

# NLO JIMWLK Evolution with Massive Quarks

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**(XVth Quark Confinement and the Hadron Spectrum)**

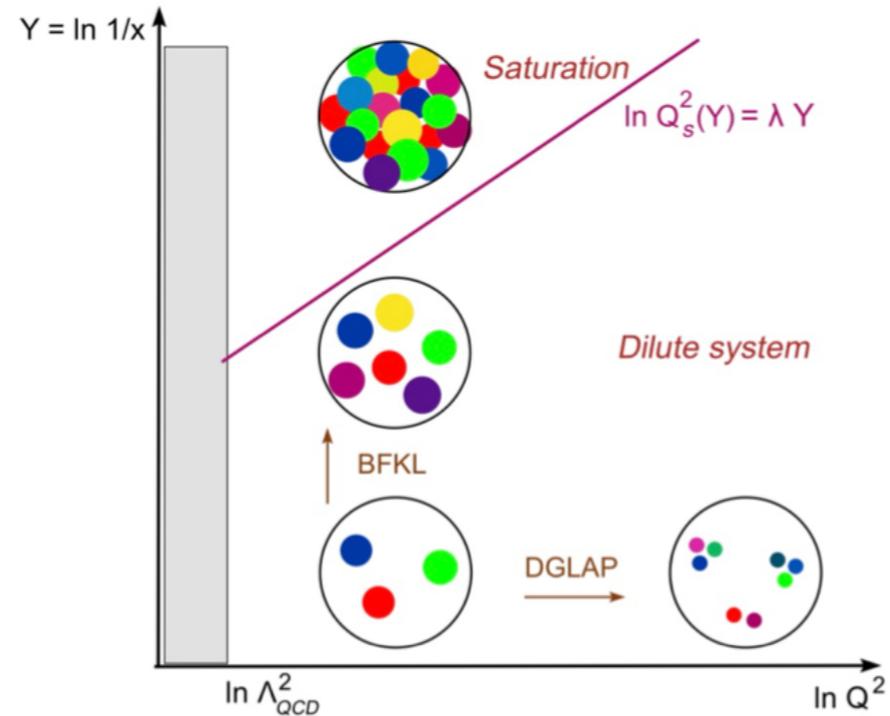
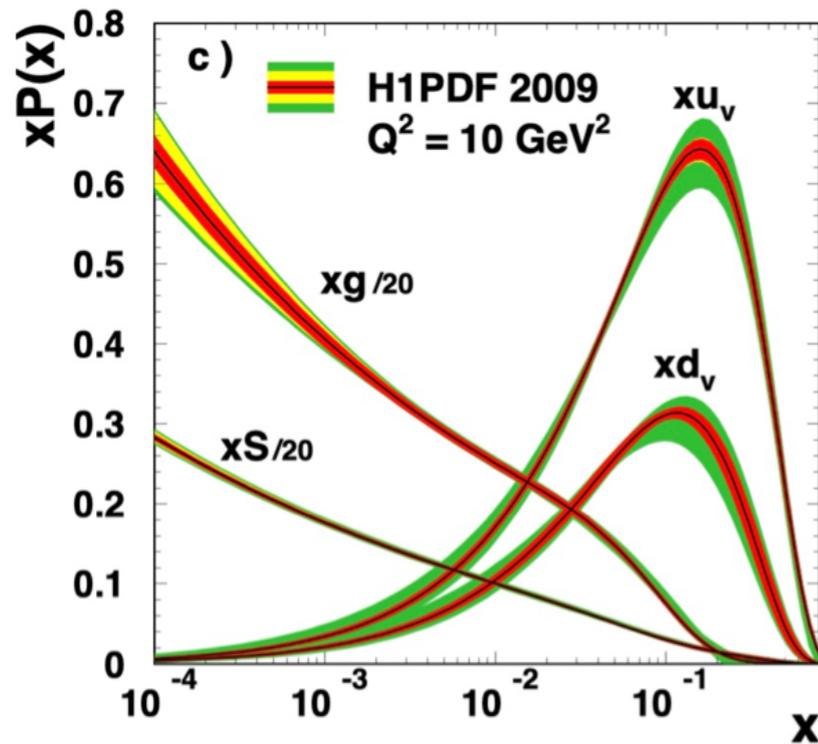
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# Outline

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- ✦ Brief Introductions (why we study JIMWLK with massive quarks)
- ✦ Light-cone wave functions approaches to JIMWLK equations, LO calculations
- ✦ NLO calculations, subtleties with finite mass included
- ✦ Summary

# Brief Introduction



- ◆ JIMWLK Eqns. describe high energy evolution of dense hadronic matter. Generalize the BFKL to nonlinear regime (in the dilute-dilute scattering limit) and is important for studying saturation effects.
- ◆ NLO Effects are known to be phenomenologically important. LO JIMWLK +NLO coupling running Effects gives better description of data than LO JIMWLK equation.
- ◆ Mass Effect are important (heavy quarks provide clean probes to hadron structures); Even at HERA, charm contributes to about 10% of  $F_2$  (at Bjorken  $x = 10^{-2} \sim 10^{-3}$ ). Future EIC heavy probes are also going to play important roles.
- ◆ Note that LO JIMWLK has no Mass Effects at all, so NLO calculations are necessary to see quark mass effects

# LC Quantization

$$H_0 \equiv \int dx^- d^2 \mathbf{x} \left[ \frac{1}{2} (\partial_i A_j^a)^2 + \psi_+^\dagger \frac{-\partial_i \partial_i + m^2}{i \partial^+} \psi_+ \right]$$

Mass dependent terms

$$H_{int} \equiv \int dx^- d^2 \mathbf{x} \left[ -g f^{abc} \partial_i A_j^a A_i^b A_j^c + \frac{g^2}{4} f^{abc} f^{ade} A_i^b A_j^c A_i^d A_j^e \right. \\ -g f^{abc} (\partial_i A_i^a) \frac{1}{\partial^+} (A_j^b \partial^+ A_j^c) + \frac{g^2}{2} f^{abc} f^{ade} \frac{1}{\partial^+} (A_i^b \partial^+ A_i^c) \frac{1}{\partial^+} (A_j^d \partial^+ A_j^e) \\ + 2g^2 f^{abc} \frac{1}{\partial^+} (A_i^b \partial^+ A_i^c) \frac{1}{\partial^+} (\psi_+^\dagger t^a \psi_+) + 2g^2 \frac{1}{\partial^+} (\psi_+^\dagger t^a \psi_+) \frac{1}{\partial^+} (\psi_+^\dagger t^a \psi_+) \\ - 2g (\partial_i A_i^a) \frac{1}{\partial^+} (\psi_+^\dagger t^a \psi_+) - g \psi_+^\dagger t^a (i \alpha_i \partial_i + m \beta) \frac{1}{i \partial^+} (\alpha_j A_j^a \psi_+) \\ \left. - g \psi_+^\dagger t^a \alpha_i A_i^a \frac{1}{i \partial^+} (i \alpha_j \partial_j + m \beta) \psi_+ - i g^2 \psi_+^\dagger t^a t^b \alpha_i A_i^a \frac{1}{\partial^+} (\alpha_j A_j^b \psi_+) \right]$$

Quantization:

$$A_i^a(x) = \int_0^\infty \frac{dk^+}{2\pi} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{1}{\sqrt{2k^+}} \left( a_i^a(k^+, \mathbf{k}) e^{-ik \cdot x} + a_i^{a\dagger}(k^+, \mathbf{k}) e^{ik \cdot x} \right)$$

$$\psi_+^\alpha(x) = \sum_{\lambda=\pm\frac{1}{2}} \chi_\lambda \int_0^\infty \frac{dk^+}{2\pi} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{1}{\sqrt{2}} \left( b_\lambda^\alpha(k^+, \mathbf{k}) e^{-ik \cdot x} + b_\lambda^{\alpha\dagger}(k^+, \mathbf{k}) e^{ik \cdot x} \right)$$

$$\left[ a_i^a(k^+, \mathbf{k}), a_j^{b\dagger}(p^+, \mathbf{p}) \right] = (2\pi)^3 \delta^{ab} \delta_{ij} \delta(k^+ - p^+) \delta^{(2)}(\mathbf{k} - \mathbf{p})$$

$$\left\{ b_{\lambda_1}^\alpha(k^+, \mathbf{k}), b_{\lambda_2}^{\beta\dagger}(p^+, \mathbf{p}) \right\} = \left\{ d_{\lambda_1}^\alpha(k^+, \mathbf{k}), d_{\lambda_2}^{\beta\dagger}(p^+, \mathbf{p}) \right\} \\ = (2\pi)^3 \delta_{\lambda_1 \lambda_2} \delta^{\alpha\beta} \delta^{(2)}(\mathbf{k} - \mathbf{p}) \delta(k^+ - p^+)$$

# LC Quantization

$$H_0 = \int_0^\infty \frac{dk^+}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left( \frac{\mathbf{k}^2}{2k^+} a_i^{a\dagger}(k^+, \mathbf{k}) a_i^a(k^+, \mathbf{k}) + \sum_\lambda \frac{\mathbf{k}^2 + m^2}{2k^+} \left[ b_\lambda^{\alpha\dagger}(k^+, \mathbf{k}) b_\lambda^\alpha(k^+, \mathbf{k}) - d_\lambda^\alpha(k^+, \mathbf{k}) d_\lambda^{\alpha\dagger}(k^+, \mathbf{k}) \right] \right)$$

Energies of free partons:

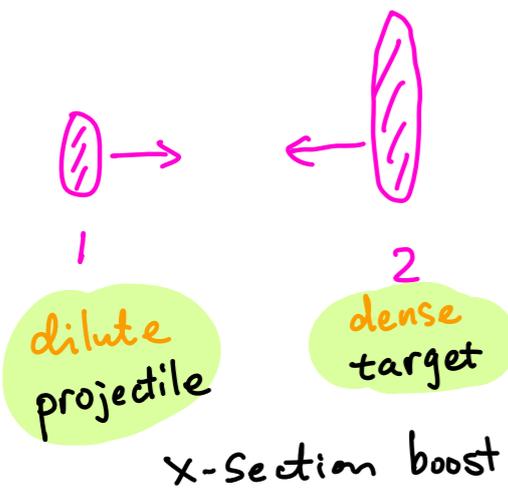
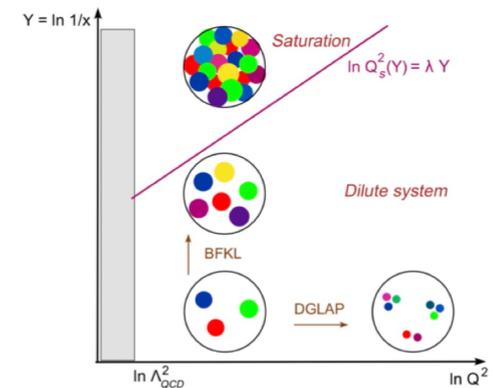
$$E_g(k) = \frac{\mathbf{k}^2}{2k^+} \quad E_q(k) = \frac{\mathbf{k}^2 + m^2}{2k^+}$$

Look like Non-Relativistic dispersion relations

# JIMWLK Equations

JIMWLK :

$$\frac{d}{dY} W_Y[\mathcal{S}] = H^{\text{JIMWLK}}[\mathcal{S}, \frac{\delta}{\delta \mathcal{S}}] W_Y[\mathcal{S}]$$



$$\sigma_{inc} = \int D\mathcal{S}_1 D\mathcal{S}_2 \sigma_{Y_1}[\mathcal{S}_1] W_{Y_2}[\mathcal{S}_2]$$

$$Y_1 + Y_2 = Y$$

$$\frac{d\sigma_{inc}}{dY_{1,2}} = 0$$

$$\frac{d\sigma_Y}{dY} = -H^{\text{JIMWLK}} \sigma_Y$$

Forward scatt amplitude.

inclusive: Optical theorem:  $\Sigma_Y \equiv \langle \Psi | S - 1 | \Psi \rangle_Y$

$$\text{Im } \Sigma_Y = \sigma_Y$$

$$\frac{d}{dY} \Sigma = -H^{\text{JIMWLK}} \Sigma$$

- Goal: Calculate  $|\Psi\rangle_Y$  up to  $\mathcal{O}(g^3)$  with massive quarks and extract  $H^{\text{JIMWLK}}$

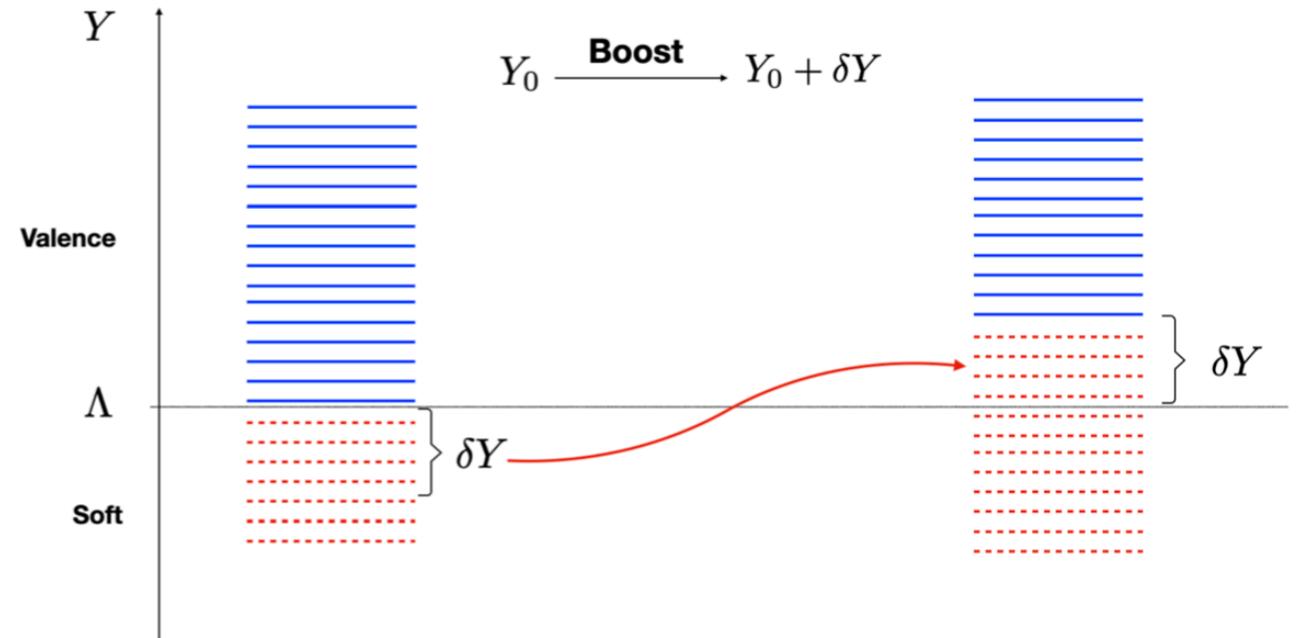
# LC wave function

## Born-Oppenheimer:

**Valence Modes:** modes above  $\Lambda$ ,  
Large  $Y$ , slow modes. Background Charge as  
a source of the soft modes.

**Soft Modes:** Modes below  $\Lambda$ . Not energetic  
Enough to participate in scattering

$\Lambda$  depends on the scattering process involved



$$|\Psi\rangle_{Y_0} = |v\rangle \longrightarrow |\Psi\rangle_{Y_0 + \delta Y} = |\psi\rangle \otimes |v\rangle$$

Consider the scattering of soft  $|\psi\rangle$  living in a small rapidity window  $\delta Y$ ,  
and to extract JIMWLK Hamiltonian using:

$$\Sigma \equiv \langle \psi | \hat{S} - 1 | \psi \rangle$$

Need to integrate out modes above  $\Lambda$ .

$$\psi_+^\alpha(x) = \underline{\psi}_+^\alpha(x) + \bar{\psi}_+^\alpha(x), \quad A_i^a(x) = \underline{A}_i^a(x) + \bar{A}_i^a(x)$$

Leading power Hamiltonian:

$$\lambda \sim p_{\text{soft}}^+ / p_{\text{valence}}^+ \quad \frac{1}{\partial^+} \left( \underline{A}_j^b \partial^+ \bar{A}_j^c \right) \sim \lambda^0 \quad \frac{1}{\partial^+} \left( \bar{A}_j^b \partial^+ \bar{A}_j^c \right) \sim \lambda^{-1}$$

# LO JIMWLK

**Leading Order interaction:**

$$H_g \equiv -g \int dx_+ d^2\mathbf{x} (\partial_i A_i^a) \left( f^{abc} \frac{1}{\partial^+} (\bar{A}_j^b \partial^+ \bar{A}_j^c) + 2 \frac{1}{\partial^+} (\bar{\psi}_+^\dagger t^a \bar{\psi}_+) \right)$$

**Modes expansion:**

$$H_g = \int_{\Lambda}^{e^{\delta Y} \Lambda} \frac{dk^+}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{g\mathbf{k}^i}{\sqrt{2} |k^+|^{3/2}} \left[ a_i^{a\dagger}(k^+, \mathbf{k}) \rho^a(-\mathbf{k}) + a_i^a(k^+, \mathbf{k}) \rho^a(\mathbf{k}) \right]$$

$$\rho_g^a(-\mathbf{p}) \equiv -i f^{abc} \int_{e^{\delta Y} \Lambda}^{\infty} \frac{dk^+}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} a_j^{\dagger b}(k^+, \mathbf{k}) a_j^c(k^+, \mathbf{k} + \mathbf{p}),$$

Non-commutative charge distribution, need to be careful when do perturbation

**Eikonal Scattering (Soft Modes)**

$$\hat{S} a_i^{a\dagger}(x^+, \mathbf{x}) |0\rangle = S_A^{ab}(\mathbf{x}) a_i^{b\dagger}(x^+, \mathbf{x}) |0\rangle, \quad S_A^{ab}(\mathbf{x}) = \left[ \mathcal{P} \exp \left\{ ig \int dx^+ T^c A^{-c}(x^+, \mathbf{x}) \right\} \right]^{ab}$$

For highly boosted partons, good approximation.

Actual form of  $S_A^{ab}$  is not important, the dense target just color rotate the projectile

# LO JIMWLK

## Eikonal Scattering (Valence Modes)

$$\rho_g^a(\mathbf{x}) \hat{S} |v\rangle = J_{R,adj}^a(\mathbf{x}) |\hat{S}v\rangle, \quad \hat{S}\rho_g^a(\mathbf{x}) |v\rangle = J_{L,adj}^a(\mathbf{x}) |\hat{S}v\rangle$$

$$J_{R,adj}^a(\mathbf{x}) \equiv -tr \left[ S_A(\mathbf{x}) T^a \frac{\delta}{\delta S_A^\dagger(\mathbf{x})} \right];$$

$$J_{L,adj}^a(\mathbf{x}) \equiv [S_A(\mathbf{x}) J_R(\mathbf{x})]^a = -tr \left[ T^a S_A(\mathbf{x}) \frac{\delta}{\delta S_A^\dagger(\mathbf{x})} \right]$$

Why the notation?

$$\rho_g^a(-\mathbf{p}) \equiv -i f^{abc} \int_{e^{\delta\gamma\Lambda}} \frac{dk^+}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} a_j^{\dagger b}(k^+, \mathbf{k}) a_j^c(k^+, \mathbf{k} + \mathbf{p}),$$

$$tr \left[ M \frac{\delta}{\delta S^T} \right] \hat{S} a^{k\dagger} = M_{ab} \frac{\delta}{\delta S_{ba}^T} S^{kl} a^{l\dagger} = M_{ab} \delta_a^k \delta_b^l a^{l\dagger} = M_{kl} a^{l\dagger}$$

$$\left[ \hat{S} \rho^a \right] a^{k\dagger} = \hat{S} T_{bc}^a a^{b\dagger} a^c a^{k\dagger}$$

$$\rightarrow \hat{S} T_{bc}^a a^{b\dagger} \delta^{ck} = T_{bk}^a S^{bl} a^{l\dagger} = -T_{kb}^a S^{bl} a^{l\dagger} = -(T^a S)_{kl} a^{l\dagger}$$

# LO JIMWLK

OFP:

$$|\psi^{LO}\rangle = \mathcal{N}^{LO} |0\rangle - |i\rangle \frac{\langle i | H_{int} | 0 \rangle}{E_i} \quad \mathcal{N}^{LO} \equiv 1 - \frac{|\langle i | H_{int} | 0 \rangle|^2}{2E_i^2}.$$

$$E_g(k) = \frac{\mathbf{k}^2}{2k^+} \quad \langle g_i^a(k) | H_g | 0 \rangle = \frac{g \mathbf{k}^i \rho^a(-\mathbf{k})}{4\pi^{3/2} |k^+|^{3/2}}.$$

$$H_g = \int_{\Lambda}^{e^{\delta Y} \Lambda} \frac{dk^+}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{g \mathbf{k}^i}{\sqrt{2} |k^+|^{3/2}} \left[ a_i^{a\dagger}(k^+, \mathbf{k}) \rho^a(-\mathbf{k}) + a_i^a(k^+, \mathbf{k}) \rho^a(\mathbf{k}) \right]$$



$$|\psi^{LO}\rangle = \mathcal{N}^{LO} |0\rangle - \int_{\Lambda}^{e^{\delta Y} \Lambda} dk^+ \int d^2\mathbf{k} \frac{g \mathbf{k}^i}{2\pi^{3/2} \sqrt{k^+} k^2} \rho^a(-\mathbf{k}) |g_i^a(k)\rangle$$

To coordinate space:

$$|\psi^{LO}\rangle = \mathcal{N}^{LO} |0\rangle + \int_{\Lambda}^{e^{\delta Y} \Lambda} \frac{dk^+}{\sqrt{k^+}} \int_{\mathbf{x}, \mathbf{z}} \frac{i g X^i}{2\pi^{3/2} X^2} \rho^a(\mathbf{x}) |g_i^a(k^+, \mathbf{z})\rangle.$$

$$X^i \equiv \mathbf{x}^i - \mathbf{z}^i$$

$$Y^i \equiv \mathbf{y}^i - \mathbf{z}^i$$

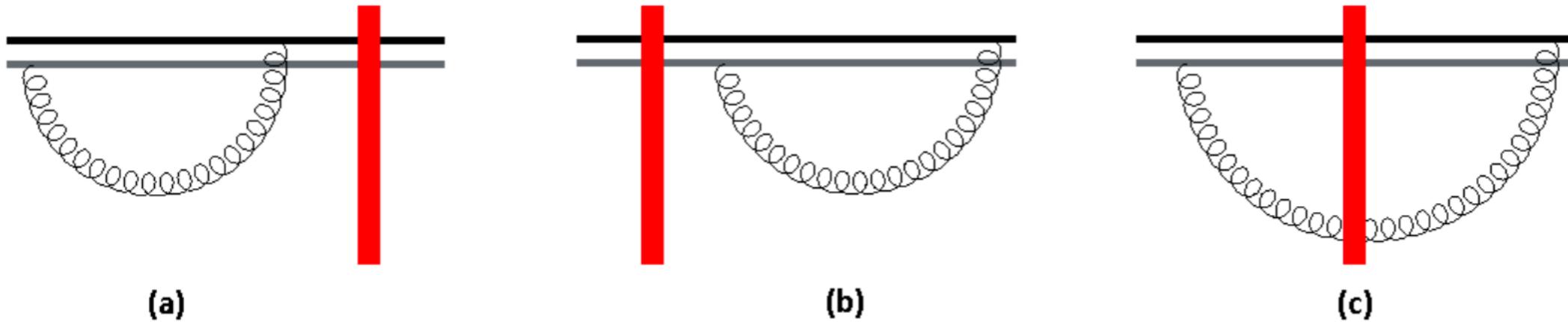
$$\mathcal{N}^{LO} = 1 - \frac{g^2}{8\pi^3} \int_{\Lambda}^{e^{\delta Y} \Lambda} \frac{dk^+}{k^+} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \rho^a(\mathbf{y}) \rho^a(\mathbf{x})$$

# LO JIMWLK

$$\hat{S} |\psi^{LO}\rangle = \left( 1 - \frac{g^2}{8\pi^3} \int_{\Lambda} e^{\delta Y \Lambda} \frac{dk^+}{k^+} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \hat{S} \rho^a(\mathbf{y}) \rho^a(\mathbf{x}) \right) |0\rangle + \int_{\Lambda} e^{\delta Y \Lambda} \frac{dk^+}{\sqrt{k^+}} \int_{\mathbf{x}, \mathbf{z}} \frac{i g X^i}{2\pi^{3/2} X^2} \hat{S} \rho^a(\mathbf{x}) |g_i^a(k^+, \mathbf{z})\rangle.$$

$$\begin{aligned} \Sigma^{LO} &\equiv \langle \psi^{LO} | \hat{S} - 1 | \psi^{LO} \rangle \\ &= -\frac{\alpha_s}{2\pi^2} \delta Y \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \left[ J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) - 2J_L^a(\mathbf{x}) S_A^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right] \end{aligned}$$

$$H_{JIMWLK}^{LO} = \frac{\alpha_s}{2\pi^2} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \left[ J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) - 2J_L^a(\mathbf{x}) S_A^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right]$$



$$K_{JSJ}^{LO}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{\alpha_s}{2\pi^2} \frac{X \cdot Y}{X^2 Y^2}$$

# NLO with Massive Quarks

$$\begin{aligned}
 H_{JIMWLK}^{NLO} = & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} K_{JSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \left[ J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) - 2J_L^a(\mathbf{x}) S_A^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right] \\
 & + \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} K_{JSSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') \left[ f^{abc} f^{def} J_L^a(\mathbf{x}) S_A^{be}(\mathbf{z}) S_A^{cf}(\mathbf{z}') J_R^d(\mathbf{y}) - N_c J_L^a(\mathbf{x}) S_A^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right] \\
 & + \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} K_{q\bar{q}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') \left[ 2 J_L^a(\mathbf{x}) \text{tr}[S^\dagger(\mathbf{z}) t^a S(\mathbf{z}') t^b] J_R^b(\mathbf{y}) - J_L^a(\mathbf{x}) S_A^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right] \\
 & + \int_{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} K_{JJSSJ}(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') f^{acb} \left[ J_L^d(\mathbf{x}) J_L^e(\mathbf{y}) S_A^{dc}(\mathbf{z}) S_A^{eb}(\mathbf{z}') J_R^a(\mathbf{w}) \right. \\
 & \left. - J_L^a(\mathbf{w}) S_A^{cd}(\mathbf{z}) S_A^{be}(\mathbf{z}') J_R^d(\mathbf{x}) J_R^e(\mathbf{y}) + \frac{1}{3} (J_L^c(\mathbf{x}) J_L^b(\mathbf{y}) J_L^a(\mathbf{w}) - J_R^c(\mathbf{x}) J_R^b(\mathbf{y}) J_R^a(\mathbf{w})) \right] \\
 & + \int_{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}} K_{JJSSJ}(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}) f^{bde} \left[ J_L^d(\mathbf{x}) J_L^e(\mathbf{y}) S_A^{ba}(\mathbf{z}) J_R^a(\mathbf{w}) \right. \\
 & \left. - J_L^a(\mathbf{w}) S_A^{ab}(\mathbf{z}) J_R^d(\mathbf{x}) J_R^e(\mathbf{y}) + \frac{1}{3} (J_L^d(\mathbf{x}) J_L^e(\mathbf{y}) J_L^b(\mathbf{w}) - J_R^d(\mathbf{x}) J_R^e(\mathbf{y}) J_R^b(\mathbf{w})) \right].
 \end{aligned}$$

Quark mass will modify this  
Two kernels

$$\begin{aligned}
 X^i & \equiv \mathbf{x}^i - \mathbf{z}^i \\
 Y^i & \equiv \mathbf{y}^i - \mathbf{z}^i \\
 Z^i & \equiv \mathbf{z} - \mathbf{z}'
 \end{aligned}$$

$$\begin{aligned}
 K_{q\bar{q}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') & = \frac{\alpha_s^2 N_f}{8\pi^4} \left( \frac{2}{Z^4} - \frac{(X')^2 Y^2 + (Y')^2 X^2 - (X - Y)^2 Z^2}{Z^4 (X^2 (Y')^2 - (X')^2 Y^2)} \left( \frac{X^2 (Y')^2}{(X')^2 Y^2} \right) \right. \\
 & \left. - \frac{I_f(\mathbf{x}, \mathbf{z}, \mathbf{z}')}{Z^2} - \frac{I_f(\mathbf{y}, \mathbf{z}, \mathbf{z}')}{Z^2} \right), \quad I_f(\mathbf{x}, \mathbf{z}, \mathbf{z}') \equiv \frac{2}{Z^2} - \frac{2\mathbf{X} \cdot \mathbf{X}'}{Z^2 (X^2 - (X')^2)} \ln \left( \frac{X^2}{(X')^2} \right)
 \end{aligned}$$

Mass relevant:  $K_{JSJ}, K_{q\bar{q}}$  ;

Note subtraction term in  $K_{q\bar{q}}$  !  $K_{JSJ} = K'_{JSJ} - \frac{1}{2} \int_{\mathbf{z}'} K_{q\bar{q}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}')$

# NLO with Massive Quarks

$$H_{gqq} = \sum_{\lambda_1, \lambda_2 = \pm \frac{1}{2}} \int_{\Lambda} e^{\delta_Y \Lambda} \frac{dk^+}{2\pi} \frac{dp^+}{2\pi} dq^+ \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{d^2\mathbf{p}}{(2\pi)^2} d^2\mathbf{q} \frac{gt_{\alpha\beta}^a}{2\sqrt{2k^+}} \delta^{(3)}(k-p-q) \Gamma_{\lambda_1\lambda_2}^i$$

$$\times \left( a_i^a(k^+, \mathbf{k}) b_{\lambda_1}^{\alpha\dagger}(p^+, \mathbf{p}) d_{\lambda_2}^{\beta\dagger}(q^+, \mathbf{q}) + h.c. \right) \quad \Gamma_{\lambda_1\lambda_2}^i = \chi_{\lambda_1}^\dagger \left[ \frac{2k_i}{k^+} - \frac{\alpha \cdot \mathbf{p} + m\beta}{p^+} \alpha_i - \alpha_i \frac{\alpha \cdot \mathbf{q} - m\beta}{q^+} \right] \chi_{\lambda_2}$$

$$H_{qq-inst} \equiv 2g^2 \int dx^- d^2\mathbf{x} \frac{1}{\partial^+} \left( \underline{\psi}_+^\dagger t^a \underline{\psi}_+ \right) \left( f^{abc} \frac{1}{\partial^+} \left( \bar{A}_j^b \partial^+ \bar{A}_j^c \right) + 2 \frac{1}{\partial^+} \left( \bar{\psi}_+^\dagger t^a \bar{\psi}_+ \right) \right)$$

$$\left\langle \bar{q}_{\lambda_2}^\delta(q) q_{\lambda_1}^\gamma(p) | H_{gqq} | g_i^a(k) \right\rangle = \frac{gt_{\gamma\delta}^a}{8\pi^{3/2}\sqrt{k^+}} \chi_{\lambda_1}^\dagger \left[ \frac{2\mathbf{k}^i}{k^+} - \frac{\alpha \cdot \mathbf{p} + m\beta}{p^+} \alpha^i - \alpha^i \frac{\alpha \cdot \mathbf{q} - m\beta}{q^+} \right] \chi_{\lambda_2}$$

$$\times \delta^{(3)}(k-p-q)$$

$$\left\langle \bar{q}_{\lambda_2}^\beta(q) q_{\lambda_1}^\alpha(p) | H_{qq-inst} | 0 \right\rangle = \frac{g^2 t_{\alpha\beta}^a \rho^a(-\mathbf{p}-\mathbf{q})}{8\pi^3 (p^+ + q^+)^2} \chi_{\lambda_1}^\dagger \chi_{\lambda_2}$$

# NLO with Massive Quarks ( $K_{JSJ}^{NLO}$ )

$$|\psi_g^{1'}\rangle \equiv - \int_{\Lambda} e^{\delta Y} dk^+ dp^+ dq^+ dr^+ \int d^2\mathbf{k} d^2\mathbf{p} d^2\mathbf{q} d^2\mathbf{r}$$

$$\times |g_j^d(r)\rangle \frac{\langle g_j^d(r) | H_{gqq} | q_{\lambda_1}^\alpha(p) \bar{q}_{\lambda_2}^\beta(q) \rangle \langle q_{\lambda_1}^\alpha(p) \bar{q}_{\lambda_2}^\beta(q) | H_{gqq} | g_i^a(k) \rangle \langle g_i^a(k) | H_g | 0 \rangle}{E_g(r) E_{q\bar{q}}(p, q) E_g(k)}$$

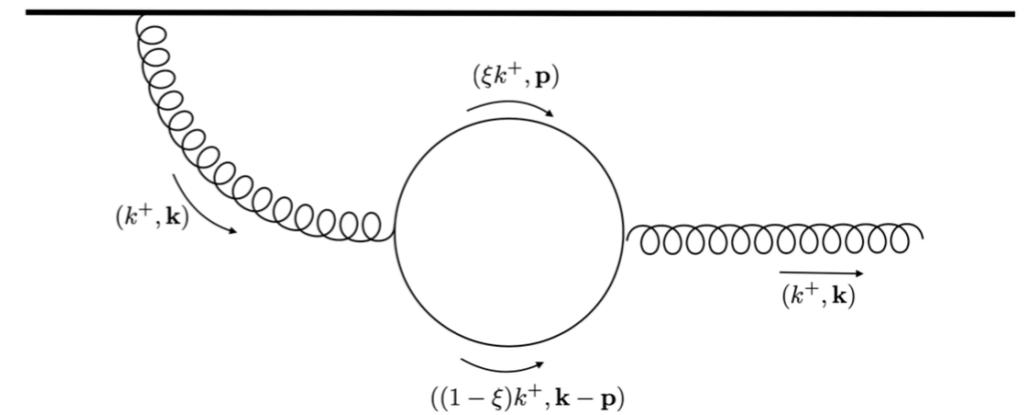
$$= \int_{\Lambda} e^{\delta Y} dk^+ dp^+ \int d^2\mathbf{k} d^2\mathbf{p} \frac{g^3 \rho^a(-\mathbf{k}) \text{tr}[t^a t^d] k_i}{32\pi^{9/2} \sqrt{k^+} \mathbf{k}^4 \left( \frac{\mathbf{p}^2 + m^2}{p^+} + \frac{(\mathbf{k}-\mathbf{p})^2 + m^2}{k^+ - p^+} \right)}$$

$$\times \chi_{\lambda_2}^\dagger \left( \frac{2k_j}{k^+} - \alpha_j \frac{\alpha \cdot \mathbf{p} + \beta m}{p^+} - \frac{\alpha \cdot (\mathbf{k} - \mathbf{p}) - \beta m}{k^+ - p^+} \alpha_j \right) \chi_{\lambda_1}$$

$$\times \chi_{\lambda_1}^\dagger \left( \frac{2k_i}{k^+} - \frac{\alpha \cdot \mathbf{p} + \beta m}{p^+} \alpha_i - \alpha_i \frac{\alpha \cdot (\mathbf{k} - \mathbf{p}) - \beta m}{k^+ - p^+} \right) \chi_{\lambda_2} |g_j^d(k)\rangle$$

$$= - \int_{\Lambda} e^{\delta Y} dk^+ \int_0^1 d\xi \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{64\pi^{7/2}} \frac{g^3 \rho^a(-\mathbf{k}) k_i}{\sqrt{k^+} \mathbf{k}^2}$$

$$\times \left[ \left( 4\xi^2 - 4\xi + 2 - \frac{4m^2}{\mathbf{k}^2} \right) \left( -\frac{2}{\epsilon} + \ln \frac{\xi(1-\xi)\mathbf{k}^2 + m^2}{\mu_{MS}^2} - 1 \right) + 2 \right] |g_i^a(k)\rangle$$

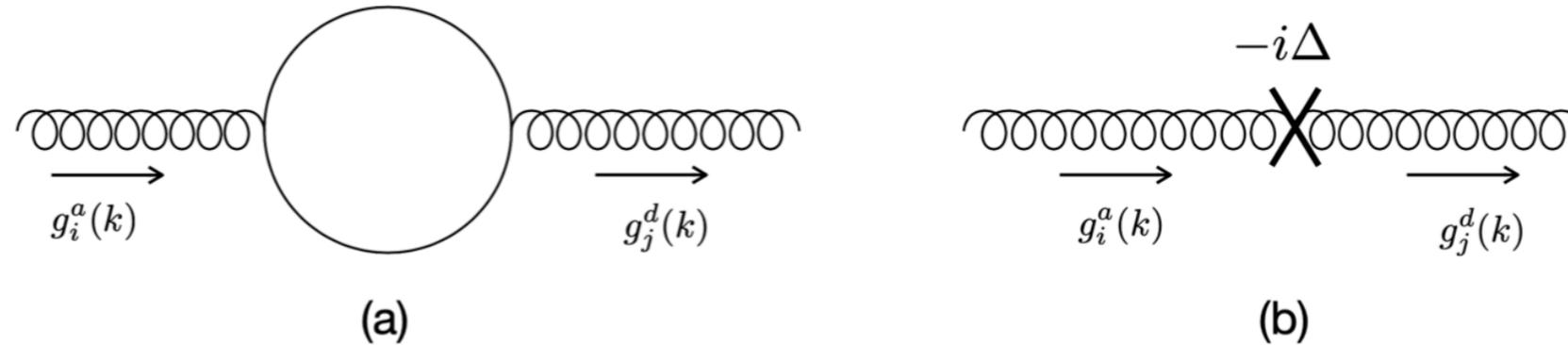


New divergence appear because of mass!  $\frac{1}{\mathbf{k}^2}$  divergence! (Not really IR, weird features of LC quantization)

Cancelled by a mass dependent counter term.

# NLO with Massive Quarks ( $K_{JSJ}^{NLO}$ )

Why need a mass dependent counter term?



LC dispersion relation:  $m^2 = k^+ k^- - \mathbf{k}_\perp^2$ . At fixed  $(k^+, k_\perp)$ :  $\delta E_g(k) = \frac{\delta m^2}{2k^+}$ .

OFP, 2nd order energy correction:

$$\begin{aligned} \delta E_g(k) &= - \int dp^+ dq^+ \int d^2\mathbf{p} d^2\mathbf{q} \langle g_j^d(r) | H_{gqq} | q_{\lambda_1}^\alpha(p) \bar{q}_{\lambda_2}^\beta(q) \rangle \langle q_{\lambda_1}^\alpha(p) \bar{q}_{\lambda_2}^\beta(q) | H_{gqq} | g_i^a(k) \rangle \\ &\quad \times \frac{1}{E_{q\bar{q}}(p, q) - E_g(k)} \\ &= - \frac{2g^2 \text{tr}[t^d t^a]}{8\pi k^+} \delta^3(r - k) \delta_{ij} \int_0^1 d\xi \frac{1}{\xi(1-\xi)d} \int \frac{d^d \tilde{\mathbf{p}}}{(2\pi)^d} \frac{(4\xi^2 - 4\xi + d) \mathbf{p}^2 + m^2 d}{\tilde{\mathbf{p}}^2 + m^2} \end{aligned}$$

**[Zhang, Harindranath:  
PhysRevD.48.4881,  
PhysRevD.48.4868,  
PhysRevD.48.4903 ]**

To ensure the vanishing of  $\delta E_g(k)$ :

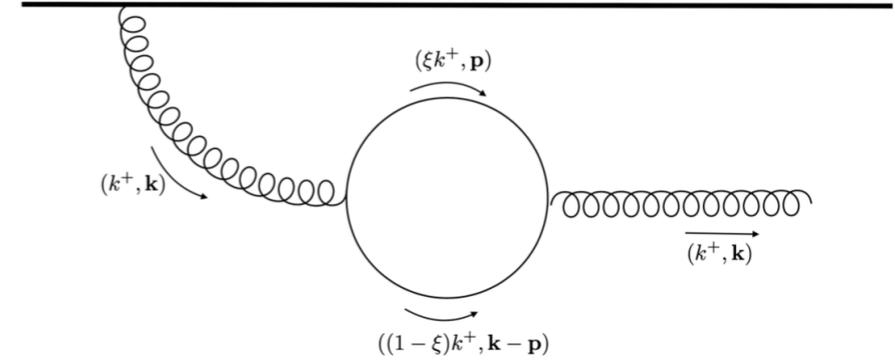
$$H_{gg}^{c.t.} = \frac{1}{2} \Delta (A_i^a)^2,$$

where

$$\Delta = \frac{g^2}{4\pi} \int_0^1 d\xi \frac{1}{\xi(1-\xi)d} \int \frac{d^d \tilde{\mathbf{p}}}{(2\pi)^d} \frac{(4\xi^2 - 4\xi + d) \tilde{\mathbf{p}}^2 + m^2 d}{\tilde{\mathbf{p}}^2 + m^2}.$$

# NLO with Massive Quarks ( $K_{JSJ}^{NLO}$ )

Adding counter term contribution to the loop diagram:



$$|\psi_g^1\rangle = - \int_{\Lambda} e^{\delta Y \Lambda} dk^+ \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{d^d \tilde{\mathbf{p}}}{(2\pi)^d} \int_0^1 d\xi \frac{g^3 \rho^a(-\mathbf{k}) k_i}{2\sqrt{\pi k^+} \mathbf{k}^4} \frac{1}{\xi(1-\xi)d} [(4\xi^2 - 4\xi + d) \tilde{\mathbf{p}}^2 + m^2 d] \\ \times \left( \frac{1}{\xi(1-\xi)\mathbf{k}^2 + \tilde{\mathbf{p}}^2 + m^2} - \frac{1}{\tilde{\mathbf{p}}^2 + m^2} \right) |g_i^a(k)\rangle.$$

How the  $1/\mathbf{k}^2$  divergence cancels:  $\left( \frac{1}{\xi(1-\xi)\mathbf{k}^2 + \tilde{\mathbf{p}}^2 + m^2} - \frac{1}{\tilde{\mathbf{p}}^2 + m^2} \right) = -\frac{\xi(1-\xi)\mathbf{k}^2}{(\xi(1-\xi)\mathbf{k}^2 + \tilde{\mathbf{p}}^2 + m^2)(\tilde{\mathbf{p}}^2 + m^2)}$ .

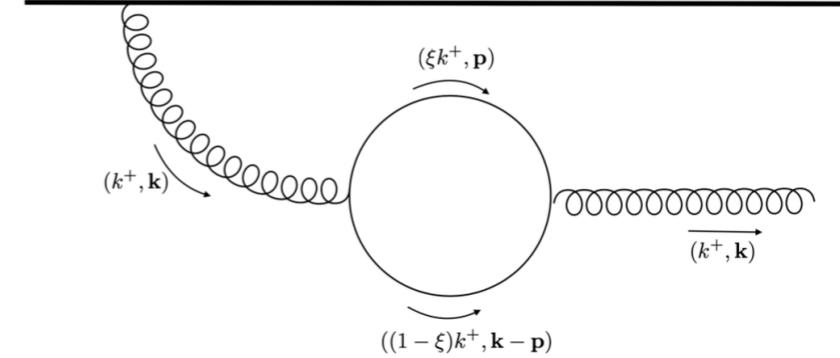
Final result:

$$|\psi_g^1\rangle = \int_{\Lambda} e^{\delta Y \Lambda} \frac{dk^+}{\sqrt{k^+}} \int_{\mathbf{xz}} \frac{g^3 \rho^a(\mathbf{x}) iX_i}{32\pi^{3/2} X^2} \left[ \frac{2}{3} \left( -\frac{2}{\epsilon} + \ln \frac{m^2}{\bar{\mu}^2} \right) + g_1(m|X|) \right] |g_i^a(k^+, \mathbf{z})\rangle$$

$$g_1(x) \equiv \int_0^1 d\xi \int_0^1 dt \left\{ 2(2\xi^2 - 2\xi + 1) K_0 \left[ \frac{x}{\sqrt{t\xi(1-\xi)}} \right] + \frac{x}{\sqrt{t\xi(1-\xi)}} K_1 \left[ \frac{x}{\sqrt{t\xi(1-\xi)}} \right] \right\}$$

# NLO with Massive Quarks ( $K_{JSJ}^{NLO}$ )

Kernel:



$$\begin{aligned}
 \Sigma_{JSJ}^{\text{NLO}} &\equiv \langle \psi_g^1 | \hat{S} | \psi_g^{\text{LO}} \rangle + \langle \psi_g^{\text{LO}} | \hat{S} | \psi_g^1 \rangle \\
 &= \int_{\Lambda} e^{\delta Y \Lambda} \frac{dk^+}{\sqrt{k^+}} \int_{\mathbf{xz}} \frac{g^3 \rho^a(\mathbf{x})}{32\pi^{7/2}} \frac{-iX_i}{X^2} \left[ \frac{2}{3} \left( -\frac{2}{\epsilon} + \ln \frac{m^2}{\bar{\mu}^2} \right) + g_1(m|X|) \right] \langle g_i^a(k^+, \mathbf{z}) | \\
 &\quad \times (\hat{S} - 1) \int_{\Lambda} e^{\delta Y \Lambda} \frac{dk'^+}{\sqrt{k'^+}} \int_{\mathbf{y}, \mathbf{z}'} \frac{igY^{i'}}{2\pi^{3/2}Y'^2} \rho^{a'}(\mathbf{y}) | g_{i'}^{a'}(k'^+, \mathbf{z}') \rangle + (\psi_g^1 \leftrightarrow \psi_g^{\text{LO}}) \\
 &= \delta Y \frac{g^4}{64\pi^5} \int_{\mathbf{xyz}} \frac{X \cdot Y}{X^2 Y^2} J_L^a(\mathbf{x}) S^{ab} J_R^b(\mathbf{y}) \left[ \frac{2}{3} \left( -\frac{2}{\epsilon} + \ln \frac{m^2}{\bar{\mu}^2} \right) + g_1(m|X|) \right] + (\psi_g^1 \leftrightarrow \psi_g^{\text{LO}}) \\
 &= \delta Y \frac{g^4}{64\pi^5} \int_{\mathbf{xyz}} \frac{X \cdot Y}{X^2 Y^2} J_L^a(\mathbf{x}) S^{ab} J_R^b(\mathbf{y}) \left[ \frac{4}{3} \left( -\frac{2}{\epsilon} + \ln \frac{m^2}{\bar{\mu}^2} \right) + g_1(m|X|) + g_1(m|Y|) \right],
 \end{aligned}$$

$$K_{JSJ}^m = \frac{g^4}{128\pi^5} \frac{X \cdot Y}{X^2 Y^2} \left[ \frac{4}{3} \ln \frac{m^2}{\bar{\mu}^2} + g_1(m|X|) + g_1(m|Y|) \right]$$

$$g_1(x) \equiv \int_0^1 d\xi \int_0^1 dt \left\{ 2(2\xi^2 - 2\xi + 1) K_0 \left[ \frac{x}{\sqrt{t\xi(1-\xi)}} \right] + \frac{x}{\sqrt{t\xi(1-\xi)}} K_1 \left[ \frac{x}{\sqrt{t\xi(1-\xi)}} \right] \right\}$$

$$g_1(x) = \frac{2}{3} (-2\gamma_E - \ln \frac{x^2}{4}) - \frac{10}{9} + \mathcal{O}(x^2)$$

# NLO with Massive Quarks

$$|\psi_{q\bar{q}}^1\rangle \equiv \sum_{\lambda_1, \lambda_2} \int_{\Lambda} e^{\delta Y \Lambda} dk^+ dp^+ dq^+ \int d^2\mathbf{k} d^2\mathbf{p} d^2\mathbf{q}$$

$$\times \left| \bar{q}_{\lambda_2}^{\beta}(q) q_{\lambda_1}^{\alpha}(p) \right\rangle \frac{\langle \bar{q}_{\lambda_2}^{\beta}(q) q_{\lambda_1}^{\alpha}(p) | H_{gq\bar{q}} | g_i^a(k) \rangle \langle g_i^a(k) | H_g | 0 \rangle}{E_{q\bar{q}}(p, q) E_g(k)}$$

$$E_{q\bar{q}}(p, q) = \frac{\mathbf{p}^2 + m^2}{2p^+} + \frac{\mathbf{q}^2 + m^2}{2q^+}.$$

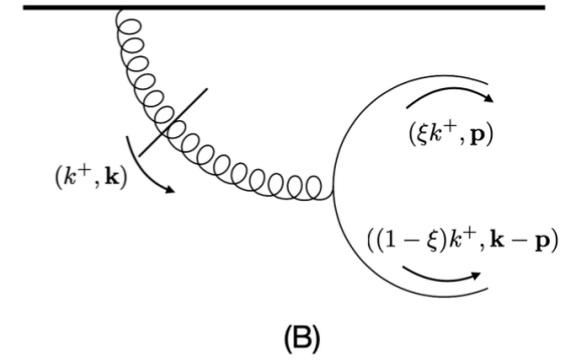
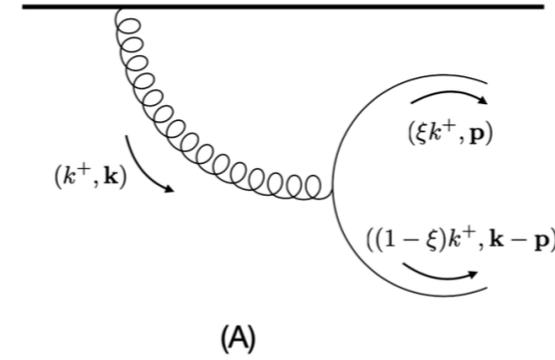
$$|\psi_{q\bar{q}}^2\rangle \equiv - \sum_{\lambda_1, \lambda_2} \int_{\Lambda} e^{\delta Y \Lambda} dk^+ dp^+ \int d^2\mathbf{k} d^2\mathbf{p} \left| \bar{q}_{\lambda_2}^{\beta}(k-p) q_{\lambda_1}^{\alpha}(p) \right\rangle \frac{\langle \bar{q}_{\lambda_2}^{\beta}(k-p) q_{\lambda_1}^{\alpha}(p) | H_{q\bar{q}\text{-inst}} | 0 \rangle}{E_{q\bar{q}}(p, k-p)}$$

$$\Sigma_{q\bar{q}}^{\text{NLO}} = \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} \int_{\Lambda} e^{\delta Y \Lambda} dk^+ \int_0^1 d\xi \int_{\mathbf{k}, \tilde{\mathbf{p}}, \mathbf{u}, \tilde{\mathbf{v}}} e^{-i\tilde{\mathbf{v}} \cdot \mathbf{Z} + i\mathbf{u} \cdot (\mathbf{Y}' - \xi \mathbf{Z}) + i\tilde{\mathbf{p}} \cdot \mathbf{Z} - i\mathbf{k} \cdot (\mathbf{X}' - \xi \mathbf{Z})}$$

$$\times \frac{2g^4 J_L^a(\mathbf{x}) \text{tr} [S(z) t^a S(z')^\dagger t^b] J_R^b(\mathbf{y})}{(2\pi)^{10} (\xi(1-\xi)\mathbf{u}^2 + \tilde{\mathbf{v}}^2 + m^2) ((1-\xi)\xi\mathbf{k}^2 + \tilde{\mathbf{p}}^2 + m^2)}$$

$$\times \left[ 4\xi^2(1-\xi)^2 + 2\xi(1-\xi)(1-2\xi) \left( \frac{\mathbf{k} \cdot \tilde{\mathbf{p}}}{\mathbf{k}^2} + \frac{\mathbf{u} \cdot \tilde{\mathbf{v}}}{\mathbf{u}^2} \right) \right.$$

$$\left. + \frac{(1-2\xi)^2 \mathbf{k} \cdot \tilde{\mathbf{p}} \mathbf{u} \cdot \tilde{\mathbf{v}} + \mathbf{u} \cdot \mathbf{k} \tilde{\mathbf{v}} \cdot \tilde{\mathbf{p}} - \mathbf{u} \cdot \tilde{\mathbf{p}} \tilde{\mathbf{v}} \cdot \mathbf{k} + m^2 \mathbf{u} \cdot \mathbf{k}}{\mathbf{u}^2 \mathbf{k}^2} \right].$$



No weird divergences like the loop diagram.

# NLO with Massive Quarks

$$\begin{aligned}
 K_{q\bar{q}}^m(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') &= \frac{\alpha_s^2}{4\pi^4} \int_0^1 d\xi \left\{ \frac{1}{Z^4 \left( (1-\xi)(X')^2 + \xi X^2 \right) \left( (1-\xi)(Y')^2 + \xi Y^2 \right)} \right. \\
 &\times \left[ \xi(1-\xi)(X'^2 - X^2)(Y'^2 - Y) H_4 \right. \\
 &+ Z^2 \left( 2Z^2 \left( 2\xi(1-\xi)(1-2\xi+2\xi^2) H_4 - \xi(1-\xi)(1-2\xi)^2 (H_2 + H_3) - 4\xi^2(1-\xi)^2 H_1 \right) \right. \\
 &- X'^2 \left( (-2\xi(1-\xi)(1-2\xi) + (1-\xi)) H_4 + 2\xi(1-\xi)(1-2\xi) H_2 \right) \\
 &- Y'^2 \left( (-2\xi(1-\xi)(1-2\xi) + (1-\xi)) H_4 + 2\xi(1-\xi)(1-2\xi) H_3 \right) \\
 &- X^2 \left( (2\xi(1-\xi)(1-2\xi) - \xi) H_4 - 2\xi(1-\xi)(1-2\xi) H_2 \right) \\
 &\left. \left. - Y^2 \left( (2\xi(1-\xi)(1-2\xi) - \xi) H_4 - 2\xi(1-\xi)(1-2\xi) H_3 \right) - (X-Y)^2 H_4 \right) \right] \\
 &\left. - 4m^2 \frac{(X' - \xi Z) \cdot (Y' - \xi Z)}{(X' - \xi Z)^2 (Y' - \xi Z)^2} H_5 \right\}.
 \end{aligned}$$

$$H_1 \equiv H_1(m\Xi, m\Upsilon)$$

$$H_1(x, y) = xy K_1(x) K_1(y)$$

$$\Xi \equiv \sqrt{\frac{X'^2}{\xi} + \frac{X^2}{1-\xi}}, \quad \Upsilon \equiv \sqrt{\frac{Y'^2}{\xi} + \frac{Y^2}{1-\xi}}$$

$$K_{JSJ}^m = K_{JSJ}'^m - \frac{1}{2} \int_{\mathbf{z}'} K_{q\bar{q}}^m(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}')$$

# NLO with Massive (subtraction)

$$\begin{aligned} \Sigma_{\bar{q}q}^{\text{NLO}} = & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} \int_{\Lambda} e^{\delta Y \Lambda} dk^+ \int_0^1 d\xi \int_{\mathbf{k}, \tilde{\mathbf{p}}, \mathbf{u}, \tilde{\mathbf{v}}} e^{-i\tilde{\mathbf{v}} \cdot \mathbf{Z} + i\mathbf{u} \cdot (\mathbf{Y}' - \xi \mathbf{Z}) + i\tilde{\mathbf{p}} \cdot \mathbf{Z} - i\mathbf{k} \cdot (\mathbf{X}' - \xi \mathbf{Z})} \\ & \times \frac{2g^4 J_L^a(\mathbf{x}) \text{tr} [S(z)t^a S(z')^\dagger t^b] J_R^b(\mathbf{y})}{(2\pi)^{10} (\xi(1-\xi)\mathbf{u}^2 + \tilde{\mathbf{v}}^2 + m^2) ((1-\xi)\xi\mathbf{k}^2 + \tilde{\mathbf{p}} + m^2)} \\ & \times \left[ 4\xi^2(1-\xi)^2 + 2\xi(1-\xi)(1-2\xi) \left( \frac{\mathbf{k} \cdot \tilde{\mathbf{p}}}{\mathbf{k}^2} + \frac{\mathbf{u} \cdot \tilde{\mathbf{v}}}{\mathbf{u}^2} \right) \right. \\ & \left. + \frac{(1-2\xi)^2 \mathbf{k} \cdot \tilde{\mathbf{p}} \mathbf{u} \cdot \tilde{\mathbf{v}} + \mathbf{u} \cdot \mathbf{k} \tilde{\mathbf{v}} \cdot \tilde{\mathbf{p}} - \mathbf{u} \cdot \tilde{\mathbf{p}} \tilde{\mathbf{v}} \cdot \mathbf{k} + m^2 \mathbf{u} \cdot \mathbf{k}}{\mathbf{u}^2 \mathbf{k}^2} \right]. \end{aligned}$$

$$\begin{aligned} & \int_{\mathbf{z}'} K_{\bar{q}q}^m(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') \\ & = \frac{g^4}{32\pi^5} \int_0^1 d\alpha \alpha(1-\alpha) \int \frac{d^2\mathbf{k}}{2\pi} \frac{d^2\mathbf{u}}{2\pi} e^{-i\mathbf{k} \cdot \mathbf{X} + i\mathbf{u} \cdot \mathbf{Y}} \left[ -\frac{\mathbf{u} \cdot \mathbf{k}}{\mathbf{u}^2 \mathbf{k}^2} \ln \frac{\alpha(1-\alpha)(\mathbf{u}-\mathbf{k})^2 + m^2}{\bar{\mu}^2} \right. \\ & \quad \left. + \frac{1}{\mathbf{u}^2} \ln \frac{\alpha(1-\alpha)(\mathbf{u}-\mathbf{k})^2 + m^2}{\alpha(1-\alpha)\mathbf{k}^2 + m^2} + \frac{1}{\mathbf{k}^2} \ln \frac{\alpha(1-\alpha)(\mathbf{u}-\mathbf{k})^2 + m^2}{\alpha(1-\alpha)\mathbf{u}^2 + m^2} \right] \\ & = \frac{g^4}{32\pi^5} \int_0^1 d\alpha \alpha(1-\alpha) \left\{ \frac{2\mathbf{X} \cdot \mathbf{Y}}{X^2 Y^2} \left( -\ln[\alpha(1-\alpha)] - 2K_0 \left[ \frac{m|X|}{\sqrt{\alpha(1-\alpha)}} \right] - \ln \frac{m^2}{\alpha(1-\alpha)\bar{\mu}^2} \right) \right. \\ & \quad \left. + 2F \left[ \frac{m}{\sqrt{\alpha(1-\alpha)}}, -X, Y \right] + \frac{1}{X^2} \ln \left[ \frac{(X-Y)^2}{Y^2} \right] \frac{m|X|}{\sqrt{\alpha(1-\alpha)}} K_1 \left[ \frac{m|X|}{\sqrt{\alpha(1-\alpha)}} \right] \right. \\ & \quad \left. + \frac{1}{Y^2} \ln \left[ \frac{(X-Y)^2}{X^2} \right] \frac{m|Y|}{\sqrt{\alpha(1-\alpha)}} K_1 \left[ \frac{m|Y|}{\sqrt{\alpha(1-\alpha)}} \right] \right\}, \quad (3.44) \end{aligned}$$

Quick Cross-Check with  
CWZ subtraction scheme:  
MS-bar for massless quarks  
And zero Mom subtraction for  
Massiv quarks ( $\mu \rightarrow m$ ),  
Heavy quarks decouple at large  
Mass limit.

$$K_{JSJ}^m = K_{JSJ}^{\prime m} - \frac{1}{2} \int_{\mathbf{z}'} K_{\bar{q}q}^m(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') \quad F(m, \mathbf{x}, \mathbf{y}) \Big|_{m \rightarrow 0} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{x}^2 \mathbf{y}^2} \ln \left[ \frac{(\mathbf{x} + \mathbf{y})^2}{\mathbf{y}^2} \right]$$

# Conclude

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- ✦ NLO JIMWLK with massive quarks, in addition to nasty integrals, new divergence appear.
- ✦ Expressions are very cumbersome, and the Kernels are expressed as 1-D integrals. Heavy and Light mass limit are hard to obtained (not simplified compared to the full expression). Can EFT methods help? (SCET, i.e., boosted HQET)



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<http://www.humboldt-foundation.de>

