

Determination of HQET matrix elements with renormalon subtraction using Fourier transform

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in collaboration with

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Some state-of-the-art perturbative QCD computations

High-order perturbative series [of order $\mathcal{O}(\alpha_s^3)$ or $\mathcal{O}(\alpha_s^4)$] become available for many QCD observables.

$\mathcal{O}(\alpha_s^4)$ series are known for

- $\sigma(e^+e^- \rightarrow \text{hadrons})$, Hadronic τ decay 2008 Baikov, Chetyrkin, Kühn
2012 Baikov, Chetyrkin, Kühn, Rittinger
- Splitting function 2022 Moch, Ruiji, Ueda, Vermaseren, Vogt
- Static QCD energy 2010 Anzai, Kiyo, Sumino
2010 Smirnov, Smirnov, Steinhauser
(Ultrasoft) 1999 Brambilla, Pineda, Soto, Vairo
- Relation between the pole and the $\overline{\text{MS}}$ mass 2015 Marquard, Smirnov, Smirnov, Steinhauser

$\mathcal{O}(\alpha_s^3)$ series are known for

- semileptonic B decay 2020 Fael, Schönwald, Steinhauser
- Action density in gradient-flow formalism 2016 Harlander, Neumann
- ...

OPE

-inclusion of nonperturbative effects-

$$D(Q^2) = Q^2 \frac{d\Pi(Q^2)}{dQ^2}$$

w/ $i \int d^4x \langle J^\mu(x) J^\nu(0) \rangle e^{iq \cdot x} = (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi(Q^2 = -q^2)$

$$D(Q^2) = \underbrace{C_1(Q^2)}_{\text{perturbative}} + \underbrace{C_{FF}(Q^2)}_{\text{nonperturbative}} \frac{\langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle}{Q^4} + \dots$$

$$\left\{ \begin{array}{l} C_1(Q^2) = \text{const.} + c_0 \alpha_s(Q^2) + c_1 \alpha_s^2(Q^2) + \dots \\ \frac{\langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle}{Q^4} = \frac{\Lambda_{\text{QCD}}^4}{Q^4} = e^{-2/(b_0 \alpha_s(Q^2))} (b_0 \alpha_s(Q^2))^{-2b_1/b_0^2} \dots \end{array} \right.$$

Importance of nonperturbative effects

OPE for semileptonic B decay

$$\Gamma_{B \rightarrow X_c \ell \nu} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3} A_{\text{EW}} (m_b^{1S})^5 \left[C_{\bar{Q}Q}(\bar{m}_c/m_b^{1S}, \alpha_s) + C_{\text{kin}} \frac{\mu_\pi^2}{(m_b^{1S})^2} + C_{\text{cm}} \frac{\mu_G^2}{(m_b^{1S})^2} + \dots \right]$$

Perturbative contribution

2022 Hayashi, Sumino, Takaura

$$C_{\bar{Q}Q} = \text{LO} - \text{NLO} - \text{NNLO} - \text{NNNLO} \\ 0.5863 - 0.0797 - 0.0294 - 0.0073 + \dots$$

Nonperturbative contribution

$$C_{\text{cm}} \frac{\mu_G^2}{(m_b^{1S})^2} \approx -0.0151$$

Nonperturbative effects need to be included.

Renormalon problem

$$D(Q^2) = \underbrace{C_1(Q^2)}_{\text{perturbative}} + \underbrace{C_{FF}(Q^2)}_{\text{nonperturbative}} \frac{\langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle}{Q^4} + \dots$$

Perturbative side

$$C_1(Q^2) = \sum_{n \geq 0} c_n \alpha_s^n(Q^2) \text{ is a divergent series due to } c_n \sim n! \left(\frac{b_0}{2} \right)^n$$

→ $\delta C_1(Q^2) \approx \frac{\Lambda_{\text{QCD}}^4}{Q^4}$ Inevitable uncertainty called renormalon

Nonperturbative side

$$\langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle \text{ is a UV divergent quantity} \rightarrow \delta(\langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle) \approx \Lambda_{\text{QCD}}^4$$

Perturbative and nonperturbative quantities are ill-defined as they are.

Renormalon cancellation

$$D(Q^2) = \underbrace{C_{1,\text{RF}}(Q^2)}_{\text{perturbative}} + \underbrace{C_{FF}(Q^2)}_{\text{nonperturbative}} \frac{\langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle_{\text{RF}}}{Q^4} + \dots$$

Perturbative side

$C_1(Q^2) = \sum_{n \geq 0} c_n \alpha_s^n(Q^2)$ is a divergent series due to $c_n \sim n! \left(\frac{b_0}{2}\right)^n$

$$C_1(Q^2) = C_{1,\text{RF}}(Q^2) + \delta C_1(Q^2) \quad \text{w/} \quad \delta C_1(Q^2) \approx \frac{\Lambda_{\text{QCD}}^4}{Q^4}$$

Nonperturbative side

$\langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle$ is a UV divergent quantity

$$\langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle = \langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle_{\text{RF}} + \delta \langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle \quad \text{w/} \quad \delta(\langle 0 | F_{\mu\nu}^a F_{\mu\nu}^a | 0 \rangle) \approx \Lambda_{\text{QCD}}^4$$

The nonperturbative quantity is now defined and can be measured.

Renormalon subtraction

In order to use the previous renormalon-free OPE, we should perform the following decomposition

$$C_1(Q^2) = C_{1,\text{RF}}(Q^2) + \delta C_1(Q^2)$$

with finite order perturbative series.

Proposed methods

2002 T. Lee

2019 Ayala, Lobregat, Pineda

2019 H. Takaura

2020 Y. Hayashi, Y. Sumino, H. Takaura

...

Renormalon subtraction
using Fourier transform (FTRS)

Application to HQET

B, D meson mass

$$\begin{cases} \overline{M}_B = \frac{M_B + 3M_{B^*}}{4} = m_b + \bar{\Lambda} + \frac{\mu_\pi^2}{2m_b} + \dots \\ \overline{M}_D = \frac{M_D + 3M_{D^*}}{4} = m_c + \bar{\Lambda} + \frac{\mu_\pi^2}{2m_c} + \dots \end{cases}$$

Renormalon uncertainties

$$\delta m_{b(c)}|_{u=1/2} \propto \Lambda_{\overline{\text{MS}}} , \quad \delta m_{b(c)}|_{u=1} \propto \Lambda_{\overline{\text{MS}}}^2 / m_{b(c)}$$

Application to HQET

B, D meson mass

$$\left\{ \begin{array}{l} \overline{M}_B = \frac{M_B + 3M_{B^*}}{4} = m_{b,\text{RF}} + \bar{\Lambda}_{\text{RF}} + \frac{(\mu_\pi^2)_{\text{RF}}}{2m_{b,\text{RF}}} + \dots \\ \overline{M}_D = \frac{M_D + 3M_{D^*}}{4} = m_{c,\text{RF}} + \bar{\Lambda}_{\text{RF}} + \frac{(\mu_\pi^2)_{\text{RF}}}{2m_{c,\text{RF}}} + \dots \end{array} \right.$$

Renormalon uncertainties

$$\delta m_{b(c)}|_{u=1/2} \propto \Lambda_{\overline{\text{MS}}} , \quad \delta m_{b(c)}|_{u=1} \propto \Lambda_{\overline{\text{MS}}}^2 / m_{b(c)}$$

We perform the following decomposition using FTRS:

$$m_{b(c)} = m_{b(c),\text{RF}} + \delta m_{b(c)}|_{u=1/2} + \delta m_{b(c)}|_{u=1}$$

We determine $\bar{\Lambda}_{\text{RF}}$, $(\mu_\pi^2)_{\text{RF}}$.

$(\mu_\pi^2)_{\text{RF}}$ can be used as a nonperturbative input for $\Gamma_{B \rightarrow X_{\text{clv}}}$.

Contents

✓ Introduction

- FTRS method and determination of HQET matrix elements
- Conclusions

FTRS

$$r(\bar{m}) \equiv \frac{m_{\text{pole}}}{\bar{m}} = 1 + c_0 \alpha_s(\mu^2) + (c_1 + c_0 b_0 \log(\mu^2/\bar{m}^2)) \alpha_s^2(\mu^2) + \dots$$

where $\bar{m} = \bar{m}(\bar{m})$

We consider dual space to \bar{m} : 2021 Hayashi, Sumino, Takaura

$$\tilde{r}(\tau) \equiv \frac{\pi}{\tau} \int_0^\infty \frac{d\bar{m}}{\bar{m}} \sin(\tau/\sqrt{\bar{m}}) r(\bar{m})$$

$\tilde{r}(\tau)$ is free from $\mathcal{O}(\Lambda_{\overline{\text{MS}}})$ and $\mathcal{O}(\Lambda_{\overline{\text{MS}}}^2)$ renormalons.

$$\begin{aligned} \delta \tilde{r}(\tau) &= \frac{\pi}{\tau} \int_0^\infty \frac{d\bar{m}}{\bar{m}} \sin(\tau/\sqrt{\bar{m}}) \delta r(\bar{m}) \\ &= \frac{\pi}{\tau} \int_0^\infty \frac{d\bar{m}}{\bar{m}} \sin(\tau/\bar{m}) \left(\frac{\Lambda_{\overline{\text{MS}}}}{\bar{m}} \right)^{2u} \\ &= 2 \left(\frac{\Lambda_{\overline{\text{MS}}}}{\tau^2} \right)^{2u} \sin(2\pi u) \Gamma(4u) = 0 \quad \text{for } u=1/2 \text{ and } u=1 \end{aligned}$$

FTRS

$$r(\bar{m}) \equiv \frac{m_{\text{pole}}}{\bar{m}} = 1 + c_0 \alpha_s(\mu^2) + (c_1 + c_0 b_0 \log(\mu^2/\bar{m}^2)) \alpha_s^2(\mu^2) + \dots$$

↓ dual

$$\begin{aligned}\tilde{r}(\tau) &\equiv \frac{\pi}{\tau} \int_0^\infty \frac{d\bar{m}}{\bar{m}} \sin(\tau/\sqrt{\bar{m}}) r(\bar{m}) \\ &= \frac{\pi^2}{\tau} [1 + \tilde{c}_0 \alpha_s(\mu^2) + (\tilde{c}_1 + \tilde{c}_0 b_0 \log(\mu^2/\tau^4)) \alpha_s^2(\mu^2) + \dots]\end{aligned}$$

$$= \frac{\pi^2}{\tau} [1 + \tilde{c}_0 \alpha_s(\tau^4) + \tilde{c}_1 \alpha_s^2(\tau^4) + \dots]$$

Free from $\mathcal{O}(\Lambda_{\overline{\text{MS}}})$ and $\mathcal{O}(\Lambda_{\overline{\text{MS}}}^2)$
renormalons

Inverse formula

$$r(\bar{m}) = -\frac{1}{2\pi^2 \sqrt{\bar{m}}} \int_0^\infty d\tau \tau \sin(\tau/\sqrt{\bar{m}}) \tilde{r}(\tau)$$

FTRS

$$r(\bar{m}) \equiv \frac{m_{\text{pole}}}{\bar{m}} = 1 + c_0 \alpha_s(\mu^2) + (c_1 + c_0 b_0 \log(\mu^2/\bar{m}^2)) \alpha_s^2(\mu^2) + \dots$$

↓ dual

$$\begin{aligned}\tilde{r}(\tau) &\equiv \frac{\pi}{\tau} \int_0^\infty \frac{d\bar{m}}{\bar{m}} \sin(\tau/\sqrt{\bar{m}}) r(\bar{m}) \\ &= \frac{\pi^2}{\tau} [1 + \tilde{c}_0 \alpha_s(\mu^2) + (\tilde{c}_1 + \tilde{c}_0 b_0 \log(\mu^2/\tau^4)) \alpha_s^2(\mu^2) + \dots]\end{aligned}$$

$$= \frac{\pi^2}{\tau} [1 + \tilde{c}_0 \alpha_s(\tau^4) + \tilde{c}_1 \alpha_s^2(\tau^4) + \dots]$$

Free from $\mathcal{O}(\Lambda_{\overline{\text{MS}}})$ and $\mathcal{O}(\Lambda_{\overline{\text{MS}}}^2)$
renormalons

Inverse formula

$$\underline{r(\bar{m})} = -\frac{1}{2\pi^2 \sqrt{\bar{m}}} \boxed{\int_0^\infty d\tau \tau \sin(\tau/\sqrt{\bar{m}}) \underline{\tilde{r}(\tau)}}$$

has renormalons

generates renormalons

doesn't have renormalons

FO: $\tau \sim 0$

RG imp: $\tau^2 \sim \Lambda_{\overline{\text{MS}}}$

FTRS

$$r(\bar{m}) \equiv \frac{m_{\text{pole}}}{\bar{m}} = 1 + c_0 \alpha_s(\mu^2) + (c_1 + c_0 b_0 \log(\mu^2/\bar{m}^2)) \alpha_s^2(\mu^2) + \dots$$

↓ dual

$$\begin{aligned} \tilde{r}(\tau) &\equiv \frac{\pi}{\tau} \int_0^\infty \frac{d\bar{m}}{\bar{m}} \sin(\tau/\sqrt{\bar{m}}) r(\bar{m}) \\ &= \frac{\pi^2}{\tau} [1 + \tilde{c}_0 \alpha_s(\mu^2) + (\tilde{c}_1 + \tilde{c}_0 b_0 \log(\mu^2/\tau^4)) \alpha_s^2(\mu^2) + \dots] \\ &= \frac{\pi^2}{\tau} [1 + \tilde{c}_0 \alpha_s(\tau^4) + \tilde{c}_1 \alpha_s^2(\tau^4) + \dots] \quad \text{Free from } \mathcal{O}(\Lambda_{\overline{\text{MS}}}) \text{ and } \mathcal{O}(\Lambda_{\overline{\text{MS}}}^2) \\ &\qquad\qquad\qquad \text{renormalons} \end{aligned}$$

Inverse formula

$$\begin{aligned} r(\bar{m})_{\text{RF}} &= -\frac{1}{2\pi^2 \sqrt{\bar{m}}} \int_{\text{PV}} d\tau \tau \sin(\tau/\sqrt{\bar{m}}) \tilde{r}(\tau) \quad \text{Avoids } \tau^2 \sim \Lambda_{\overline{\text{MS}}} \\ &= \int_{0, \text{PV}}^\infty du B_r(u) e^{-u/(b_0 \alpha_s)} \quad \text{w/ some assumptions} \end{aligned}$$

Finite mass effects

$$\overline{M}_B = m_b + \boxed{\bar{\Lambda}} + \frac{\boxed{\mu_\pi^2}}{2m_b} + \dots$$

$$\overline{M}_D = m_c + \boxed{\bar{\Lambda}} + \frac{\boxed{\mu_\pi^2}}{2m_c} + \dots$$

The same nonperturbative parameters for B and D

First renormalon uncertainties of m_b and m_c should be the same.

Treating both b and c as heavy quarks Using 2020 Fael, Schönwald, Steinhauser
 (finite charm mass effects for m_b and non-decoupling of bottom for m_c)

$$m_b/\overline{m}_b = 1 + 0.424413\alpha_s^{(3)}(\overline{m}_b) + 1.03744(\alpha_s^{(3)}(\overline{m}_b))^2 + 3.74358(\alpha_s^{(3)}(\overline{m}_b))^3$$

$$m_c/\overline{m}_c = 1 + 0.424413\alpha_s^{(3)}(\overline{m}_c) + 1.04375(\alpha_s^{(3)}(\overline{m}_c))^2 + 3.75736(\alpha_s^{(3)}(\overline{m}_c))^3$$

Perturbative coefficients are close $\mathcal{O}(\alpha_s^4)$ coefficient set to 17.438

Naive treatment (setting $n_{\text{massless}}=4$ for m_b and $n_{\text{massless}}=3$ for m_c)

$$m_b/\overline{m}_b = 1 + 0.42441\alpha_s^{(4)}(\overline{m}_b) + 0.94005(\alpha_s^{(4)}(\overline{m}_b))^2 + 3.0385(\alpha_s^{(4)}(\overline{m}_b))^3 + 12.647(\alpha_s^{(4)}(\overline{m}_b))^4$$

$$m_c/\overline{m}_c = 1 + 0.42441\alpha_s^{(3)}(\overline{m}_c) + 1.0456(\alpha_s^{(3)}(\overline{m}_c))^2 + 3.7509(\alpha_s^{(3)}(\overline{m}_c))^3 + 17.438(\alpha_s^{(3)}(\overline{m}_c))^4$$

Determination of the nonperturbative matrix elements

$$\left\{ \begin{array}{l} \overline{M}_B = (m_b)_{\text{PV}} + \bar{\Lambda}_{\text{PV}} + \frac{(\mu_\pi^2)_{\text{PV}}}{2(m_b)_{\text{PV}}} + \dots \\ \overline{M}_D = (m_c)_{\text{PV}} + \bar{\Lambda}_{\text{PV}} + \frac{(\mu_\pi^2)_{\text{PV}}}{2(m_c)_{\text{PV}}} + \dots \end{array} \right.$$

Inputs: $\overline{M}_{B,\text{exp}} = 5.313 \text{ GeV}$, $\overline{M}_{D,\text{exp}} = 1.971 \text{ GeV}$

$\overline{m}_b = 4.18^{+0.03}_{-0.02} \text{ GeV}$ $\overline{m}_c = 1.27 \pm 0.02 \text{ GeV}$ $\Lambda_{\overline{\text{MS}}} = 0.332 \pm 0.015 \text{ GeV}$

$$\left\{ \begin{array}{l} \bar{\Lambda}_{\text{PV}} = 0.495(15)_\mu(49)_{\overline{m}_b}(12)_{\overline{m}_c}(13)_{\alpha_s}(0)_{\text{f.m.}} \text{ GeV} \\ (\mu_\pi^2)_{\text{PV}} = -0.12(13)_\mu(15)_{\overline{m}_b}(11)_{\overline{m}_c}(4)_{\alpha_s}(0)_{\text{f.m.}} \text{ GeV}^2 \end{array} \right.$$

The first errors indicate higher order uncertainty.

We estimate the perturbative error for $(\mu_\pi^2)_{\text{PV}}$ will be reduced at higher order.

Comparison with other works

Our determination (subtracting $O(\Lambda)$ and $O(\Lambda^2/m)$ renormalons)

$$\begin{cases} \bar{\Lambda}_{\text{PV}} = 0.495 \pm 0.053 \text{ GeV} \\ (\mu_\pi^2)_{\text{PV}} = -0.12 \pm 0.23 \text{ GeV}^2 \end{cases}$$

2018 Bazavov et al. (subtracting $O(\Lambda)$ renormalon)

$$\begin{cases} \bar{\Lambda}_{\text{PV}} = 0.435(31) \text{ GeV} \\ \mu_\pi^2 = 0.05(22) \text{ GeV}^2 \end{cases}$$

2019 Ayala, Lobregat, Pineda (subtracting $O(\Lambda)$ renormalon)

$$\bar{\Lambda}_{\text{PV}} = 477(\mu)^{-8}_{+17}(Z_m)^{+11}_{-12}(\alpha_s)^{-8}_{+9}(\mathcal{O}(1/m_h))^{+46}_{-46} \text{ MeV}$$

Conclusions

- As high-order perturbative results are available, it is becoming important to include nonperturbative effects in the OPE.
- To this end renormalon problem needs to be resolved and we propose renormalon subtraction using Fourier transform (FTRS).
- We determined HQET nonperturbative parameters $\bar{\Lambda}_{\text{PV}}$ and $(\mu_\pi^2)_{\text{PV}}$ subtracting $\mathcal{O}(\Lambda_{\overline{\text{MS}}})$ and $\mathcal{O}(\Lambda_{\overline{\text{MS}}}^2)$.

Advantages of FTRS method

- General method already applied to the Adler function, semileptonic decay
- Multiple renormalons are simultaneously eliminated
- Normalization constants need not to be estimated
- Free from unphysical singularity of running coupling

Backup

Application to general observables

2020 Hayashi, Sumino, Takaura

General observable $X(Q)$

“Coordinate” space $r = Q^{-1/a}$

$C_1^X(Q)$ has renormalon uncertainties.

“Momentum” space τ

$$\tilde{X}(\tau) \equiv \int d^3x e^{-i\vec{\tau} \cdot \vec{x}} r^{2a u'} X(r^{-a})$$

We choose parameters (a, u') so that $\tilde{X}(\tau)$ becomes free of renormalon uncertainties.

Ex. Adler function

$$\tilde{D}(\tau) \equiv \int d^3x e^{-i\vec{\tau}\cdot\vec{x}} r^{2\textcolor{blue}{au'}} D(r^{-a}) \quad \text{w/} \quad r = Q^{-1/\textcolor{blue}{a}}$$

$$\delta C_1^D(Q) = \{(\Lambda_{\text{QCD}}/Q)^4, (\Lambda_{\text{QCD}}/Q)^6, \dots\} = \{\Lambda^4 r^{4a}, \Lambda^6 r^{6a}, \dots\}$$

$$\int d^3x e^{-i\vec{\tau}\cdot\vec{x}} \{r^{2au'+4a}, r^{2au'+6a}, \dots\} = \int d^3x e^{-i\vec{\tau}\cdot\vec{x}} \{r^0, r^2, \dots\} = \{0, 0, \dots\}$$

Choosing $(a, u') = (1, -2)$ we can eliminate renormalons.

Ex. Adler function

In the large- β_0 approximation, Borel transform is given by

$$B_D(u) = \left(\frac{\mu^2}{Q^2} \right)^u f(u)$$

$f(u)$ is singular at $u=2, 3, \dots$

“Momentum” space τ $(a, u') = (1, -2)$

$$\begin{aligned} B_{\tilde{D}}(u) &= \int d^3x r^{2au'} (\mu^2 r^2)^u f(u) \\ &= -4\pi \left(\frac{\mu^2}{\tau^2} \right)^u \frac{1}{\tau} \Gamma(2u - 2) \sin(\pi u) f(u) \end{aligned}$$

$\sin(\pi u)|_{u=2,3,\dots} = 0$ suppresses original singularity of $f(u)$.

OPE in HQET

B (D) meson: spin 0

$$M_{B(D)} = m_{b(c)} + \bar{\Lambda} + \frac{\mu_\pi^2}{2m_{b(c)}} - C_{\text{cm}}^M \frac{\mu_G^2}{2m_{b(c)}} + \dots$$

$m_{b(c)}$: Bottom (charm) quark pole mass

Renormalon uncertainties of $\mathcal{O}(\Lambda_{\overline{\text{MS}}})$, $\mathcal{O}(\Lambda_{\overline{\text{MS}}}^2/m_b)$

$\bar{\Lambda}$: Leading nonperturbative effect of $\mathcal{O}(\Lambda_{\overline{\text{MS}}})$

μ_π^2, μ_G^2 : Nonperturbative effects of $\mathcal{O}(\Lambda_{\overline{\text{MS}}}^2)$

$$\left\{ \begin{array}{l} \mu_\pi^2 = \frac{\langle B(p) | \bar{b}_v D_\perp^2 b_v | B(p) \rangle}{2m_B} \\ \mu_G^2 = -\frac{\langle B(p) | \bar{b}_v \frac{g_s}{2} \sigma_{\mu\nu} G^{\mu\nu} b_v | B(p) \rangle}{2m_B} \end{array} \right.$$

OPE in HQET

B^* (D^*) meson: spin 1

$$M_{B^*(D^*)} = m_{b(c)} + \bar{\Lambda} + \frac{\mu_\pi^2}{2m_{b(c)}} + \frac{1}{3}C_{\text{cm}}^M \frac{\mu_G^2}{2m_{b(c)}} + \dots$$

$m_{b(c)}$: Bottom (charm) quark pole mass

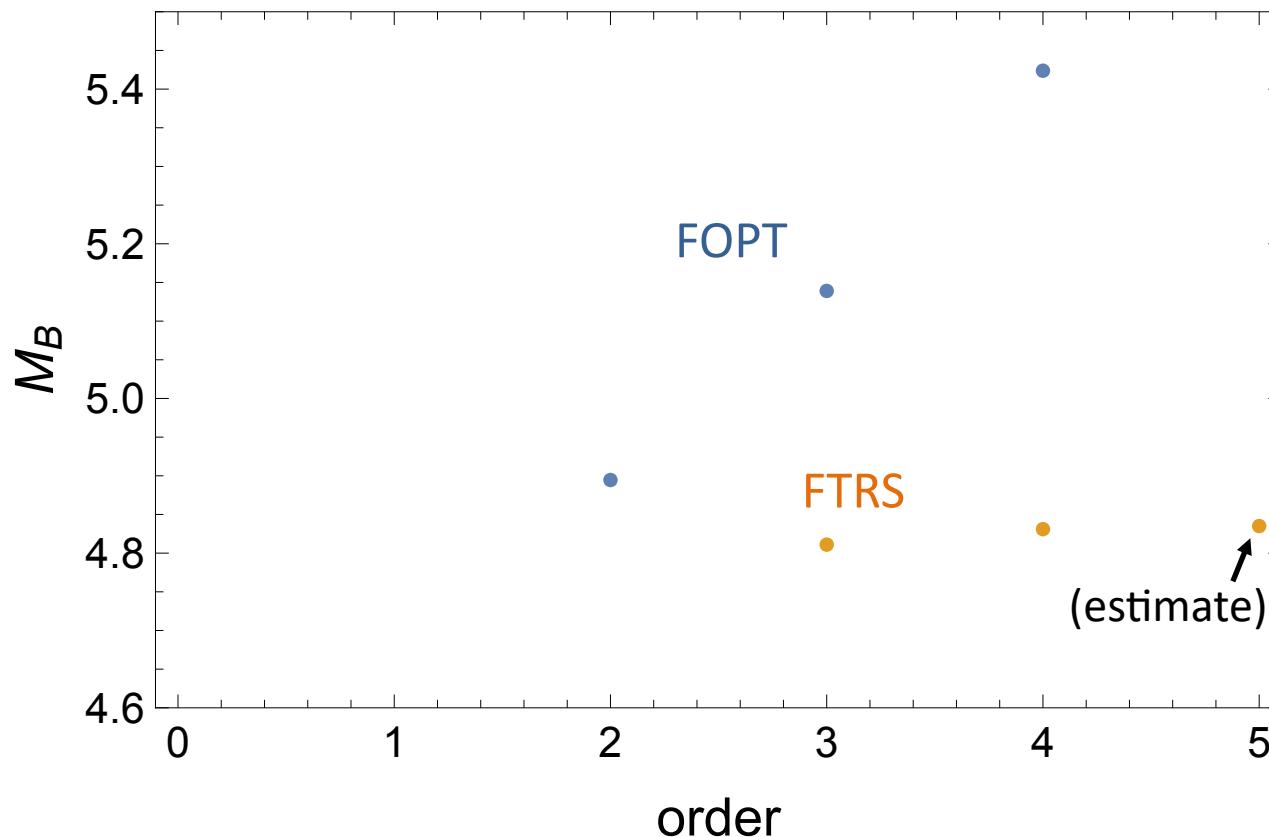
Renormalon uncertainties of $\mathcal{O}(\Lambda_{\overline{\text{MS}}})$, $\mathcal{O}(\Lambda_{\overline{\text{MS}}}^2/m_b)$

$\bar{\Lambda}$: Leading nonperturbative effect of $\mathcal{O}(\Lambda_{\overline{\text{MS}}})$

μ_π^2, μ_G^2 : Nonperturbative effects of $\mathcal{O}(\Lambda_{\overline{\text{MS}}}^2)$

$$\left\{ \begin{array}{l} \mu_\pi^2 = \frac{\langle B(p) | \bar{b}_v D_\perp^2 b_v | B(p) \rangle}{2m_B} \\ \mu_G^2 = -\frac{\langle B(p) | \bar{b}_v \frac{g_s}{2} \sigma_{\mu\nu} G^{\mu\nu} b_v | B(p) \rangle}{2m_B} \end{array} \right.$$

Convergence in FTRS



Extended formula

$$\delta X(Q^2) = f(\alpha_s(Q^2)) \left(\frac{\Lambda_{\text{QCD}}}{Q} \right)^d$$

w/ $f(\alpha_s(Q^2)) = N \alpha_s(Q^2)^{\gamma_0/b_0} (1 + \mathcal{O}(\alpha_s(Q^2)))$

Instead of $X(Q^2)$, we consider $X(Q^2)/f(\alpha_s(Q^2))$.

→ $\delta X(Q^2)/f(\alpha_s(Q^2)) = (\Lambda_{\overline{\text{MS}}}/Q)^d$