The semi-classical topological susceptibility at high temperature revisited

Alexander Boccaletti and Daniel Nogradi

2001.03383 – JHEP 03 (2020) 045

Eotvos University Budapest

Motivation

Peccei-Quinn scenario for solving strong CP-problem

Axion mass purely QCD quantity: $f_A^2 m_A^2(T) = \chi(T)$

Scale of $U(1)_{PQ}$ breaking: f_A, large

Lattice determination of $\chi(T)$: constraints on axion mass

Motivation

Lattice calculation of $\chi(T)$ very challenging

This conference:

- Axion round table today: Claudio Bonati, Francesco D'Eramo, Guido Martinelli, DN

- Parallel talk 4:30 PM today: Claudio Bonati

Useful: comparison with semi-classical results at high T

Motivation

Goal: get reliable semi-classical results

You would think: this was already done a long time ago

You would be mostly right

But not completely

Summary

Within semi-classical approximation

- More precise temperature dependence than previously
- Correct over-all prefactor for $\chi(T)$ in $\overline{\text{MS}}$
- Error budget from scale

And some historical remarks for your amusement :)

Semi-classical calculation

Reliable at high T

Temperature dependence of instanton size distribution

$$n(\varrho, T) = n(\varrho) e^{-S(\pi\varrho T)}$$

- $n(\varrho)$: $T = 0$ instanton calculation
- $S(\lambda)$: determined by function $A(\lambda)$, $\lambda = \pi\varrho T$

$\rightarrow$ Need two ingredients: $T = 0$ results and $T > 0$ dependence

Semi-classical calculation $T = 0$

Zero temperature 1-loop with light fermions, $m_i/T, m_i/\Lambda \ll 1$

$$n(\varrho) = C \left(\frac{16\pi^2}{g^2(\mu)}\right)^{2N} e^{-\frac{8\pi^2}{g^2(\mu)}} \frac{1}{\varrho^5} (\varrho\mu)^{\beta_1} \prod_{i=1}^{N_f} (\varrho m_i(\mu))$$

$g(\mu)$ running coupling, $m_i(\mu)$ running masses

Over-all constant coefficient C **is scheme-dependent**, because renormalization is defined in a particular scheme

Frequently used schemes: Pauli-Villars, MS, $\overline{\text{MS}}$, etc.

Semi-classical calculation $T = 0$

$$n(\varrho) = C \left(\frac{16\pi^2}{g^2(\mu)}\right)^{2N} e^{-\frac{8\pi^2}{g^2(\mu)}} \frac{1}{\varrho^5} (\varrho\mu)^{\beta_1} \prod_{i=1}^{N_f} (\varrho m_i(\mu))$$

Result for C in Pauli-Villars and $SU(2)$:

G. 't Hooft
Phys. Rev. D 14, 3432 (1976)
erratum: Phys. Rev. D 18, 2199 (1978)

Result for C in Pauli-Villars and $SU(N)$

C. W. Bernard
Phys. Rev. D 19, 3013 (1979)

Semi-classical calculation $T = 0$

$$n(\varrho) = C \left(\frac{16\pi^2}{g^2(\mu)}\right)^{2N} e^{-\frac{8\pi^2}{g^2(\mu)}} \frac{1}{\varrho^5} (\varrho\mu)^{\beta_1} \prod_{i=1}^{N_f} (\varrho m_i(\mu))$$

More frequently used schemes: MS and $\overline{\text{MS}}$

Need to convert C to these schemes

$$C_1 = C_2 \left(\frac{\Lambda_2}{\Lambda_1}\right)^{\beta_1}$$

Need to know Λ-parameter ratios

Scheme change

Needed: $\Lambda_{PV}/\Lambda_{\mathsf{MS}}$, first given in original

G. 't Hooft, Phys. Rev. D 14, 3432 (1976)

Unfortunately incorrect (not in erratum either...)

Correct result

$$\frac{\Lambda_{\mathsf{PV}}}{\Lambda_{\mathsf{MS}}} = e^{\frac{1}{2}(\log(4\pi)-\gamma)+\frac{1}{22}}$$

A. Hasenfratz and P. Hasenfratz, Phys. Lett. 93B, 165 (1980)

Confirmed in G. 't Hooft, Phys. Rept. 142, 357 (1986)

Scheme change

Note: incorrect Λ-parameter ratios in

P. Weisz, Phys. Lett. 100B, 331 (1981)

R. F. Dashen and D. J. Gross, Phys. Rev. D 23, 2340 (1981)

No erratums...

Scheme change

In any case, MS result correct since Hasenfratz-Hasenfratz 1980

Most frequently used: $\overline{\mathrm{MS}}$

Conversion MS $\rightarrow$ $\overline{\mathrm{MS}}$ should be straightforward

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_{\mathrm{MS}}} = e^{\frac{1}{2}(\log(4\pi)-\gamma)} \qquad\qquad \frac{\Lambda_{\mathrm{PV}}}{\Lambda_{\overline{\mathrm{MS}}}} = e^{\frac{1}{22}}$$

$\overline{\text{MS}}$ scheme

Explicitly reported in $\overline{\text{MS}}$

A. Ringwald and F. Schrempp, Phys. Lett. B 438, 217 (1998) [hep-ph/9806528]

Unfortunately incorrect, never corrected before

$$C = \frac{e^{c_0 + c_1 N + c_2 N_f}}{(N-1)!(N-2)!}$$

c_0 and c_1 correct, but c_2 reported incorrectly

$\overline{\text{MS}}$ scheme

$$C = \frac{e^{c_0 + c_1 N + c_2 N_f}}{(N-1)!(N-2)!}$$

Problem: MS $\to$ $\overline{\text{MS}}$ conversion involves β_1 which depends on N_f, conversion used pure Yang-Mills β_1: c_2 incorrect

Mismatch: $\frac{1}{33} = \frac{2}{3} \cdot \frac{1}{22}$ where $\frac{2}{3}$ from N_f-dependence of β-function, $\frac{1}{22}$ from MS-$\overline{\text{MS}}$ Λ-parameter ratio

Ringwald-Schrempp used in most (all?) lattice comparisons

$\overline{\text{MS}}$ scheme

Furthermore, another wrong c_2 reported in

S. Moch, A. Ringwald and F. Schrempp, Nucl. Phys. B 507, 134 (1997) [hep-ph/9609445]

I. I. Balitsky and V. M. Braun, Phys. Rev. D 47, 1879 (1993)

First correct $\overline{\text{MS}}$ result

$$C_{\overline{\text{MS}}} = \frac{e^{c_0+c_1N+c_2N_f}}{(N-1)!(N-2)!}$$

$$\begin{aligned}
c_0 &= \frac{5}{6}+\log 2-2\log\pi &&= -0.76297926\\
c_1 &= 4\zeta'(-1)+\frac{11}{36}-\frac{11}{3}\log 2 &&= -2.89766868\\
c_2 &= -4\zeta'(-1)-\frac{67}{396}-\frac{1}{3}\log 2 &&= \;\;0.26144360
\end{aligned}$$

Ringwald-Schrempp: $c_2 = 0.291746$

Moch-Ringwald-Schrempp, Balitsky-Braun: $c_2 = 0.153$

First correct $\overline{\mathsf{MS}}$ result

$$n(\varrho) = C_{\overline{\mathsf{MS}}} \left(\frac{16\pi^2}{g^2(\mu)}\right)^{2N} e^{-\frac{8\pi^2}{g^2(\mu)}} \frac{1}{\varrho^5} (\varrho\mu)^{\beta_1} \prod_{i=1}^{N_f} (\varrho m_i(\mu))$$

Finally $T = 0$ instanton size distribution in $\overline{\mathsf{MS}}$ at 1-loop

Once $C_{\overline{\mathsf{MS}}}$ okay, (partial) 2-loop result from literature can be taken over

Semi-classical calculation $T > 0$

$$n(\varrho, T) = n(\varrho)e^{-S(\lambda)} \qquad \lambda = \pi\varrho T$$

$$S(\lambda) = \frac{1}{3}\lambda^2(2N + N_f) + 12A(\lambda)\left(1 + \frac{N - N_f}{6}\right)$$

D. J. Gross, R. D. Pisarski and L. G. Yaffe
Rev. Mod. Phys. 53, 43 (1981)

Semi-classical calculation $T > 0$

$$12A(\lambda) = \frac{1}{16\pi^2}\left[\int_{S^1\times R^3}\left(\frac{\partial_\mu \Pi \partial_\mu \Pi}{\Pi^2}\right)^2 - \int_{R^4}\left(\frac{\partial_\mu \Pi_0 \partial_\mu \Pi_0}{\Pi_0^2}\right)^2\right]$$

- Π_0 from 1-insanton solution on R^4: $\Pi_0 = 1 + \frac{\varrho^2}{t^2+r^2}$
- Π is from Harrington-Sheppard 1-instanton solution on $S^1 \times R^3$

Semi-classical calculation $T > 0$

Because of spherical symmetry, $A(\lambda)$ is a 2-dimensional integral

Analytically not possible, numerical form from Gross-Pisarski-Yaffe:

$$12A_{GPY}(\lambda) = -\log\left(1 + \frac{\lambda^2}{3}\right) + \frac{12\alpha}{\left(1 + \gamma\lambda^{-3/2}\right)^8}$$

$$\alpha = 0.01289764 \qquad\qquad \gamma = 0.15858$$

Claimed absolute numerical uncertainty: $6 \cdot 10^{-4}$

Once $A(\lambda)$ is known, the full $\chi(T)$ is known semi-classically

Above A_{GPY} used in **all** works

New results for $A(\lambda)$

Main motivation was to understand the peculiar form of $A(\lambda)$

In Gross-Pisarski-Yaffe no details are given

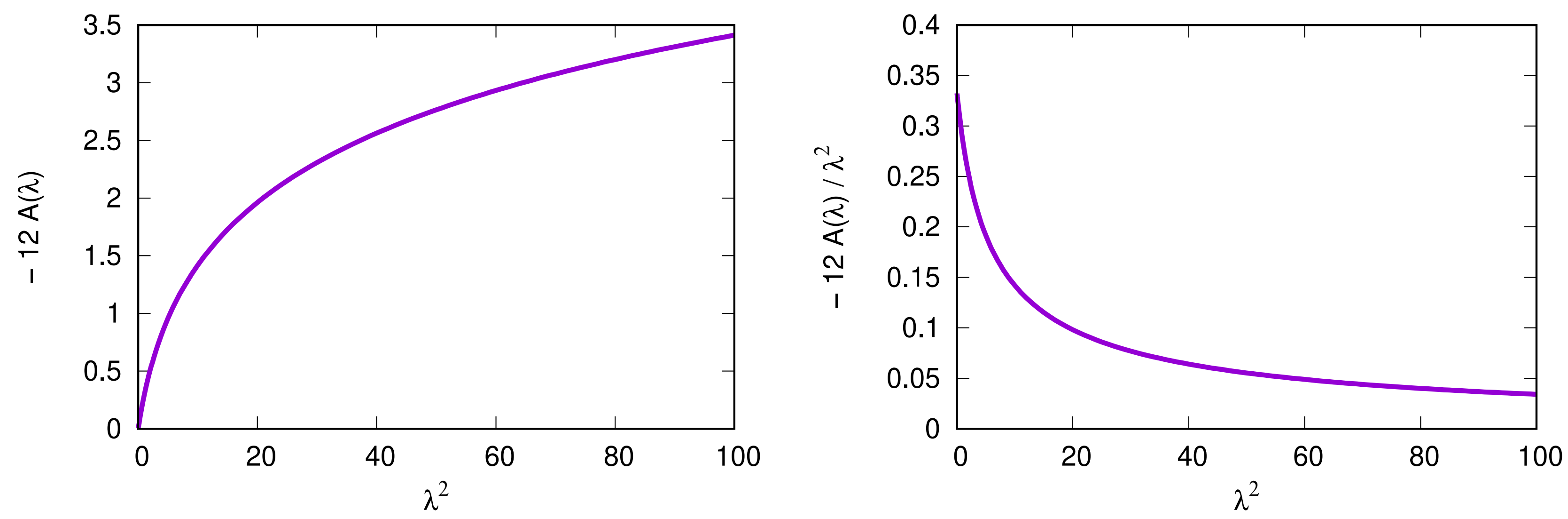
Technically: difference of two 2D integrals, both are divergent, difference finite

We do three things:

- Evaluate numerically to high precision
- Obtain analytic $\lambda \ll 1$ and $\lambda \gg 1$ series
- Fit numerical result with simple function

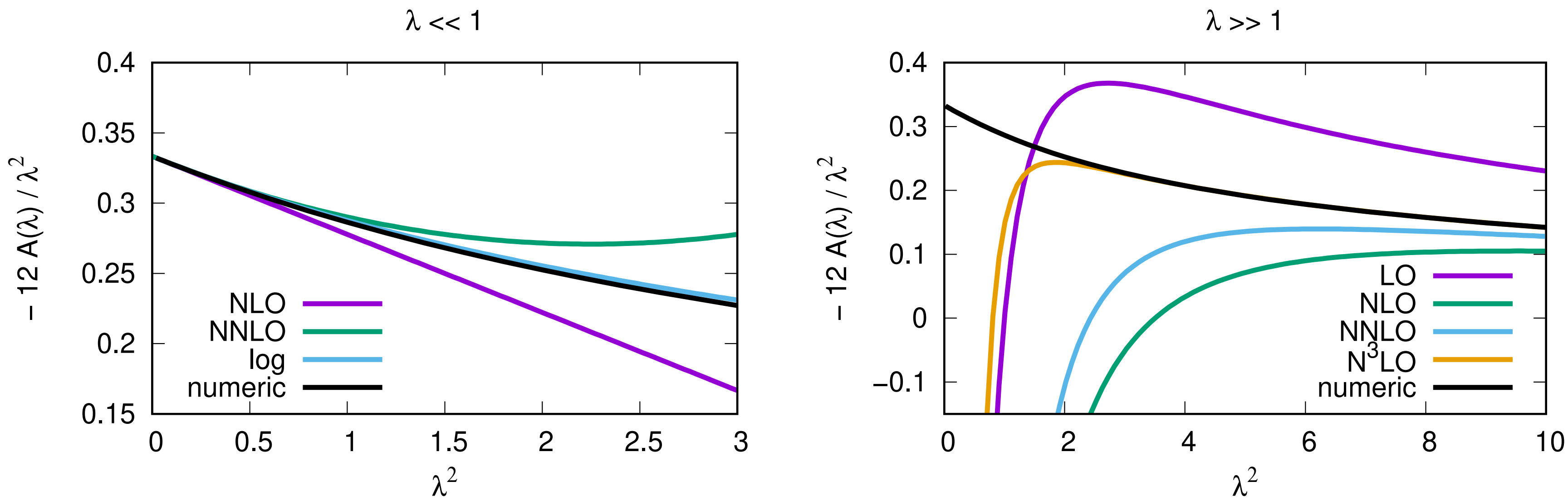
New results for $A(\lambda)$

Numerical evaluation, $O(100)$ significant digits



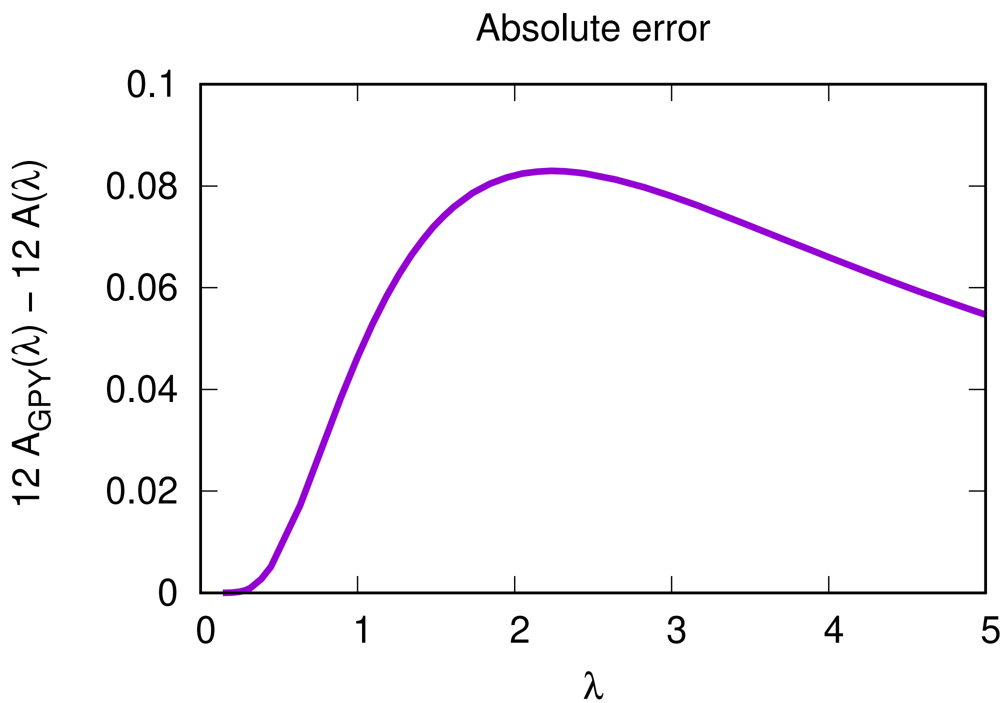
Work out small λ and large λ asymptotics

New results for $A(\lambda)$ - asymptotics



These look good - let's compare with Gross-Pisarski-Yaffe

New results for $A(\lambda)$ - comparison with GPY



$8 \cdot 10^{-2}$, two orders of magnitude worse than claimed!

GPY: 2D integral numerically on computers of 80's ...

New results for $A(\lambda)$ - useful parametrization

$$-12A_{param}(\lambda) = p_0 \log(1 + p_1\lambda^2 + p_2\lambda^4 + p_3\lambda^6 + p_4\lambda^8)$$

$$p_0 = 0.247153244, \quad p_1 = 1.356391323$$

$$p_2 = 0.675021523, \quad p_3 = 0.145446632, \quad p_4 = 0.008359667$$

Absolute precision $2 \cdot 10^{-4}$

Biggest deviation from GPY: $\lambda = O(1)$ because of large cancellations inside $I(r) \rightarrow$ the most sensitive region for ϱ-integral in $\chi(T)$ $\rightarrow$ potentially large effect

Absolute and relative precision

Absolute precision on $A(\lambda) \rightarrow$

Relative precision on $n(\varrho, T) \sim e^{-12A(\lambda)\left(1+\frac{N-N_f}{6}\right)} \rightarrow$

Relative precision on $\chi(T)$

Discrepancy A_{GPY} vs. our A_{param} in $\chi(T)$:

- $SU(3)$ $N_f = 0, 2, 3, 4$: 10%, 7%, 6%, 4%
- $SU(10)$ pure Yang-Mills: 22%
- $SU(20)$ pure Yang-Mills: 40% (scales with N)

Accounting for $T = 0$ and $T > 0$ discrepancies in QCD

$T = 0$ from $C_{\overline{\mathsf{MS}}}$: approx 5% (correct smaller)

$T > 0$ from $A(\lambda)$: approx 5% (correct larger)

But in opposite directions ... nearly cancel

Eventually very small effect in QCD

But at least now the semi-classical result is fully correct

One last thing...

Semi-classical result analytic but still has uncertainty

- μ-dependence, choice for 3, 4, 5-loop running, etc. $\rightarrow$ very small

- Scale $\rightarrow$ dominant

Scale in semi-classical result

Pure Yang-Mills

Units in T_c $\rightarrow$ need $T_c/\Lambda_{\overline{\mathrm{MS}}} = 1.26(7)$ $\rightarrow$ came from lattice $\rightarrow$ has uncertainty $\rightarrow$ surprisingly large in χ because of high power of T

$$\begin{aligned} T/T_c &= 4.1 \\ \log(\chi/T_c^4)\Big|_{lat} &= -12.47(12) \\ \log(\chi/T_c^4)\Big|_{inst} &= -13.80(10)(40) \end{aligned}$$

lat: Jahn Moore Robaina, Phys.Rev.D 98 (2018) 5, 054512, 1806.01162

Deviation within $3\,\sigma$

Scale in semi-classical result

QCD

Units in $\Lambda_{\overline{\mathrm{MS}}} = 292(16)MeV$ PDG, even higher power of T in χ

$$\begin{aligned} T &= 2\,GeV \\ \log(\chi/MeV^4)\Big|_{lat} &= 3.99(68) \\ \log(\chi/MeV^4)\Big|_{inst} &= 1.15(3)(46) \end{aligned}$$

lat: Borsanyi et al., Nature 539, no. 7627, 69 (2016), 1606.07494

Deviation about $3.5\,\sigma$

Summary

- Obtained $n(\varrho, T)$ at high temperature semi-classically in $\overline{\text{MS}}$
- Makes $\chi(T)$ comparison with lattice possible
- Dominant uncertainty from scale
- Exactly **zero** new or original idea :)

Thank you for your attention!