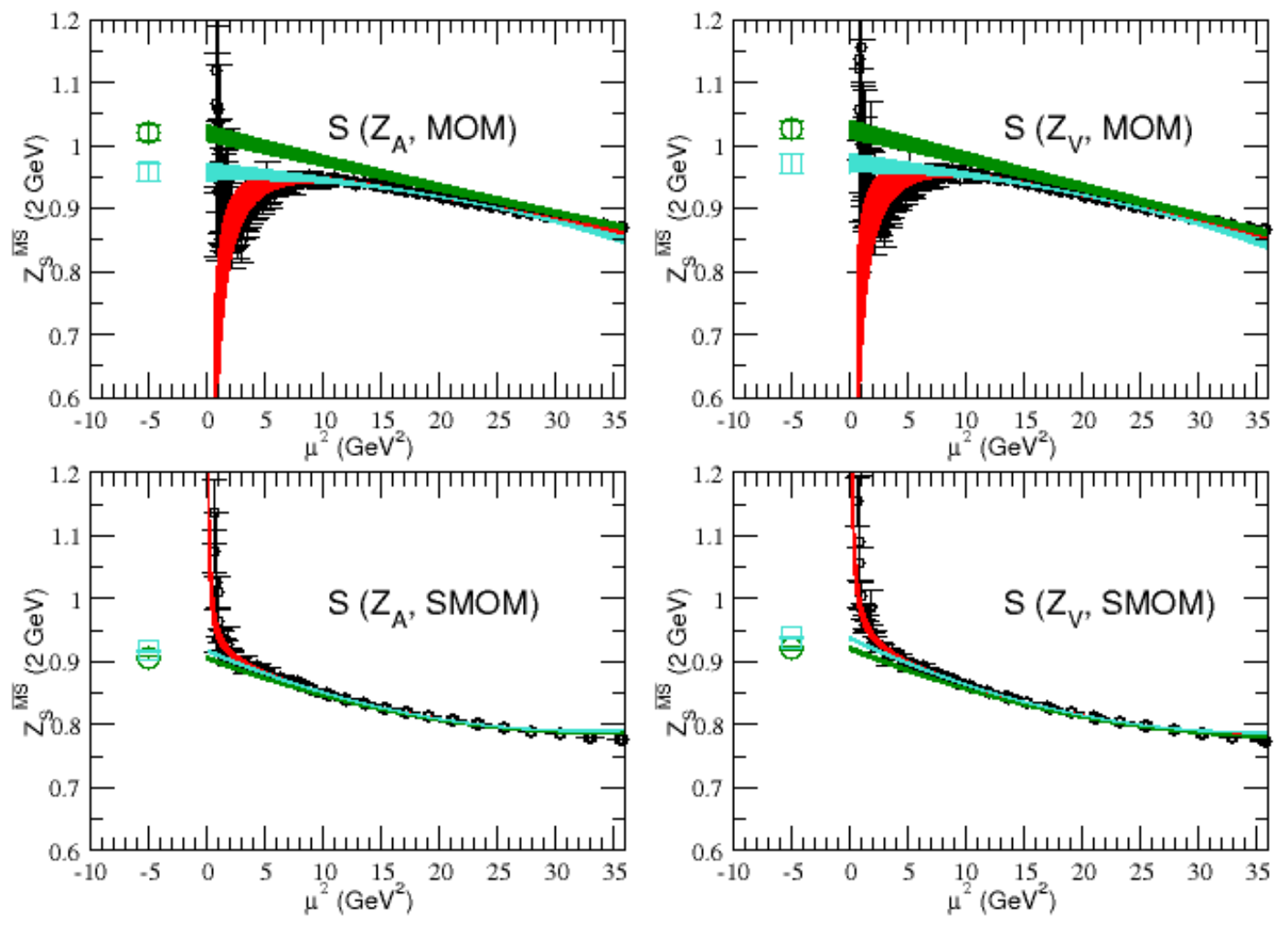
| Contribution to error | Sta. | [i] | [ii] | [iii] | Total |
|---|---|---|---|---|---|
| $Z_S = 0.910(3)_{\rm sta}(14)_{[i]}(13)_{[ii]}(14)_{[iii]}$ | 0.34% | 1.49% | 1.40% | 1.50% | 2.6% |
| $Z_T = 1.011(1)_{\rm sta}(19)_{[i]}(4)_{[ii]}(3)_{[iii]}$ | 0.12% | 1.84% | 0.36% | 0.32% | 1.9% |

Preliminary

# RI/MOM and RI/SMOM

**Scalar** operator = suffer from chiral symmetry breaking strongly

= $Z_S$ depends on how we treat IR strongly




Extract constant with

- Pole + quadratic including IR data
- Quadratic truncating IR data

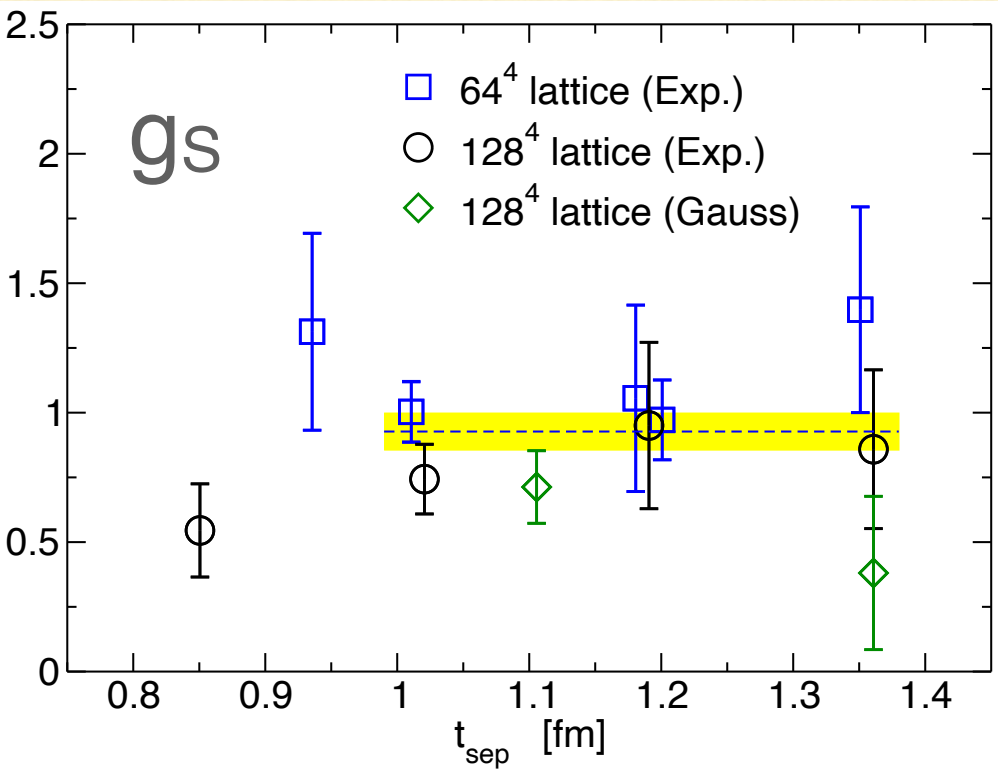
The discrepancy are

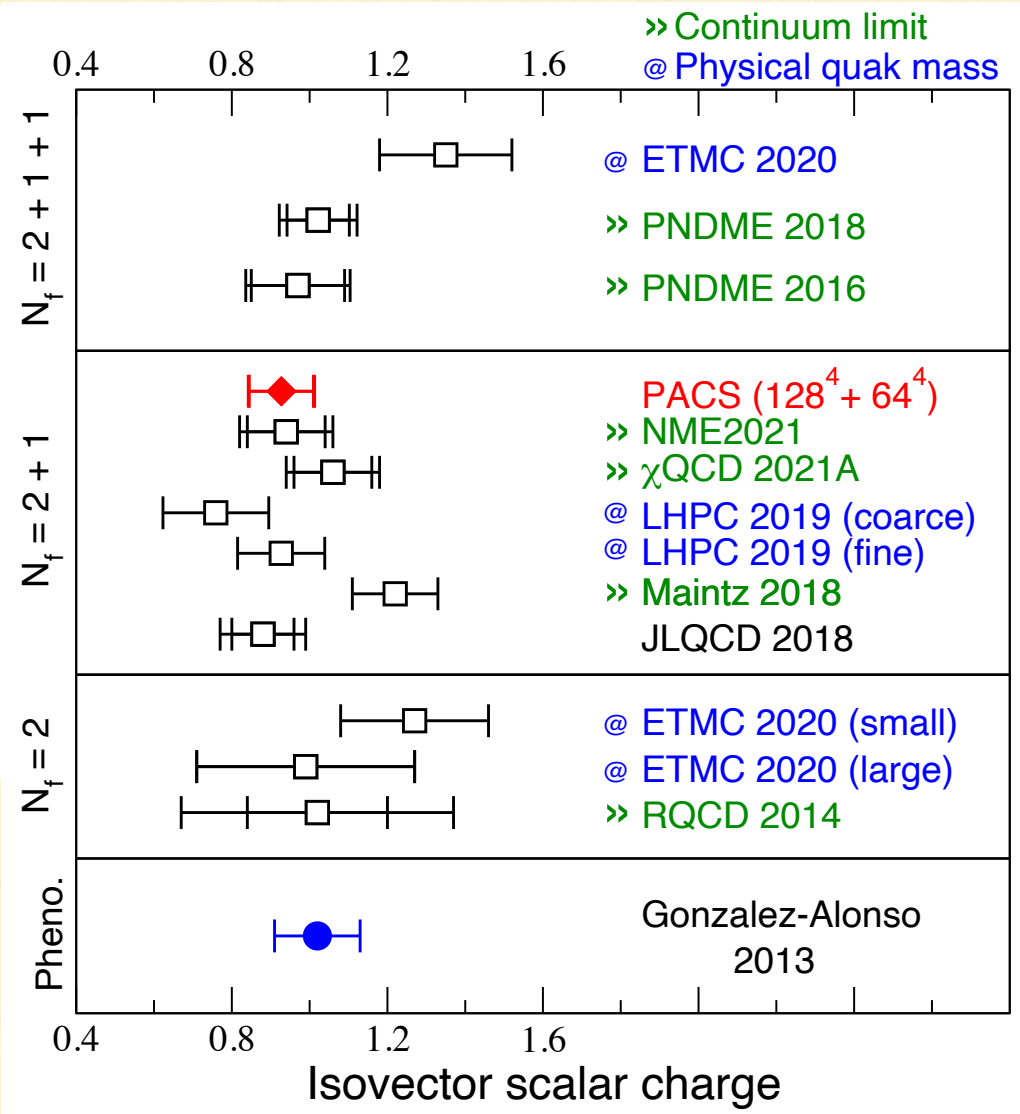
~6 % for MOM
~2 % for SMOM

Systematic error is better under control in SMOM scheme

RI/SMOM:C. Sturm, Y. Aoki, N. H. Christ, T. Izubuchi, C. T. C. Sachrajda and A, Soni, Phys. Rev. D **80,** 014501(2009).

# Renormalized scalar couplings



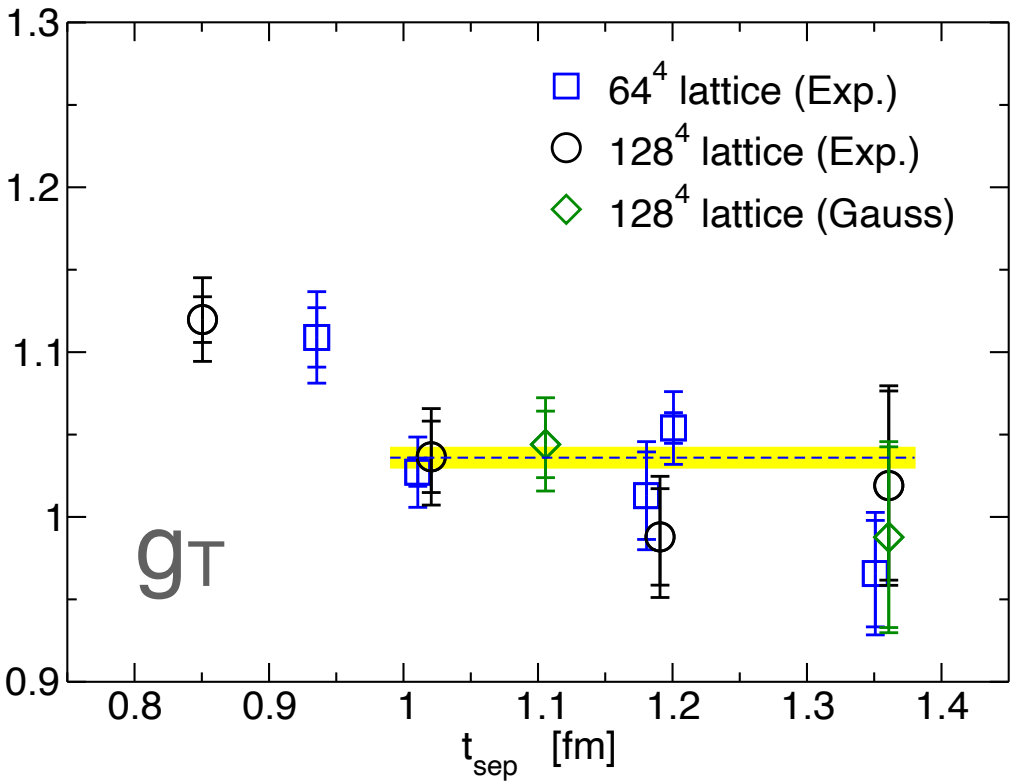



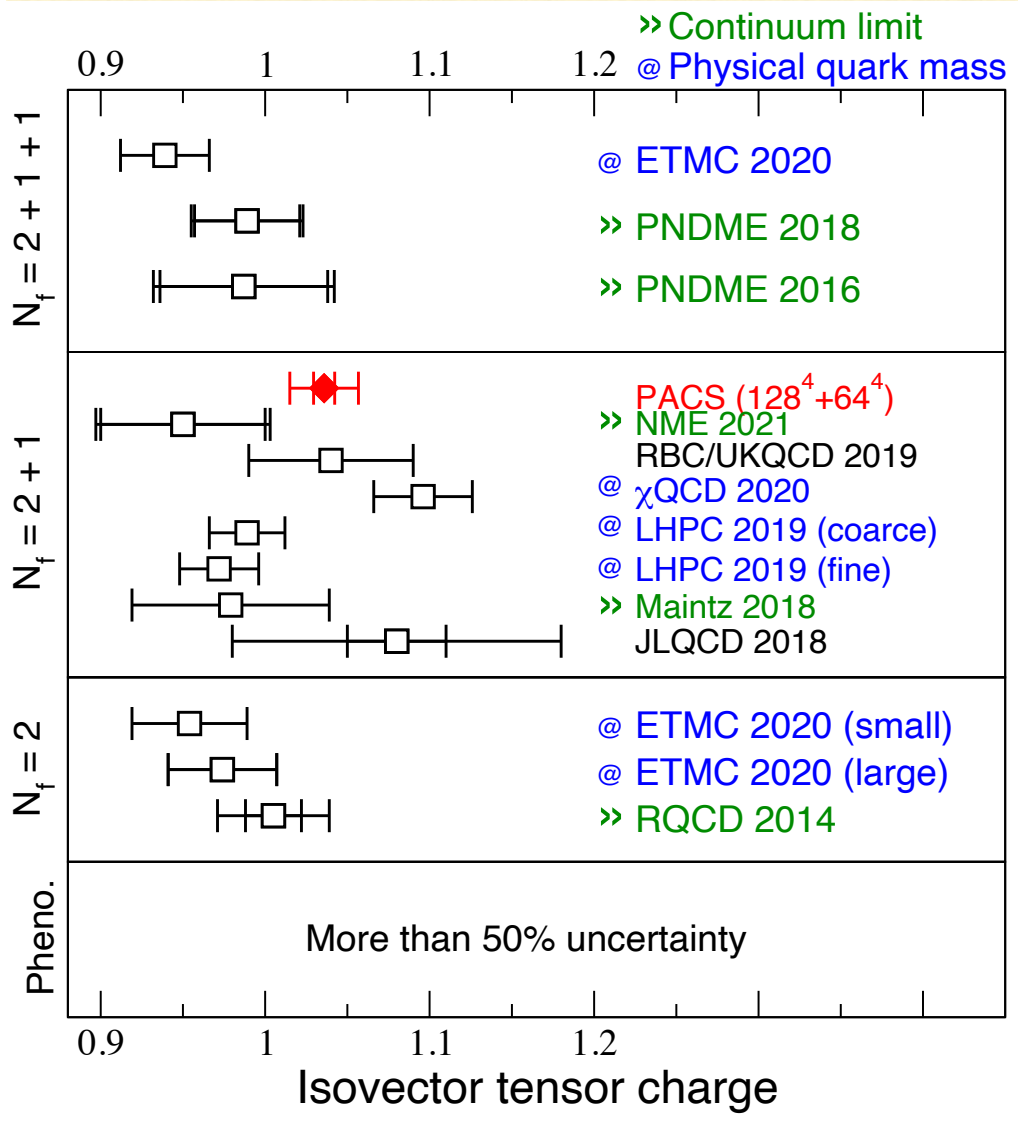

$t_{sep} > 1$ (fm) seems large enough for g.s. saturation

$$g_S = 0.927(71)_{\text{sta.}}(22)_{\text{sys.}}$$

[FLAG2019] Aoki. S et al., Eur. Phys. J. C. 80, 113 (2020).
[PNDME2018] R. Guputa et al., Phys. Rev. **D98** (2018) 034503.
[PNDME2016] T. Bhattacharya et al., Phys. Rev. **D94** (2016) 054508.
[ETMC2020] C. Alexandrou et al., Phys. Rev. **D102** (2020) 054517.
[LHPC2019] N. Hasan et al., Phys. Rev. **D99** (2019) 114505.
[Mainz2018] K. Ottnad et al., in Proceedings, Lattice2018.
[JLQCD2018] N. Yamanaka et al., Phys. Rev. **D98** (2018) 054516.
[RQCD2014] G. S. Bali et al., Phys. Rev. **D91** (2015) 054501.
[NME2021] S. Park et al., Phys. Rev **D105** (2021) 054505.
[Pheno,] M. Gonzalez-Alonso et al., Phys. **Lett** 112 (2014) 04501.

# Renormalized tensor couplings




$t_{sep} > 1$ (fm) seems large enough for g.s. saturation

$$g_T = 1.036(6)_{\text{sta.}}(20)_{\text{sys.}}$$


[FLAG2019] Aoki. S et al., Eur. Phys. J. C. 80, 113 (2020).
[χQCD2020] D. Horkel et al., arXiv:2002.06699v1 (2020).
[PNDME2018] R. Guputa et al., Phys. Rev. **D98** (2018) 034503.
[PNDME2016] T. Bhattacharya et al., Phys. Rev. **D94** (2016) 054508.
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[NME2021] S. Park et al., Phys. Rev **D105** (2021) 054505.
[RQCD2014] G. S. Bali et al., Phys. Rev. **D91** (2015) 054501.

# Summary and perspectives

- Conclusion of this talk
- Future works

# Summary and Perspectives

High-precision and high-accuracy determination:

$^{*}g_S = 0.927(71)_{sta.}(22)_{sys.}$ and $g_T = 1.036(6)_{sta.}(20)_{sys.}$

NEXT!

Towards continuum limit keeping with high precision/accuracy

Another cut-off -Preliminary

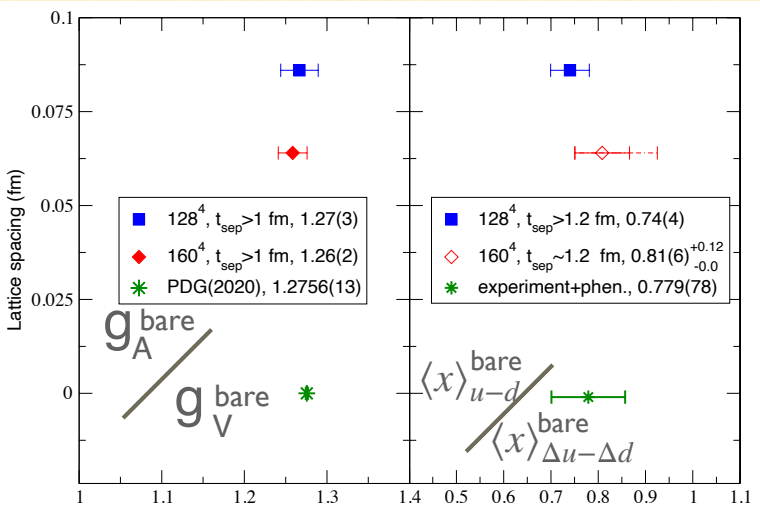



Fig. from RT at LATTICE2021

- High-precision calculation
  All-mode-averaging techniques can reduce the statistical noise.
- High-accuracy calculation
  RI/SMOM scheme enables us to keep the systematic error under control.
  How about other operators?

In future → High-precision and high-accuracy Parton physics

*We can also use these for searching the BSM such as quark chromo-EDM or beta decay (Intensity frontier).

# BACKUPS

# Neutron and Beyond S.M.

- Neutron EDM

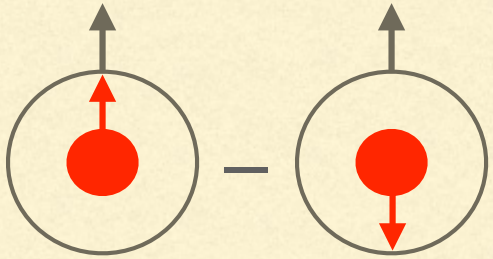
  → estimate CPV induced from quark chromo-EDM $d_q$

exp. bounds

Lat. calculation

$$d_n = \delta u d_u + \delta d d_d + \delta s d_s$$

$$\delta q(Q^2) = \int_0^1 dx[h_1^q(x,Q^2) - h_1^{\bar{q}}(x,Q^2)] : \; h_1^q =$$



Constraints on CPV quark chromo-EDM

- Neutron decay

  → $\beta$ decay NOT with V-A channel = Beyond S.M.

  e.g.) Scalar, Tensor channel coupling

# Nucleon = Transdisciplinary target

① Particle physics (Intensity frontier)

New physics (Beyond S.M.) = Experiment - Theory (S.M. of particle physics)

→ Some ambiguities stem from QCD

High-precision calculation of nucleon structure with QCD can give constraints on New physics

② Nuclear physics based on QCD


Nucleon structure in QCD is the first step towards the nuclear physics based on QCD

# Chiral S.S.B. at low energy QCD

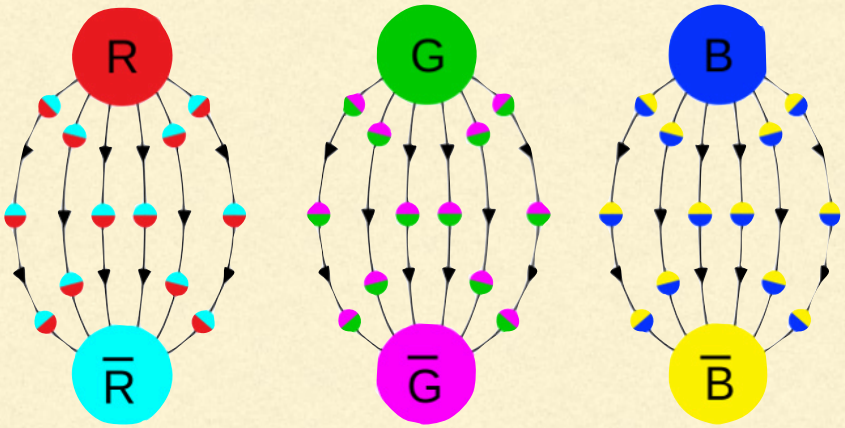
$\overline{\mathrm{MS}}$ "running" quark mass at renormalization scale of 2 GeV:

$$m_u = 2.32(10)\ (\mathrm{MeV}),\ m_d = 4.71(9)\ (\mathrm{MeV}),\ m_s = 92.9(7)\ (\mathrm{MeV})$$

$$< \ \Lambda_{\mathrm{QCD}} \sim \ 300\ (\mathrm{MeV})\ \text{or}\ M_N \sim 940\ (\mathrm{MeV})$$

$\rightarrow$ *u,d,s*-quak masses are essentially negligible

At low energy QCD, $q\bar{q}$ condensation occurs = Chiral S.S.B.




Quarks obtain "mass" dynamically

$$\mathscr{L}_{\mathrm{int.}} \supset g\langle q\bar{q}\rangle \bar{q}q$$

This "mass" is called as constituent quark mass:

$$m_u \sim 400\ (\mathrm{MeV}),\ m_d \sim 400\ (\mathrm{MeV}),\ m_s \sim 500\ (\mathrm{MeV})$$

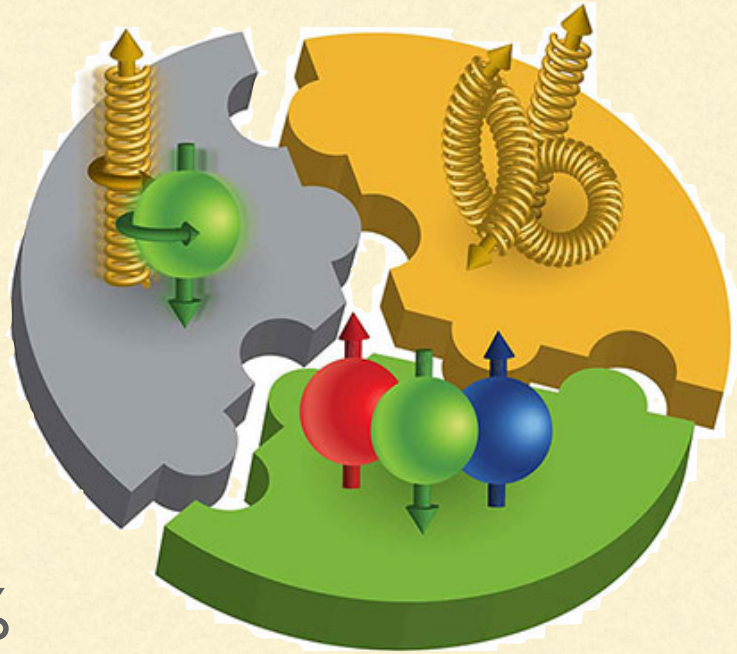
# Proton spin crisis "SPIN ORIGIN"

Quarks' spin are NOT enough to describe the proton spin

Gluon TAM

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + L_q + J_g$$

Quark SPIN and OAM



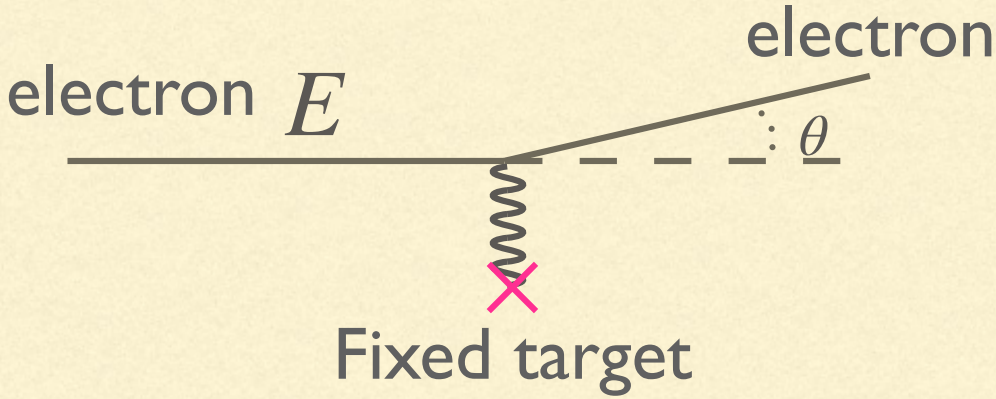
Experiment → $\Delta\Sigma$ carries 20-30 %

$$\Delta\Sigma = \sum_q \int_0^1 dx \Delta q(x)$$

: High-precise calculation of pol. PDF $\Delta q(x)$ can determine Quark SPIN

# Mott scattering

Relativistic scattering between the electron and fixed target is formulated as the Mott scattering.




Electron and fixed target have vector-type interaction

$\theta$ : the lab scattering angle
$E$ : incident electron's energy

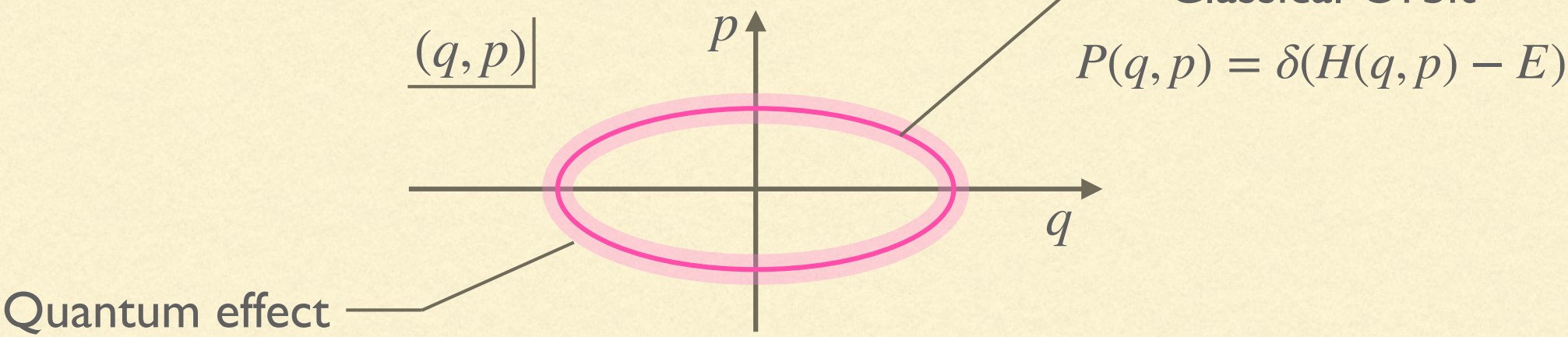
In Lab frame, where the fixed target is rest, the scattering cross section:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{Mott}} = \frac{\alpha^2 \cos^2 \frac{\theta}{2}}{4E^2 \sin^2 \frac{\theta}{2}}$$

This reproduces the Rutherford scattering in non-relativistic limit.

# Wigner distribution

e.g.) Harmonic Oscillator




Wave func. of H.O. is $\psi_n(q) = \left(\frac{m\omega}{2^{2n}(n!)^2\pi\hbar}\right) H_n\left(q\sqrt{\frac{m\omega}{\hbar}}\right) \mathrm{e}^{-q^2\frac{m\omega}{2\hbar}}$

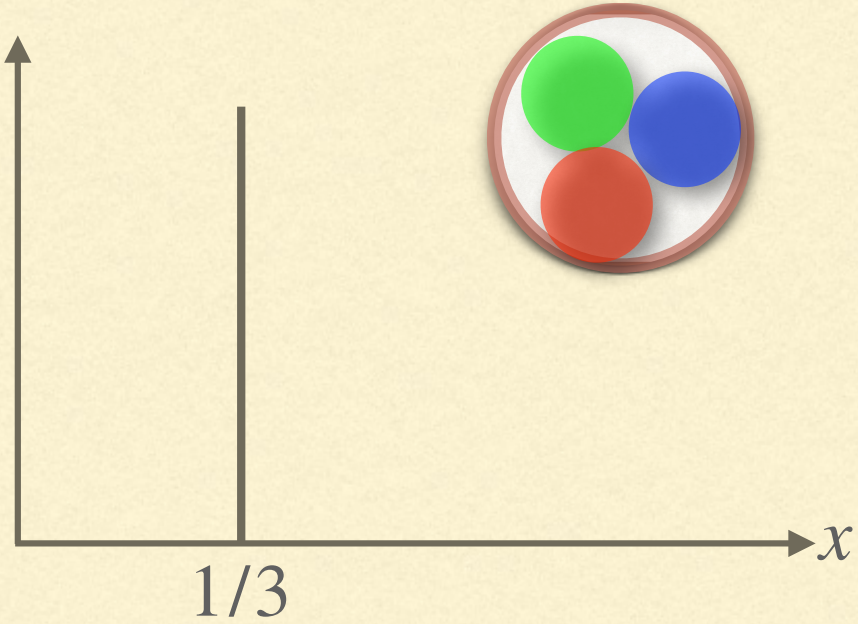
Then, Wigner dis.: $W(q,p) \equiv \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi\hbar} \mathrm{e}^{-ip\xi/\hbar} \psi^*\left(q - \frac{\xi}{2}\right) \psi\left(q + \frac{\xi}{2}\right)$

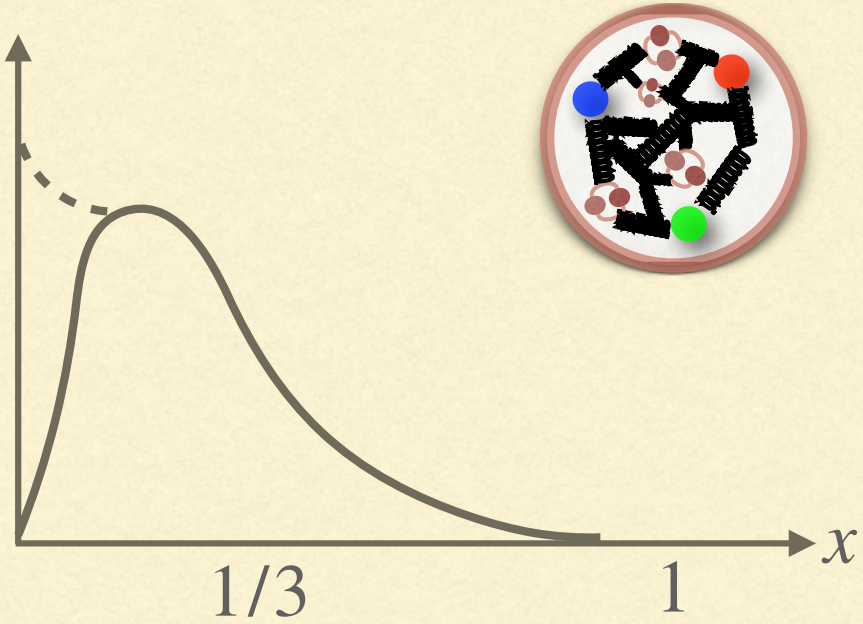
$$\rightarrow W_n(q,p) = \frac{(-1)^n}{\pi\hbar} \mathrm{e}^{-\frac{2H(q,p)}{\hbar\omega}} L_n\left(\frac{4H(q,p)}{\hbar\omega}\right) \xrightarrow[\hbar\to 0, n\gg 0]{} \delta(H(q,p) - E_n)$$

| Quantum Mechanism | | Classical Mechanism |
|---|---|---|
| $\langle A(q,p)\rangle_Q = \int dq \int dp W(q,p) A(q,p)$ | $\longleftrightarrow$ | $\langle A(q,p)\rangle_C = \int dq \int dp P(q,p) A(q,p)$ |

# Quark momentum fraction

Gluons reduce relative contribution from quarks



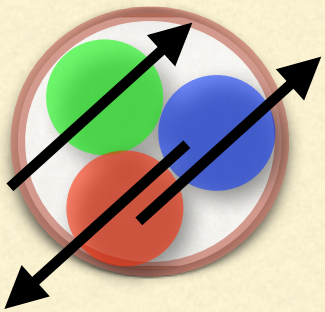




- Quark momentum fraction

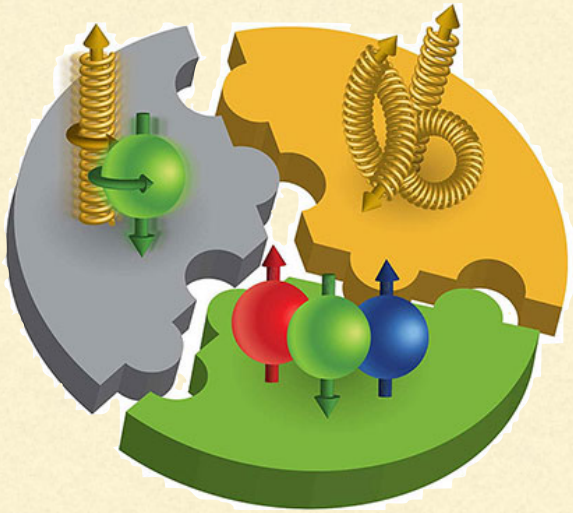
$$\int_0^1 dx x F_1(x, Q^2) \to \langle x \rangle_q = \langle p, S | \bar{q} \left[ \gamma_4 \overleftrightarrow{D}_4 - \frac{1}{3} \sum_k \gamma_k \overleftrightarrow{D}_k \right] | p, S \rangle$$

# Quark helicity fraction

Quarks' spin are NOT enough to describe the proton spin



$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$$



→ Nucleon has spin 1/2

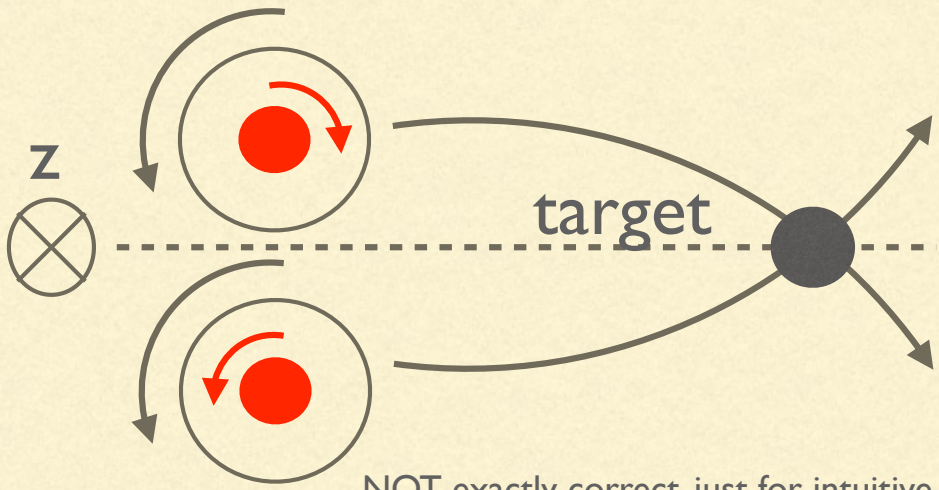
$$\rightarrow \frac{\text{Quarks' contribution}}{\text{Nucleon's spin}} = ?$$

◉Quark helicity fraction

$$\int_0^1 dx x g_1(x, Q^2) \rightarrow \langle x \rangle_{\Delta q} = i \langle p, S | \bar{q} \gamma_5 \left[ \gamma_3 \overleftrightarrow{D}_4 + \gamma_4 \overleftrightarrow{D}_3 \right] | p, S \rangle$$

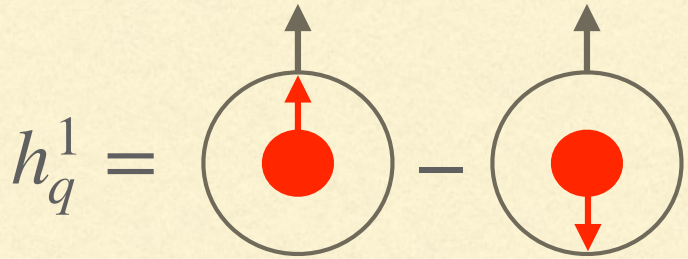
# SSA and lattice contributions

TMD PDF is essential but difficult to obtain with experiments.




NOT exactly correct, just for intuitive

One of TMD-Collins func. $h_q^1$:




Use Lattice to constrain Exp.

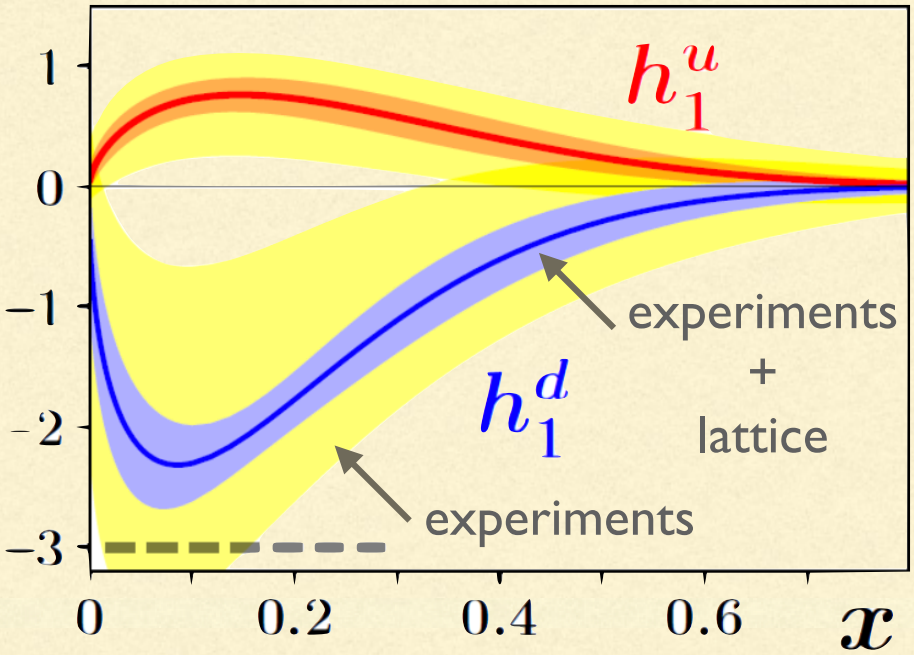
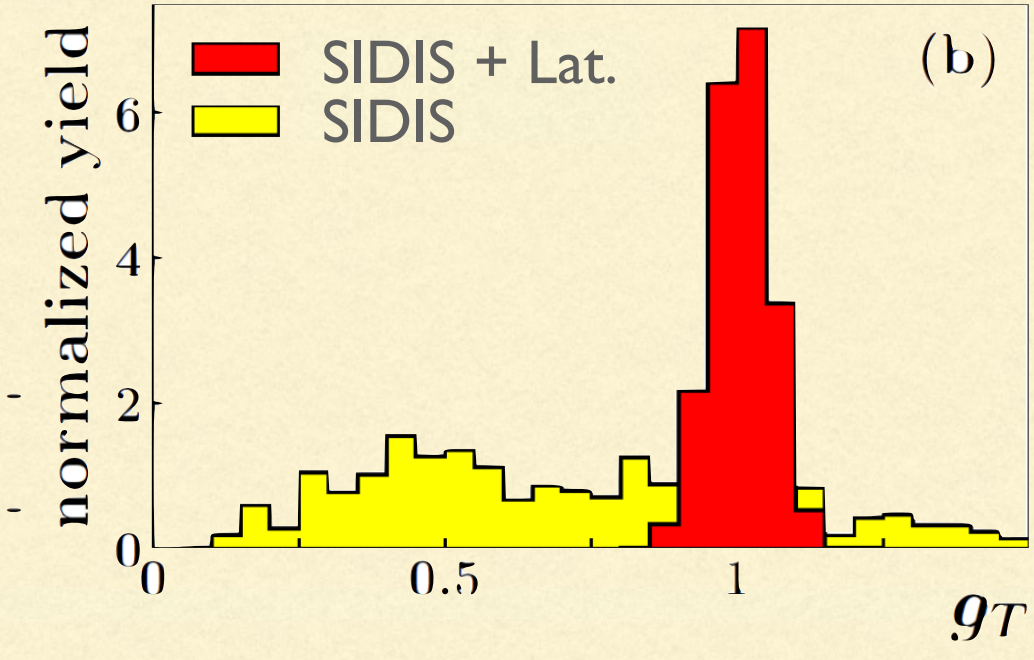
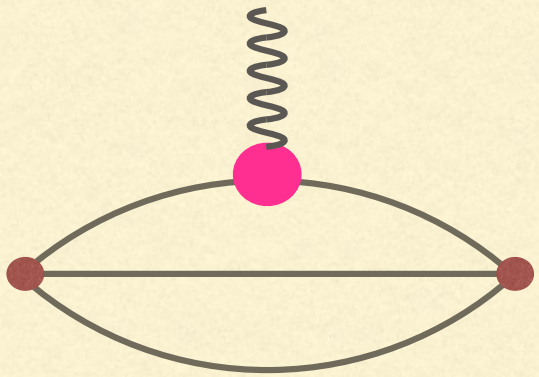



fig. from H.-W. Lin @QCD Evolution Workshop 2021

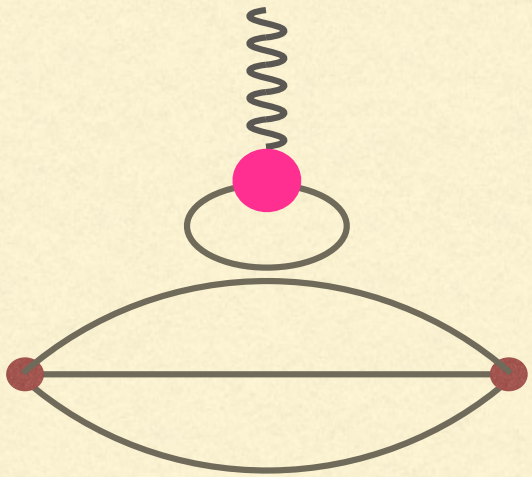


# Iso-vector quantities : $O_\Gamma = u\Gamma d$ under the iso-spin symmetry

$\langle N | O(t') | N \rangle$ : composed of two contributions.



Connected contribution



Disconnected contribution
= High computational costs

Iso-vector does NOT suffer from disconnected contribution.

# Lattice QCD -Monte Carlo integration

Path integration of QCD = High-dimensional integrals

$$\langle O \rangle = \frac{1}{Z} \int \mathscr{D}[U] \mathscr{D}\left[\overline{\psi}\right] \mathscr{D}\left[\psi\right] O\left[U, \overline{\psi}, \psi\right] \mathrm{e}^{-J_{\mathrm{QCD}}\left[U, \overline{\psi}, \psi\right]}$$

→ Estimate stochastically = Monte Carlo integration

◉ Importance sampling ⋯ Generate points only in important regions

e.g.) Suppose the integration : $I = \int_V d^d x f(x) = \int_V d^d x \frac{f(x)}{g(x)} g(x)$

1. Generate points $\{x_1, \cdots, x_n\}$ in $V$ with probability : $g(x)$
2. Compute $h_i = \frac{f(x_i)}{g(x_i)}$ for $i = 1, \cdots, n$
3. Monte Carlo theory gives : $I = \int_V d^d x h(x) g(x) = \frac{1}{n} \sum_{i=1}^{n} h_i$

# Error reduction techniques

◉ All-Mode-Averaging (AMA)

For original operator $O^{(\mathrm{org})}$, improved one $O^{(\mathrm{imp})}$ is defined as

$$O^{(\mathrm{imp})} = \frac{1}{N_{\mathrm{org}}} \sum_{f \in G}^{N_{\mathrm{org}}} \left( O^{(\mathrm{org})f} - O^{(\mathrm{approx})f} \right) + \frac{1}{N_G} \sum_{g \in G}^{N_G} O^{(\mathrm{approx})g}$$

where $G$ is the lattice symmetry. (e.g. translation symmetry)
$O^{(\mathrm{approx})}$ is relaxed CG solution.

$$O^{(\mathrm{approx})} = O^{(\mathrm{approx})} \left[ S_{\mathrm{AM}} = \sum_{i=1}^{N_{\mathrm{eig}}} v_i \frac{1}{\lambda_i} v_i^\dagger + P_n(\lambda) \left( 1 - \sum_{i=1}^{N_{\mathrm{eig}}} v_i v_i^\dagger \right) \right]$$

$P_n(\lambda)$ is polynomial approximation of $1/\lambda$

→ Low computational cost
Improved error : $\mathrm{err}^{\mathrm{imp}} \simeq \mathrm{err}/\sqrt{N_G}$

T. Blum et al., Phys.Rev. D **88,** 094503(2013)

# Interpolating operator of Nucleon

The interpolating operator of Nucleon used here are following:

$$N_X(t,\vec{p}) = \sum_{\vec{x}\vec{x_1}\vec{x_2}\vec{x_3}} \mathrm{e}^{-i\vec{p}\cdot\vec{x}} \varepsilon_{abc} \left[u_a^T(t,\vec{x_1}) C\gamma_5 d_b(t,\vec{x_2})\right] u_c(t,\vec{x_3})$$

$$\times \phi_X(\vec{x_1}-\vec{x})\phi_X(\vec{x_2}-\vec{x})\phi_X(\vec{x_3}-\vec{x})$$

where

$$\phi_X(\vec{x_i}-\vec{x}) = \begin{cases} \phi_L(\vec{x_i}-\vec{x}) = \delta(\vec{x_i}-\vec{x}) & \text{:Local type} \\ \phi_S(\vec{x_i}-\vec{x}) = A\exp(-B|\vec{x_i}-\vec{x}|) & \text{:exp. smeared} \end{cases}$$

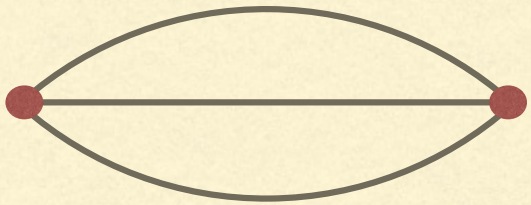
and the parameters described here are $(A,B) = (1.2, 0.16)$

# 2pt. and 3pt. function

- 2pt. function

$$\langle N(t)N^\dagger(0)\rangle = \sum_i |\langle 0|N(0)|i\rangle|^2 e^{-E_i t}$$

$$\xrightarrow[t \gg 0]{} |\langle 0|N(0)|N\rangle|^2 e^{-M_N t}$$

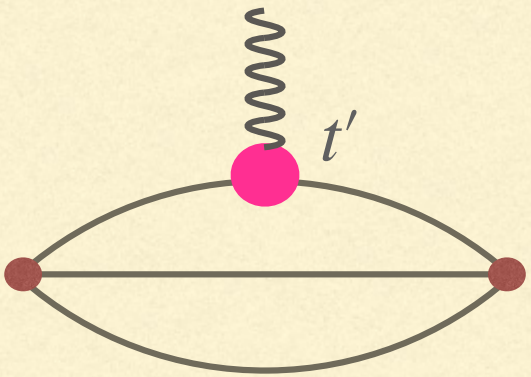


tsink = t

tsrc = 0

- 3pt. function

$$\langle N(t)\underline{O(t')}N^\dagger\rangle = \sum_{ij} \langle 0|N(0)|i\rangle\langle i|O(0)|j\rangle\langle j|N^\dagger|0\rangle e^{-E_i t}$$



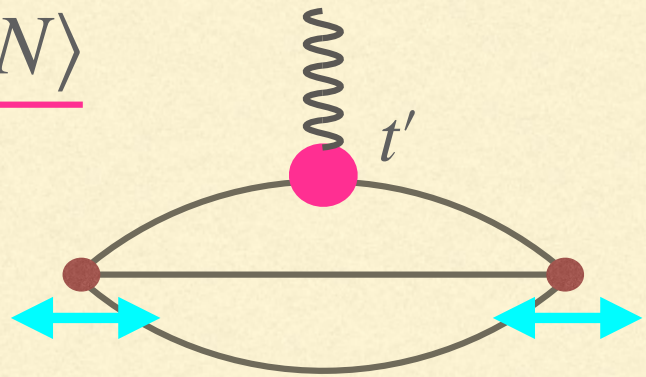

$$\xrightarrow[t \gg t' \gg 0]{} |\langle 0|N(0)|N\rangle|^2 \underline{\langle N|O(0)|N\rangle} e^{-M_N t}$$

Nucleon matrix element

# Extracting matrix elements and Ratio method

Nucleon matrix elements can be extracted from the ratio of 3pt. function to 2pt. function

$$\frac{\langle N(t)O(t')N(0)^\dagger\rangle}{\langle N(t)N(0)^\dagger\rangle} \underset{t \gg t' \gg 0}{\rightarrow} \langle N|O(0)|N\rangle$$




N.B. Excited state contamination

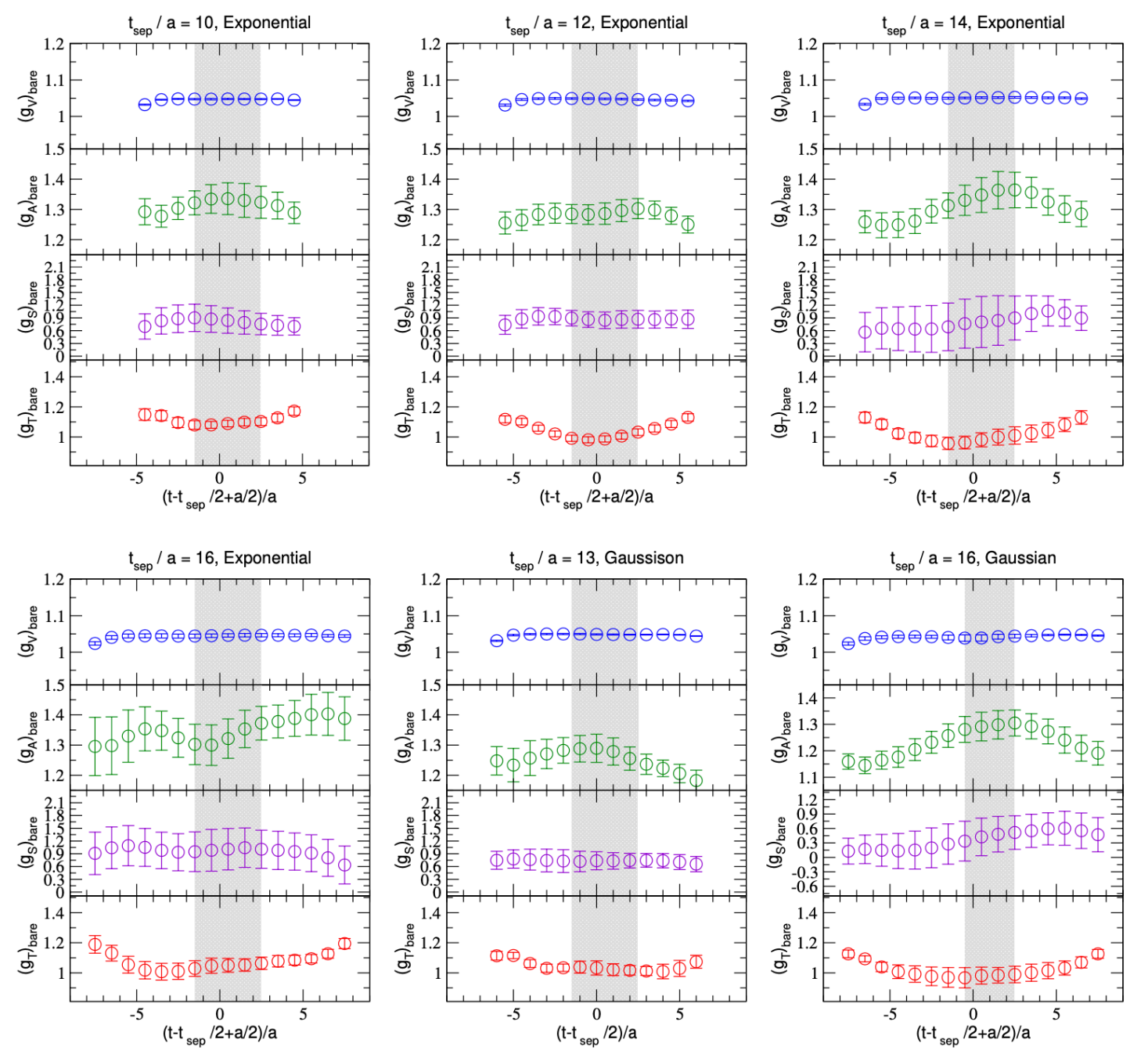
Interpolating operator also creates excited states.

$$\frac{\langle N(t)O(t')N(0)^\dagger\rangle}{\langle N(t)N(0)^\dagger\rangle} \underset{\cancel{t \gg t' \gg 0}}{\rightarrow} \langle N|O(0)|N\rangle + Ae^{-E(t'-t)} + \cdots$$

→ Gaze $|t_{sink}-t_{src}|$ independence = confirm no contamination

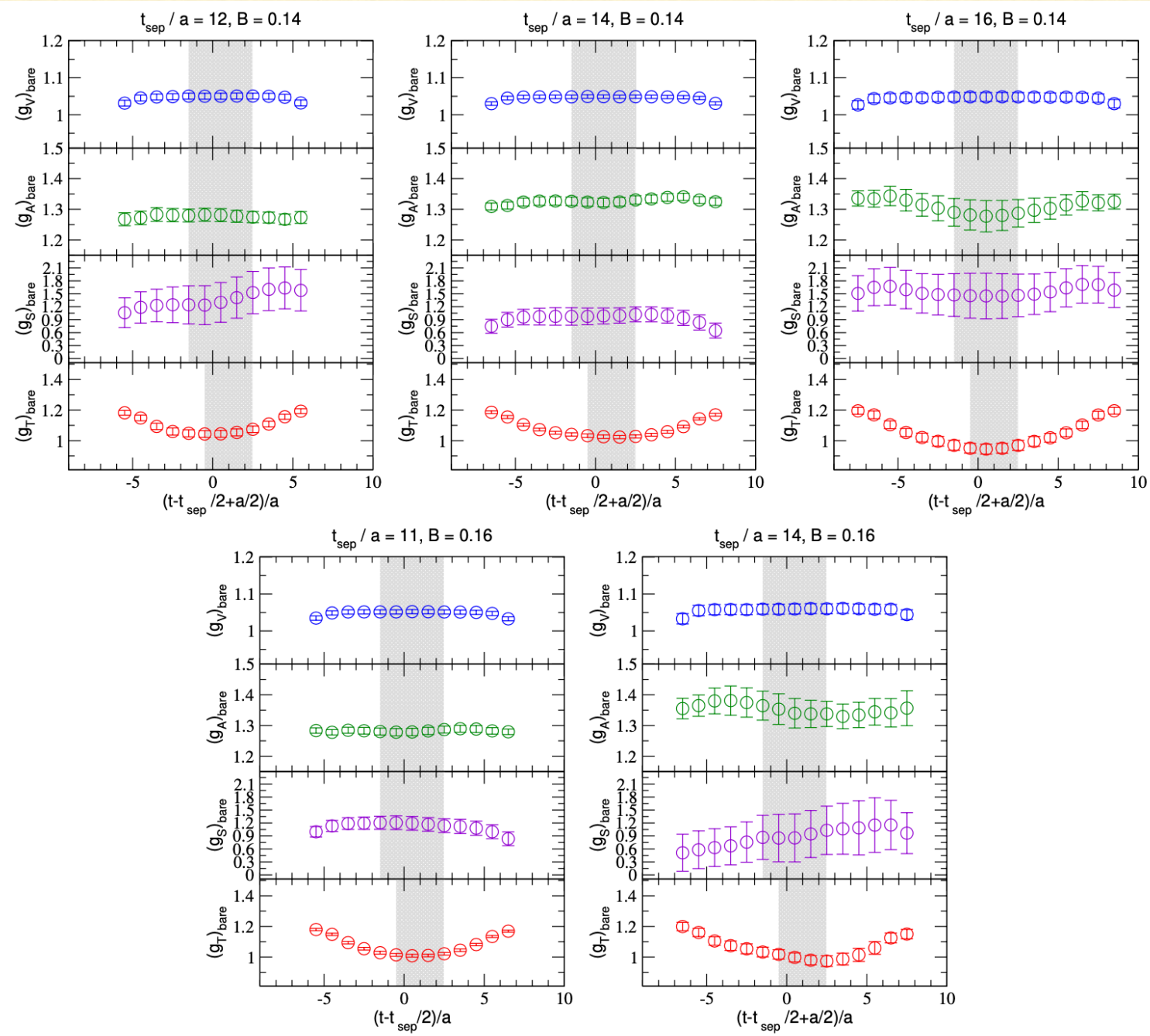
Preliminary

# Bare matrix elements -$128^4$ lattice



Preliminary

# Bare matrix elements -$64^4$ lattice

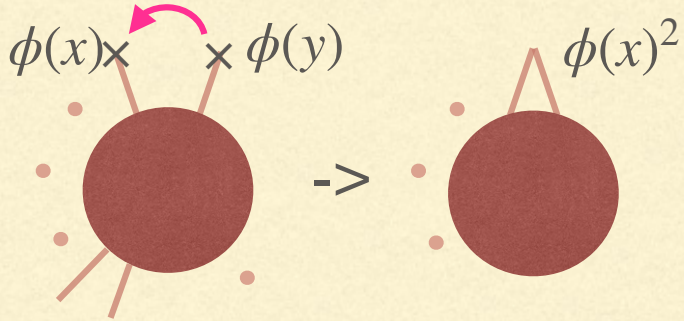


# Renormalization on Lattice

Composite operator ~ New interaction vertex = Renormalization

○ Lattice perturbation theory

Bad convergence of the perturbation




◉ Regularization Independent MOmentum Subtraction scheme (RI/MOM)

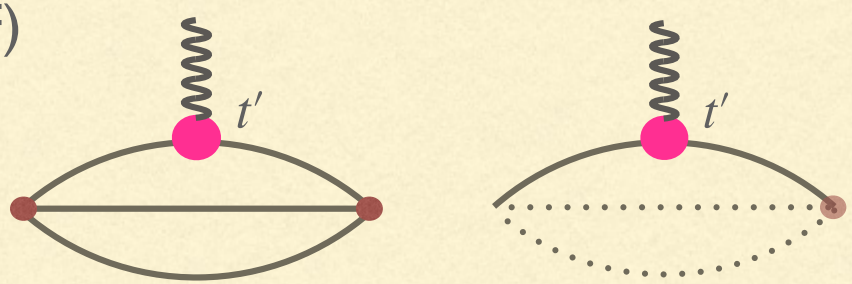
Calculation cost ··· LOW

Non-perturbative calculation

◉[1] Schrödinger Functional scheme (SF)

Calculation cost ···[2] HIGH

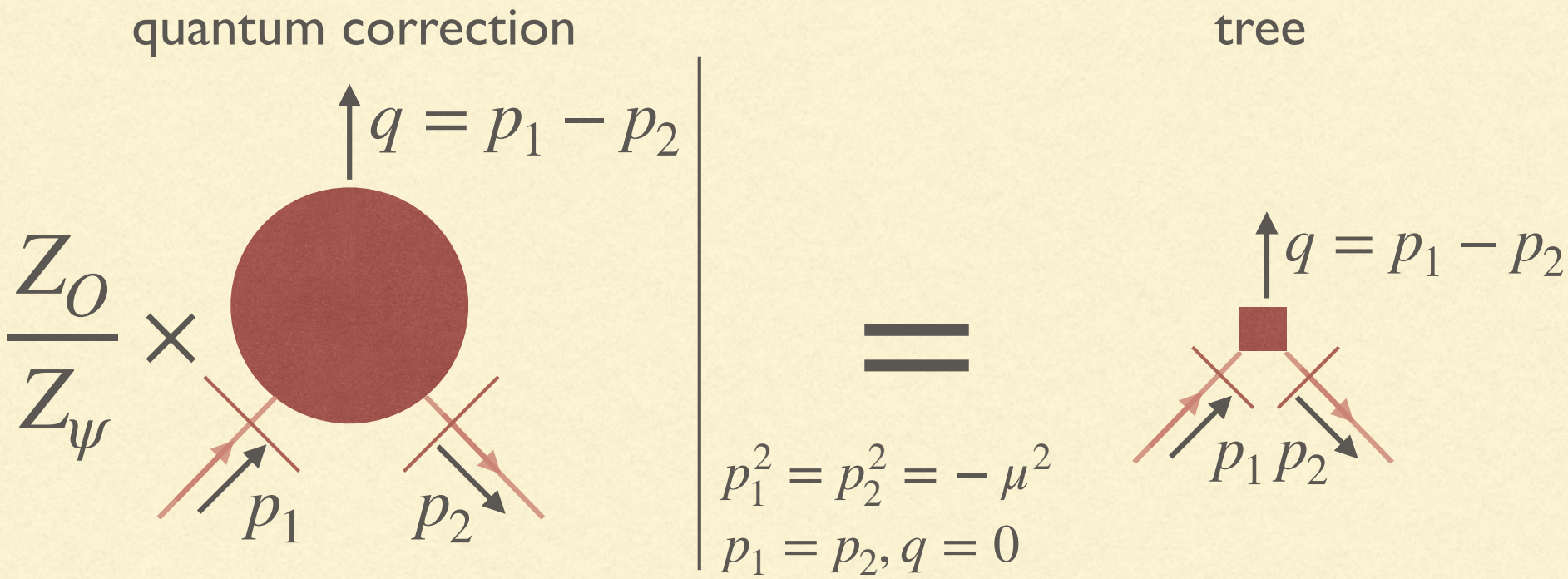
Non-perturbative calculation




Matrix elements Vertex functions

1. We use this for input calculation. 2. Necessary to calculate operator-dependently.

# RI/MOM scheme




- Impose the condition on amputated Green function
- Introduce the scale by external momentum

Non-perturbative determination of renormalization

G. Martinelli et al., Nucl. Phys. B 316, 355-372 (1989).
Instead of getting the wave function renormalization factor, we often normalize it by another current.

# Discretization error

Discretized time-space brings error = Discretization error

e.g.) Dispersion relation

Continuum : $E^2 = m^2 + p^2$

Lattice : $E^2 = m^2 + \frac{4}{a^2}\sin^2\left(\frac{ap}{2}\right) = m^2 + p^2\left\{1 - \frac{1}{3!}(ap)^2 + \cdots\right\}$

Renormalization scale is determined by the quarks' external momenta in RI/MOM scheme

= Vertex function, renormalization constant, has $(a\mu)$-dependence

$$Z_O^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) \supset c_1(a\mu)^2 + c_2(a\mu)^4 + \cdots$$

1.Odds power don't appear, since vertex functions don't change their sign under $p \to -p$ conversion.By the way, there are several types of discretization error, however the one stems from the dispersion relation is especially called as ordinary lattice artifacts and it is often that they are fitted by polynomials.

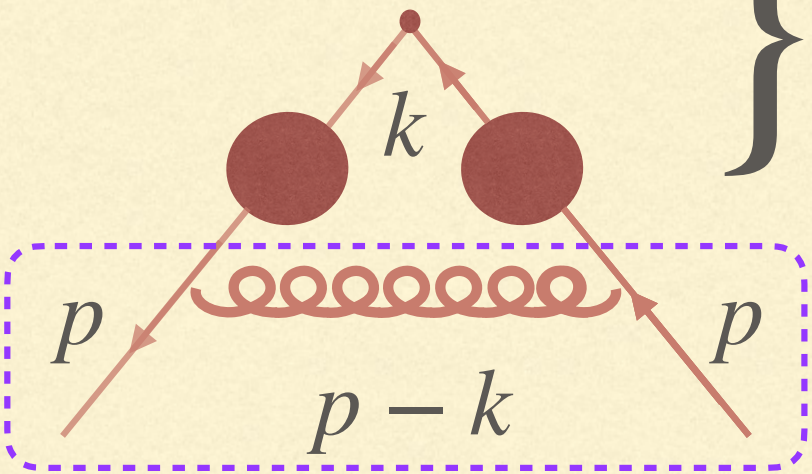
# Window problems and $\mu$-dependence

→ Condition $\overset{1.}{\Lambda_{QCD}} \overset{2.}{\ll} \mu \overset{2.}{\ll} a^{-1}$ is imposed on the scale

- $\overset{3.}{\Lambda_{QCD}} \ll \mu$ : suppress unnecessary non-perturbative effects

e.g.)

$$\bar{\psi}\Gamma\psi(q=0)$$




Easy to have low momentum by just one gluon exchange.
→ suffer from non-perturbative effects due to S.Chiral Sym. B.
Large Systematic error

Vertex func. has pole: $\boxed{\frac{m_{val}^2}{p^2}, \frac{\langle q\bar{q}\rangle^2}{p^2}, \frac{\langle A_\mu^2\rangle}{p^2}}$

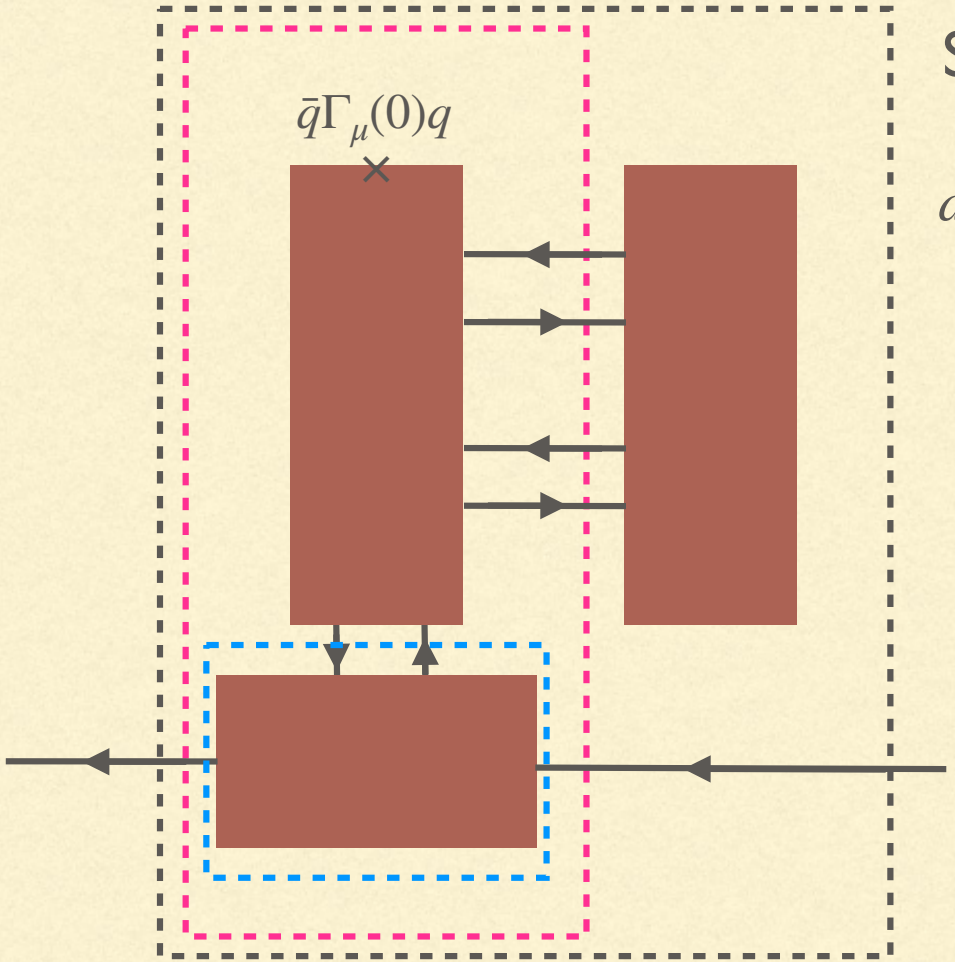
$$d_G = 4 - \frac{3}{2}\cdot 4 + \left(2\cdot\frac{3}{2} - 4\right) = -2$$

$$\rightarrow Z_O^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) \supset \frac{c_{-1}}{(a\mu)^2}$$

- $\Lambda_{QCD}$ is so-called QCD scale where non-perturbative IR divergence is NOT negligible. Therefore, $\Lambda_{QCD} \ll \mu$ also has a role to make perturbative matching to other scheme work.
- $\mu \ll a^{-1}$ is the condition for decreasing the discretization effect.

# Non-perturbative effects

- Exceptional momenta = momentum transfer is ZERO ($q = 0$)
  = invalidate naive power counting




Superficial d.o.d. is given by:

$$d_G = 4 - \frac{3}{2}n_f + \left(2 \cdot \frac{3}{2} - 4\right) \to 0$$ Naive

If S.S.B occurs, ($V - A \neq 0$)

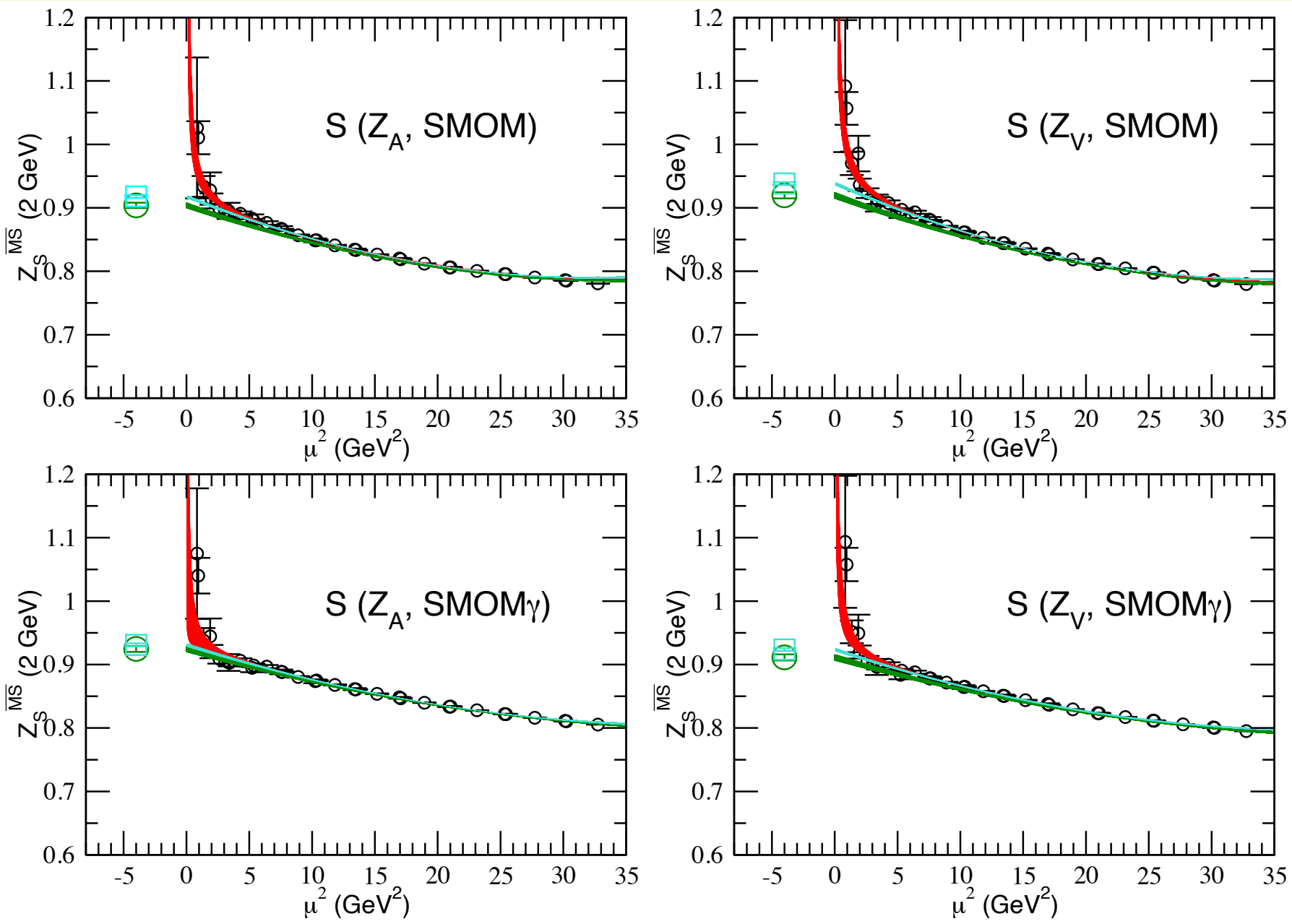
| Exceptional | | Non-exceptional |
|---|---|---|
| 4 | $n_f$ | 6 |
| -2 | $d_G^2$ | -6 |
| $\frac{m_{val}^2}{p^2}, \frac{\langle q\bar{q}\rangle^2}{p^2}$ | Effects on Vertex | $\frac{\langle q\bar{q}\rangle^2}{p^6}$ |

Vertex function has pole structure

1. Non-exceptional momentum configuration is available in RI/SMOM scheme 2. $d_G$ for 3 flavor case

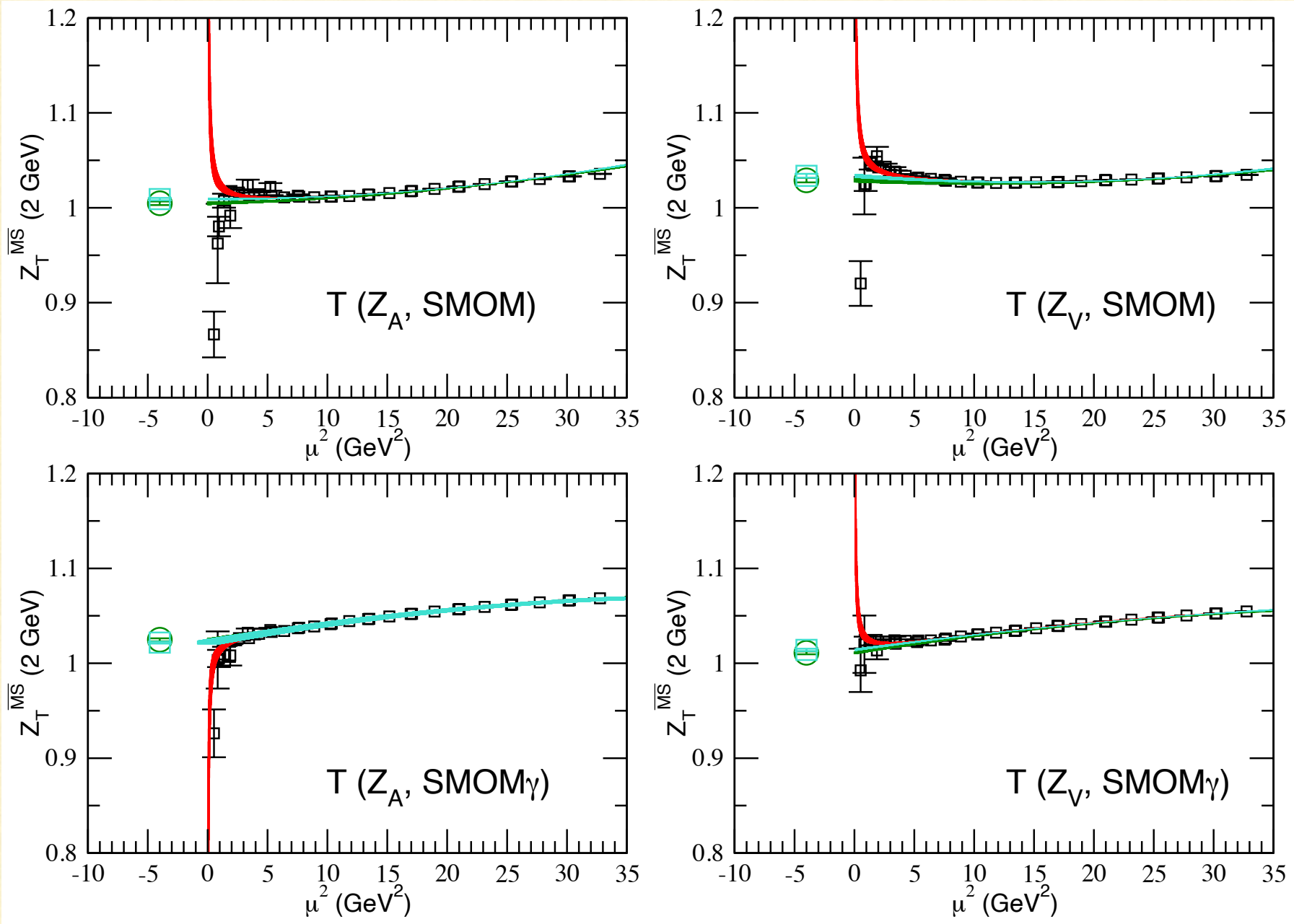
# Renormalization constants

e.g.) [1:5] Quadratic ansatz



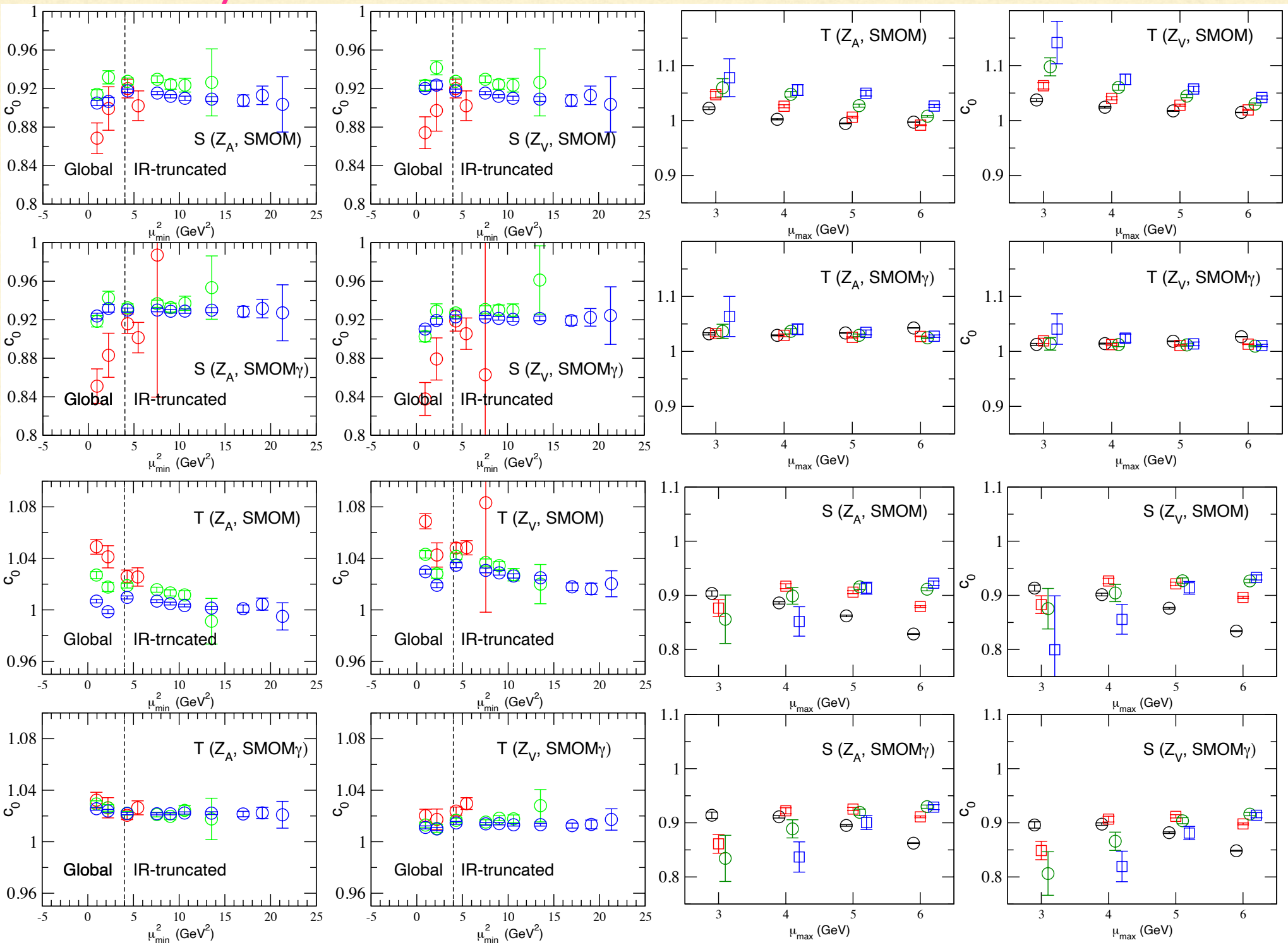
# Renormalization constants

e.g.) [1:5] Quadratic ansatz



# Preliminary

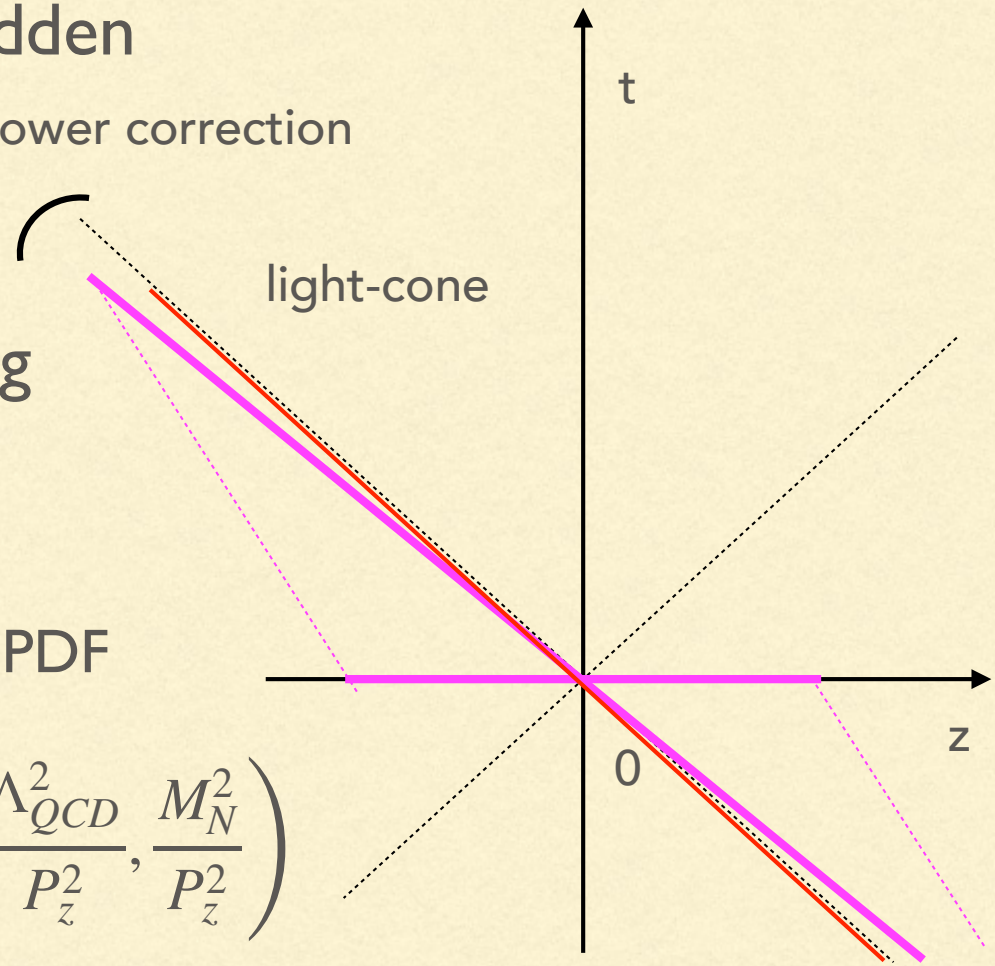
# Scalar & Tensor channel



# Future perspectives : PDFs on Lattice

PDFs are defined as non-local correlation on light-cone
→By definition, naively forbidden

However ,
we can access it
via Lorentz boost & Matching




e.g) Unpolarized Quark DF

$\tilde{q}(x, \mu, P_z)$ Accessible (Lattice QCD) → quasi-PDF

$$= \int_0^1 \frac{dy}{y} Z\left(\frac{x}{y}, \frac{\mu}{P_z}\right) \underbrace{q(y, \mu)}_{\text{PDF}} + O\left(\frac{\Lambda_{QCD}^2}{P_z^2}, \frac{M_N^2}{P_z^2}\right)$$

Xiangdong Xi, arXiv:1305.01539 (2013)

# Root-mean-square error

## Problem: Model comparison / Model selection

**Step1.** Prepare training data(-N 47 pts)

→ Gauge conf. k ( k = 1~101)
Fix FIT range

E.g.) $1 < \mu < 3$: 14 pts $\rightarrow N_{\text{train}} = 14$
$1 < \mu < 4$: 20 pts $\rightarrow N_{\text{train}} = 20$
$1 < \mu < 5$: 26 pts $\rightarrow N_{\text{train}} = 26$

**Step2.** FIT with various ansatz for conf. k

$$f^A(\mu^2; \mathbf{c}^{M,k}_{\mu_{\max}}) = \frac{c_{-1}}{\mu^2} + c_0^k + c_1^k(\mu^2) + \cdots = \frac{c_{-1}^k}{\mu^2} + \sum_{i=0}^{M} c_i^k(\mu)^{2i}$$

→ Obtain trained fit parameter $\mathbf{c}^{\star M,k}_{\mu_{\max}}$ for certain FIT range and FIT ansatz for conf. k.

N.B. Assuming that gauge configurations are independent, we can consider different l ≠ k carries different random noises.

**Step3.** Evaluate RMS error with training and test data(-R 101 pts)

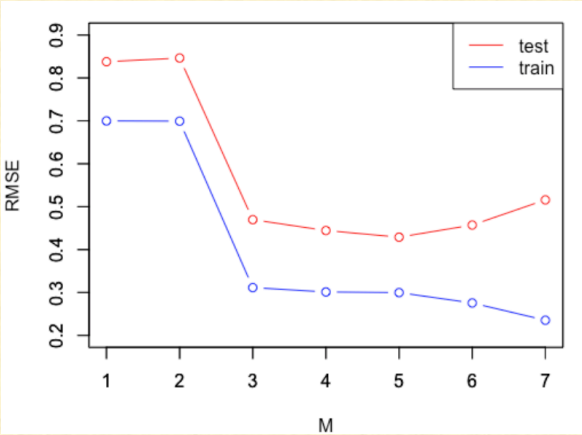
→ Gauge conf. l≠k ( l = 1~101)
Fix FIT range

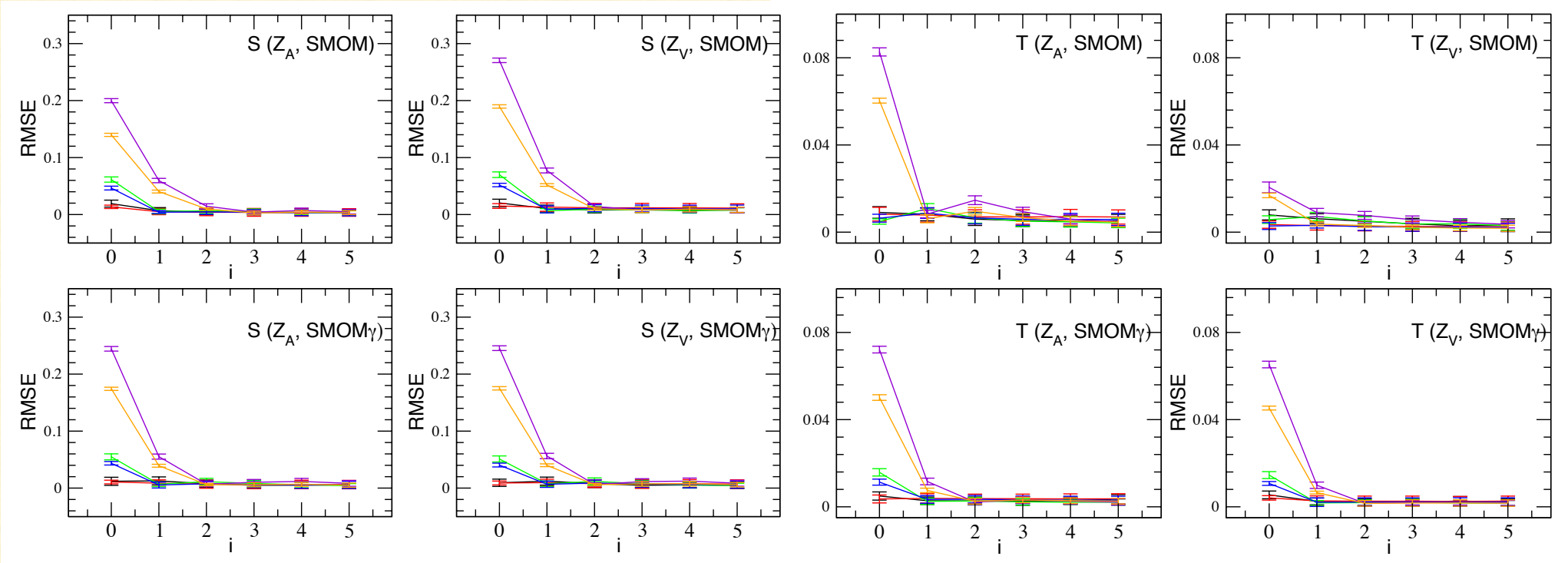
E.g.) $1 < \mu < 3$: 32 pts $\rightarrow N_{\text{test}} = 32$
$1 < \mu < 4$: 47 pts $\rightarrow N_{\text{test}} = 47$
$1 < \mu < 5$: 61 pts $\rightarrow N_{\text{test}} = 61$

$$E_{\text{RMS},\mu_{\max}}^{M,l,k} = \sqrt{\frac{2E(\mathbf{c}^{\star M,l,k}_{\mu_{\max}})}{N}}$$

**Step4.** Compare RMS error

E.g.) Sample for fixed range






{color, $\mu_{\max}$,data} = {black, 3, test}, {red, 3, training}, {green, 4, test}, {blue, 4, training}, {violet, 5, test}, {orange, 5, training}

Candidates suggested by RMS error analysis are

| | (1) | (2) | (3) | (4) | (5) |
|---|---|---|---|---|---|
| $\mu_{\max}$ (GeV) | 4 | 5 | 5 | 3 | 4 |
| Polynomial | 2 | 2 | 3 | 0 | 1 |