

Decays of Heavy Quark Hybrids



XVth Quark Confinement and the Hadron Spectrum

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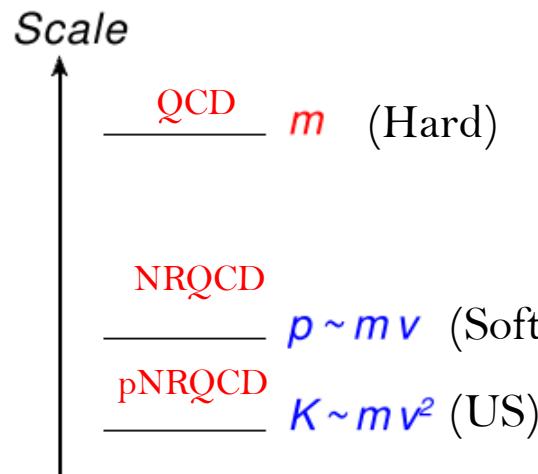
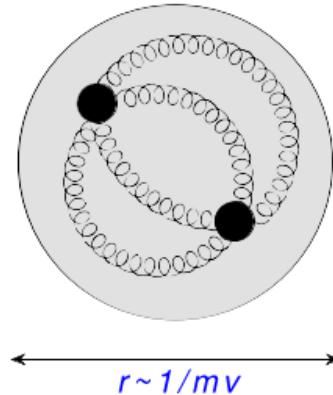


Outline

- **Introduction to X Y Z mesons**
- **EFT for Quarkonium Hybrids**
 - BO-EFT effective theory
 - Quarkonium Hybrid Spectrum
- **Decay Rates for hybrid**
- **Summary and Outlook**

Introduction

- Quarkonium: Color singlet bound state of $Q\bar{Q}$ ($Q = c, b$).⁴



- Hierarchy of Energy Scales in $Q\bar{Q}$:

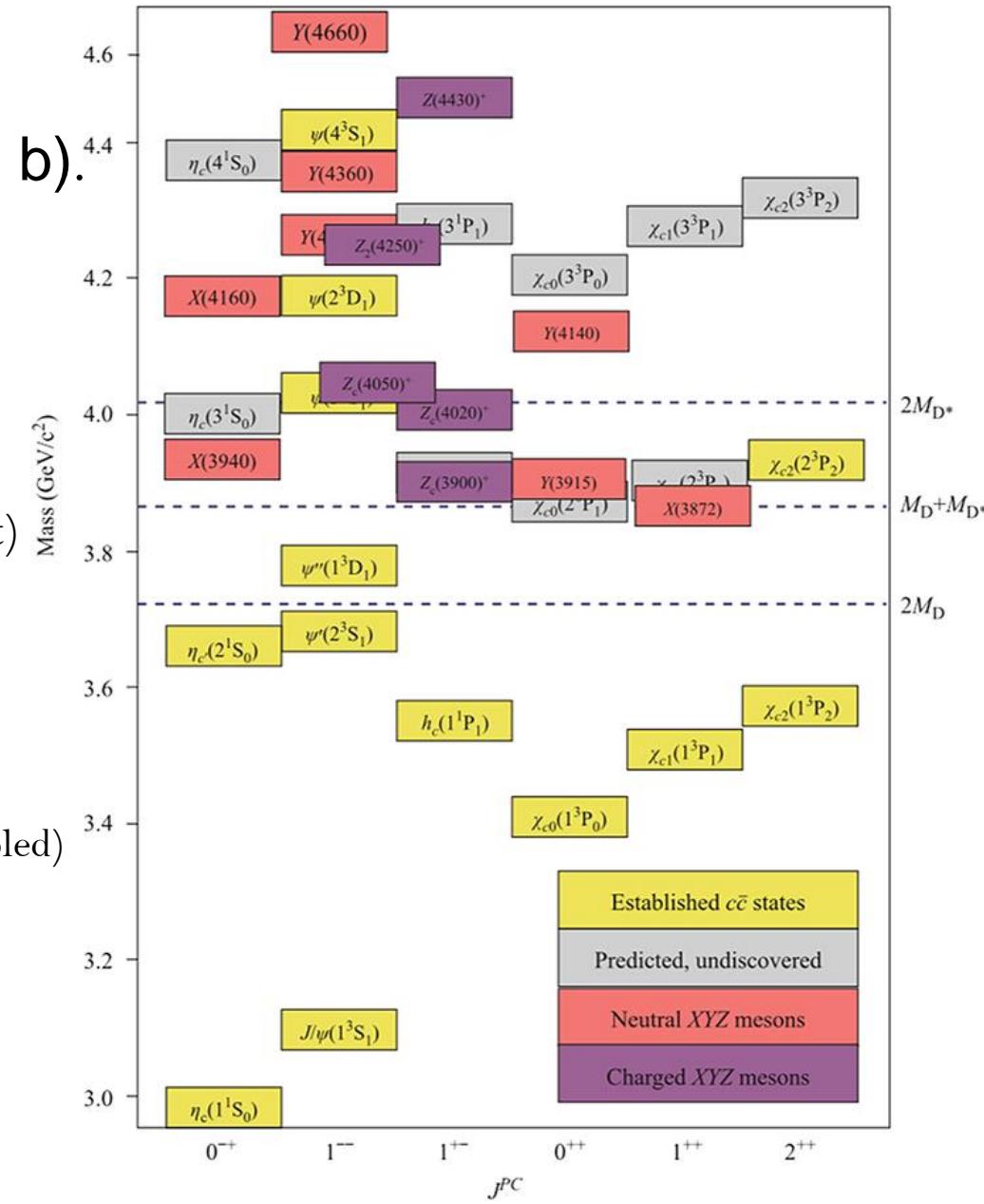
$$m \gg mv \gg mv^2, \Lambda_{\text{QCD}}$$
 (perturbative dynamics)

$$m \gg mv, \Lambda_{\text{QCD}} \gg mv^2$$
 (nonperturbative dynamics
Coupled)
 - pNRQCD: Relevant EFT for Quarkonium.

Bodwin, Braaten & Lepage (1994), Mehen and Fleming, Phys. Rev. D73, 034502 (2005)

Brambilla, Vairo, and Rosch, Phys. Rev. D72, 034021 (2005)

Luke, Manohar & Rothstein (2000)



Introduction

- Quark Model:

Mesons: quark-antiquark states

Baryons: 3-quark states

- QCD spectrum also allows for more complex structures called as **Exotics**.

- Exotic states: XYZ mesons

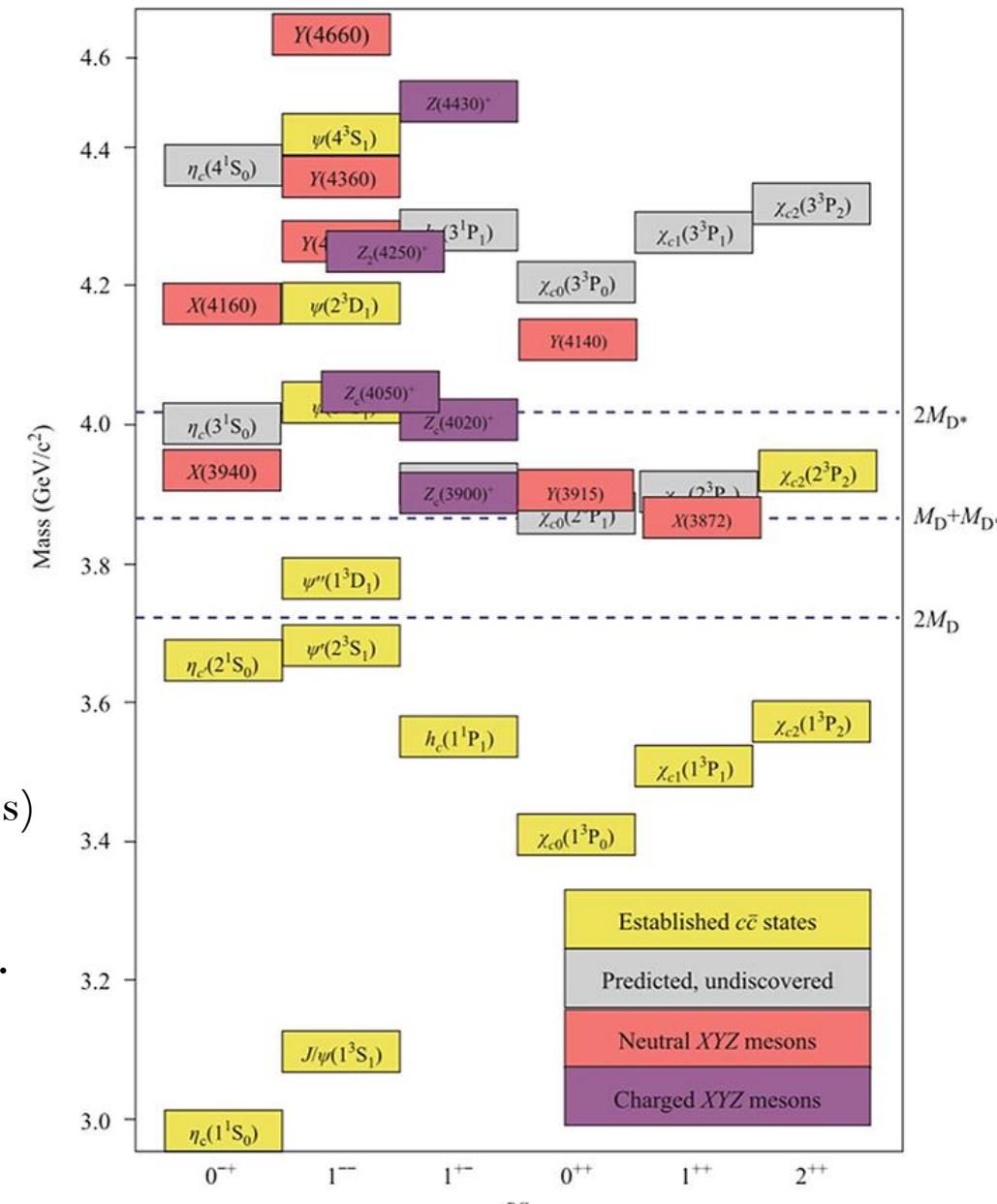
- ✓ Quarkonium-like states that don't fit traditional $Q\bar{Q}$ spectrum.
- ✓ In some cases exotic quantum numbers (charged Z_c and Z_b states)

For review see Brambilla et al. *Phys. Reports.* **873** (2020)

- $X(3872)$: First exotic state discovered in 2003 by Belle.

Phys. Rev. Lett. **91**, 262001 (2003)

- Several new heavy quark exotic states have been discovered since 2003 (masses & decay rates measured in various channels).



Introduction

- Exotics broadly classified as
 - ❖ Structures with active gluons
 - ❖ Multiquark states

- Several interpretations of Exotics:

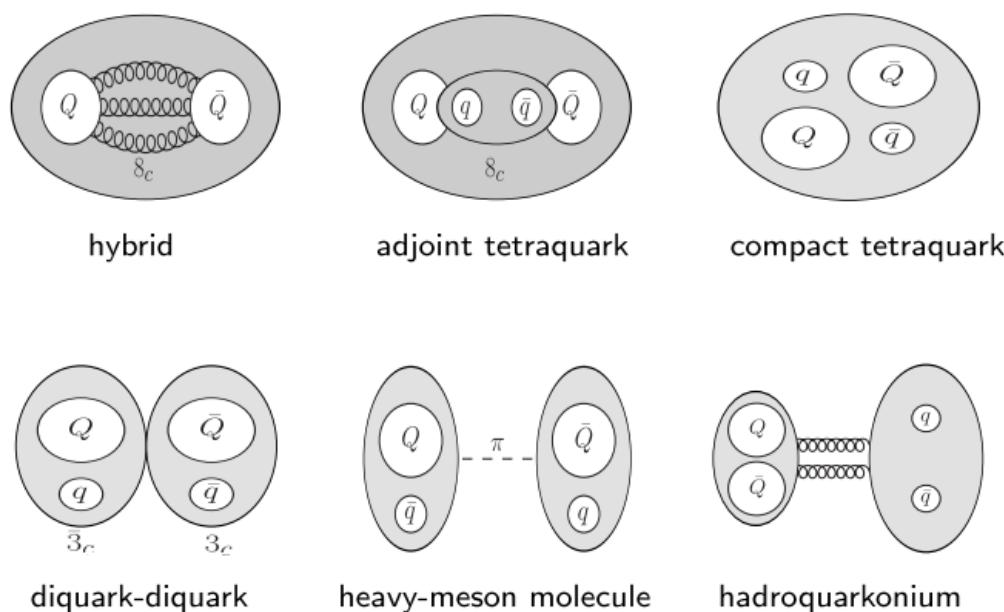
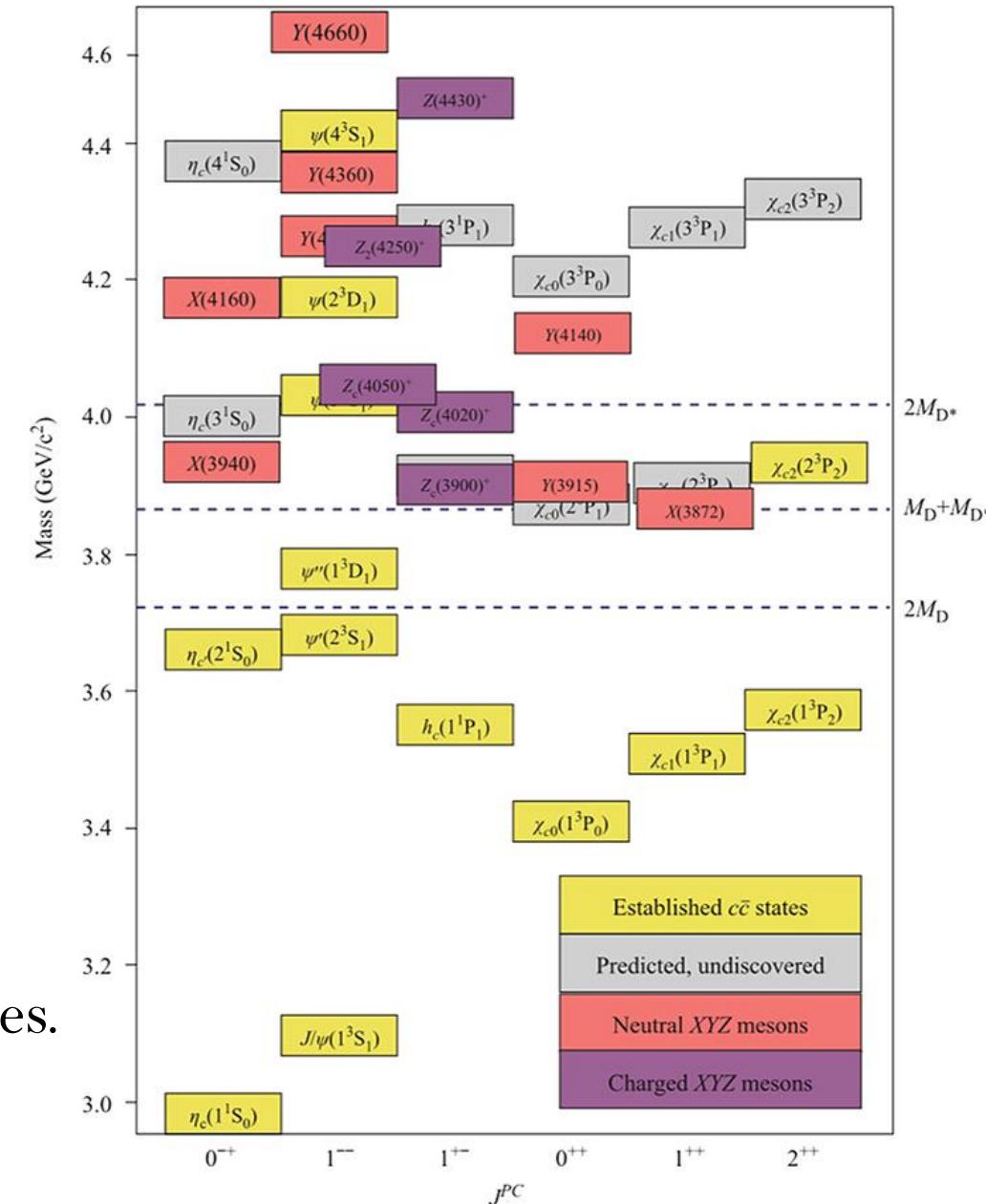


Figure from
W.K.Lai talk

- No single model completely describes all the XYZ states.
- **Hybrids ($Q\bar{Q}g$):** Focus of this talk. Use EFT + lattice to have model independent description.



Quarkonium hybrids: EFT

- Hybrids ($Q\bar{Q}g$): Color singlet combination of color octet $Q\bar{Q}$ + gluonic excitations.

- Separation of scales in hybrids:

$$m \gg mv \gg \Lambda_{QCD} \gg mv^2$$

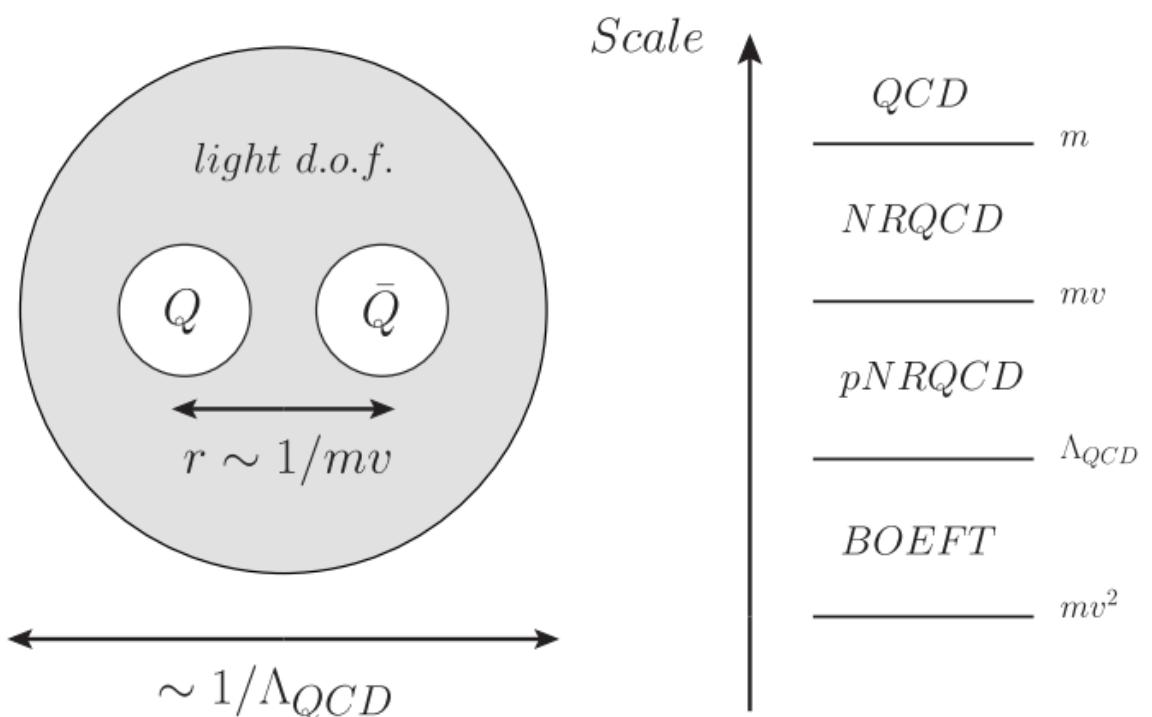
- ❖ light d.o.f: Λ_{QCD}
- ❖ Relative separation between heavy quarks: $r \sim 1/mv$
- ❖ Heavy Quark K.E scale: mv^2

- Time-scale for dynamics of $Q\bar{Q}$: $\sim \frac{1}{mv^2} \gg \frac{1}{\Lambda_{QCD}}$

Born-Oppenheimer Approximation

$$\frac{1}{mv^2} \gg \frac{1}{\Lambda_{QCD}}$$

Braaten, Langmack, Smith
Phys. Rev. D. 90, 014044 (2014)



- Appropriate EFT framework for Hybrids: **Born-Oppenheimer EFT (BOEFT)**

QCD → NRQCD → pNRQCD → BOEFT

Brambilla, Krein, Castellà , Vairo Phys. Rev. D. 97, (2018)

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Quarkonium hybrids: BOEFT

- Static limit ($m \rightarrow \infty$): Quantum #'s for hybrid

Irreducible representations of $D_{\infty h}$

- \mathbf{K} : angular momentum of light d.o.f.
 $\lambda = \hat{\mathbf{r}} \cdot \mathbf{K} = 0, \pm 1, \pm 2, \pm 3, \dots$
 $\Lambda = |\lambda| = 0, 1, 2, 3, \dots$ ($\Sigma, \Pi, \Delta, \Phi, \dots$)
- Eigenvalue of CP : $\eta = +1(g), -1(u)$
- σ : eigenvalue of reflection about a plane containing $\hat{\mathbf{r}}$ (only for Σ states)

- Static Energies (Σ, Π, Δ): Eigenvalue of NRQCD Hamiltonian in the static limit.
- For $r \rightarrow 0$: static energies are degenerate.
Characterized by $O(3) \times C$ symmetry group.

Labelled by: (K^{PC}, Λ^σ)

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Gluonic static energies

M. Foster and C. Michael, Phys. Rev. D59 (1999)

K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

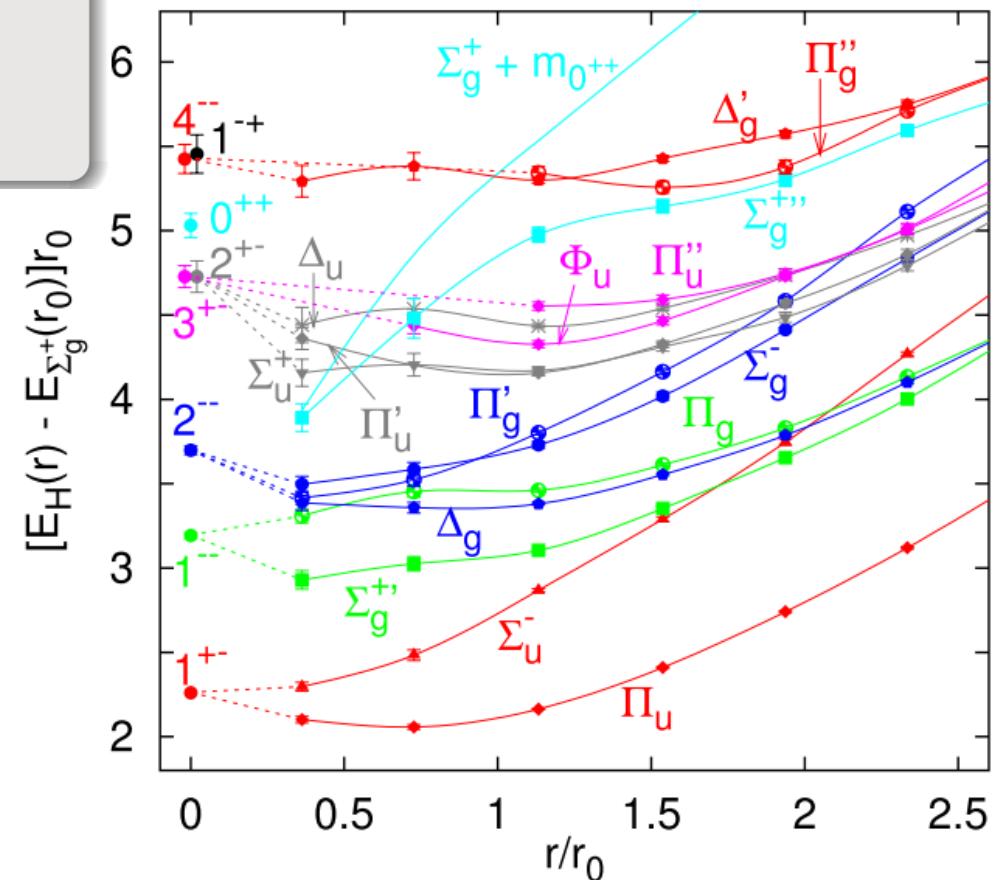


Fig from G. S. Bali and A. Pineda,
Phys. Rev. D69 (2004)

Quarkonium hybrids: BOEFT

- Static limit ($m \rightarrow \infty$): Quantum #'s for hybrid

Irreducible representations of $D_{\infty h}$

- \mathbf{K} : angular momentum of light d.o.f.
 $\lambda = \hat{\mathbf{r}} \cdot \mathbf{K} = 0, \pm 1, \pm 2, \pm 3, \dots$
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- Eigenvalue of CP : $\eta = +1(g), -1(u)$
- σ : eigenvalue of reflection about a plane containing $\hat{\mathbf{r}}$ (only for Σ states)

Gluonic operators characterizing
Hybrids in Wilson loop



- Static Energies (Σ, Π, Δ): Eigenvalue of NRQCD Hamiltonian in the static limit.

- For $r \rightarrow 0$: static energies are degenerate.
Characterized by $O(3) \times C$ symmetry group.

Labelled by: $(K^{PC}, \Lambda_\eta^\sigma)$

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Focus on these two for low lying hybrids

Λ_η^σ	K^{PC}	O_n
Σ_u^-	1^{+-}	$\hat{\mathbf{r}} \cdot \mathbf{B}, \hat{\mathbf{r}} \cdot (\mathbf{D} \times \mathbf{E})$
Π_u	1^{+-}	$\hat{\mathbf{r}} \times \mathbf{B}, \hat{\mathbf{r}} \times (\mathbf{D} \times \mathbf{E})$
$\Sigma_g^{+'}$	1^{--}	$\hat{\mathbf{r}} \cdot \mathbf{E}, \hat{\mathbf{r}} \cdot (\mathbf{D} \times \mathbf{B})$
Π_g	1^{--}	$\hat{\mathbf{r}} \times \mathbf{E}, \hat{\mathbf{r}} \times (\mathbf{D} \times \mathbf{B})$
Σ_g^-	2^{--}	$(\hat{\mathbf{r}} \cdot \mathbf{D})(\hat{\mathbf{r}} \cdot \mathbf{B})$
Π'_g	2^{--}	$\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} \cdot \mathbf{D})\mathbf{B} + \mathbf{D}(\hat{\mathbf{r}} \cdot \mathbf{B}))$
Δ_g	2^{--}	$(\hat{\mathbf{r}} \times \mathbf{D})^i (\hat{\mathbf{r}} \times \mathbf{B})^j + (\hat{\mathbf{r}} \times \mathbf{D})^j (\hat{\mathbf{r}} \times \mathbf{B})^i$
Σ_u^+	2^{+-}	$(\hat{\mathbf{r}} \cdot \mathbf{D})(\hat{\mathbf{r}} \cdot \mathbf{E})$
Π'_u	2^{+-}	$\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} \cdot \mathbf{D})\mathbf{E} + \mathbf{D}(\hat{\mathbf{r}} \cdot \mathbf{E}))$
Δ_u	2^{+-}	$(\hat{\mathbf{r}} \times \mathbf{D})^i (\hat{\mathbf{r}} \times \mathbf{E})^j + (\hat{\mathbf{r}} \times \mathbf{D})^j (\hat{\mathbf{r}} \times \mathbf{E})^i$

Quarkonium hybrids: BOEFT

- BOEFT d.o.f involve color singlet fields $\hat{\Psi}_{\kappa\lambda}(\mathbf{r}, \mathbf{R}, t) \propto P_{\kappa\lambda}^i O^{a\dagger}(\mathbf{r}, \mathbf{R}, t) G_{\kappa}^{ia}(\mathbf{R}, t)$
 - $O^{a\dagger}(\mathbf{r}, \mathbf{R}, t) G_{\kappa}^{ia}(\mathbf{R}, t)$: Gluelump operator. Eigenvector of NRQCD Hamiltonian in ($m \rightarrow \infty$):
- $$H^{(0)} O^{a\dagger}(\mathbf{r}, \mathbf{R}, t) G_{\kappa}^{ia}(\mathbf{R}, t) |0\rangle = (V_0(r) + \Lambda_{\kappa}) O^{a\dagger}(\mathbf{r}, \mathbf{R}, t) G_{\kappa}^{ia}(\mathbf{R}, t) |0\rangle \quad \Lambda_{\kappa} : \text{Gluelump energy}$$
- $P_{\kappa\lambda}^i$: Projection operators of light d.o.f along heavy quark-antiquark axis.
 - BOEFT Lagrangian:
- $$L_{\text{BOEFT}} = \int d^3 R d^3 r \sum_{\kappa} \sum_{\lambda\lambda'} \hat{\Psi}_{\kappa\lambda}^{\dagger}(\mathbf{r}, \mathbf{R}, t) \left\{ i\partial_t - V_{\kappa\lambda\lambda'}(r) + P_{\kappa\lambda}^{i\dagger} \frac{\nabla_r^2}{m} P_{\kappa\lambda'}^i \right\} \hat{\Psi}_{\kappa\lambda'}(\mathbf{r}, \mathbf{R}, t) + \dots$$
- Schrödinger Eq: Dynamics of $Q\bar{Q}$ at scale $mv^2 \ll \Lambda_{\text{QCD}}$
- Schrödinger equation

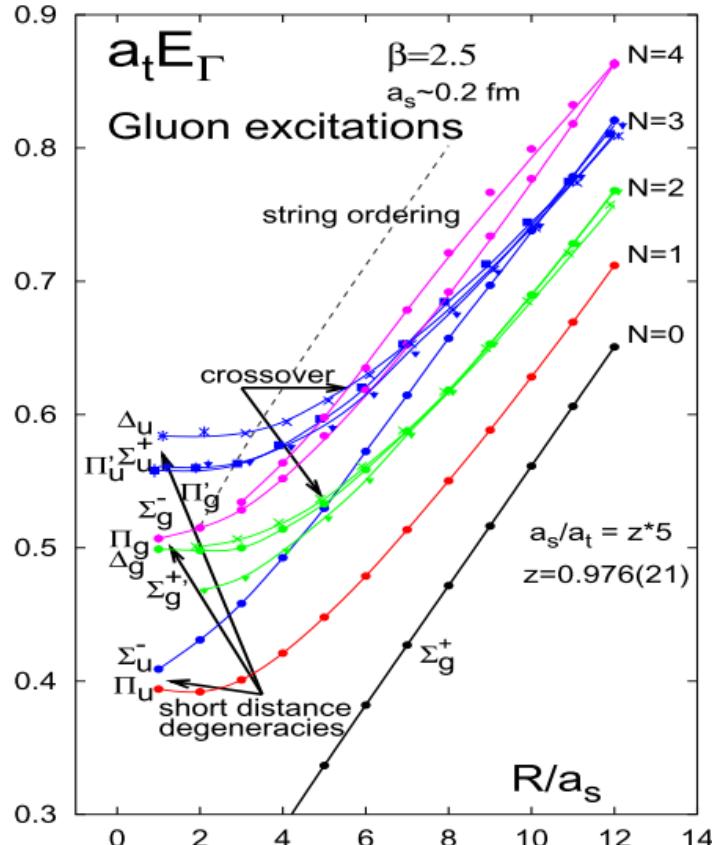
$$\left[-P_{\kappa\lambda}^{i\dagger} \frac{\nabla_r^2}{m} P_{\kappa\lambda'}^i + V_{\kappa\lambda\lambda'}(r) \right] \Psi_{\kappa\lambda'}^n(\mathbf{r}) = E_n \Psi_{\kappa\lambda}^n(\mathbf{r})$$
- Hybrid wf
- Coupled Eq. due to projection operators. Mixes Σ_u and Π_u states.

Quarkonium hybrids: Spectrum

- Lattice potentials for solving the Schrödinger Eq:

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Gluonic Static energies from lattice:



K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

$$m_c^{RS} = 1.477(40) \text{ GeV}$$

$$m_b^{RS} = 4.863(55) \text{ GeV}$$

Quarkonium Potential:

$$V_{\Sigma_g^+}(r) = -\frac{\kappa_g}{r} + \sigma_g r + E_g^{Q\bar{Q}}$$

$$\kappa_g = 0.489, \quad \sigma_g = 0.187 \text{ GeV}^2$$

$$E_g^{c\bar{c}} = -0.254 \text{ GeV}, \quad E_g^{b\bar{b}} = -0.195 \text{ GeV},$$

RS-Scheme Hybrid Potential:

$$E_n^{(0)}(r) = \begin{cases} V_o^{RS}(\nu_f) + \Lambda_H^{RS}(\nu_f) + b_n r^2, & r < 0.25 \text{ fm} \\ \frac{a_1}{r} + \sqrt{a_2 r^2 + a_3} + a_4, & r > 0.25 \text{ fm} \end{cases}$$

$$a_1^\Sigma = 0.000 \text{ GeV fm},$$

$$a_1^\Pi = 0.023 \text{ GeV fm},$$

$$b_\Sigma = 1.246 \text{ GeV/fm}^2,$$

$$a_2^\Sigma = 1.543 \text{ GeV}^2/\text{fm}^2, \quad a_3^\Sigma = 0.599 \text{ GeV}^2, \quad a_4^\Sigma = 0.154 \text{ GeV},$$

$$a_2^\Pi = 2.716 \text{ GeV}^2/\text{fm}^2, \quad a_3^\Pi = 11.091 \text{ GeV}^2, \quad a_4^\Pi = -2.536 \text{ GeV},$$

$$b_\Pi = 0.000 \text{ GeV/fm}^2$$

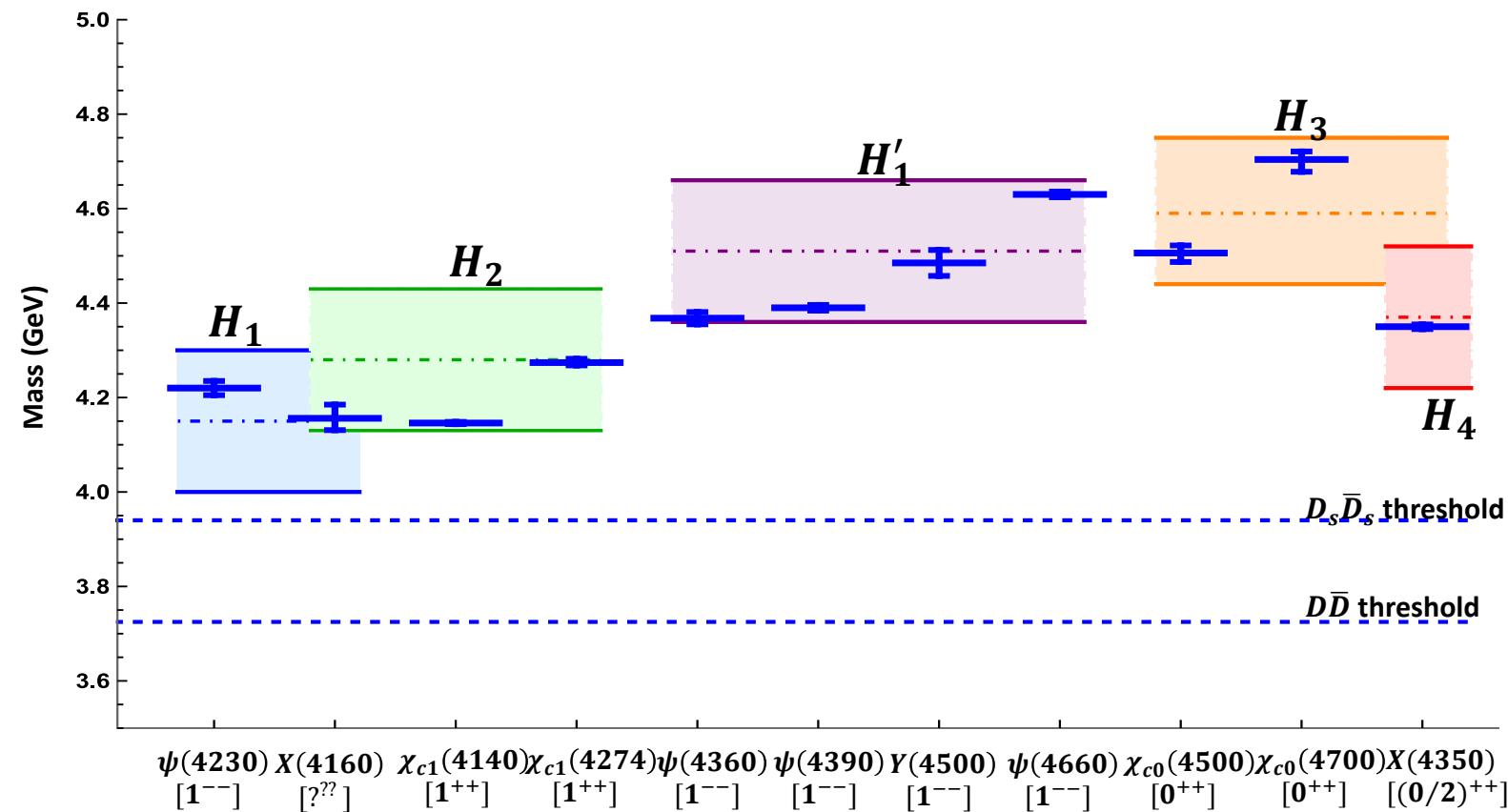
✓ Perturbative RS-scheme potentials V_o^{RS} upto order α_s^3 .

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015) Bali and Pineda Phys. Rev. D. 69, (2004)

Kniehl, Penin, Schroder , Smirnov, Steinhauser Phys. Lett. B 607, (2005)

Quarkonium hybrids: Spectrum

- Charmonium hybrids: comparison with experimental results:



Other notation of hybrid states

	l	$J^{PC}\{s = 0, s = 1\}$	$E_n^{(0)}$
$N(s/d)_1$	1	$\{1^{--}, (0, 1, 2)^{-+}\}$	Σ_u^-, Π_u
Np_1	1	$\{1^{++}, (0, 1, 2)^{+-}\}$	Π_u
Np_0	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
$N(p/f)_2$	2	$\{2^{++}, (1, 2, 3)^{+-}\}$	Σ_u^-, Π_u
Nd_2	2	$\{2^{--}, (1, 2, 3)^{-+}\}$	Π_u

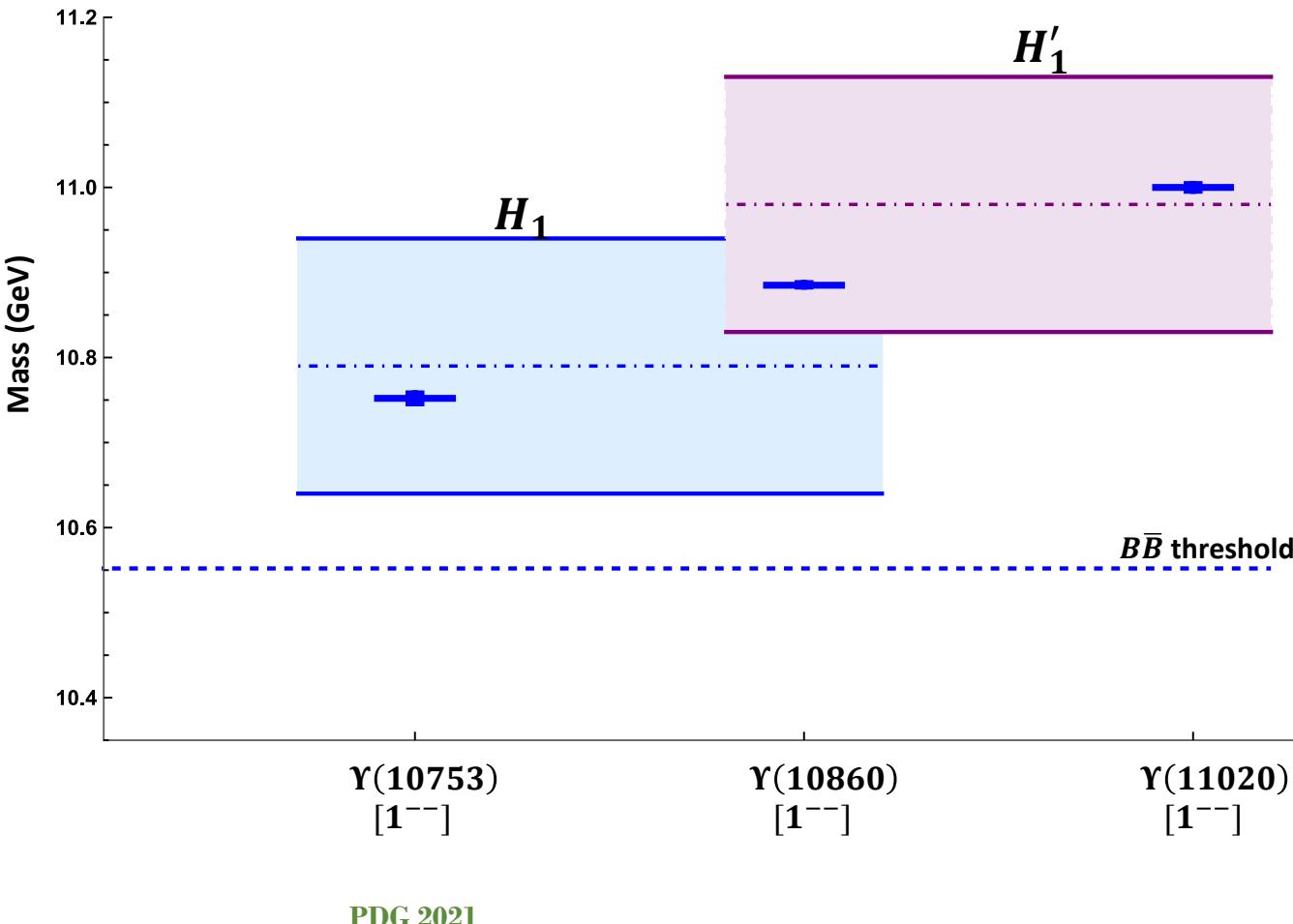
Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Quarkonium hybrids: Spectrum

- Bottomonium hybrids: comparison with experimental results:



Other notation of hybrid states

	l	$J^{PC}\{s = 0, s = 1\}$	$E_n^{(0)}$
$N(s/d)_1$	1	{1 ⁻⁻ , (0, 1, 2) ⁻⁺ }	Σ_u^- , Π_u
Np_1	1	{1 ⁺⁺ , (0, 1, 2) ⁺⁻ }	Π_u
Np_0	0	{0 ⁺⁺ , 1 ⁺⁻ }	Σ_u^-
$N(p/f)_2$	2	{2 ⁺⁺ , (1, 2, 3) ⁺⁻ }	Σ_u^- , Π_u
Nd_2	2	{2 ⁻⁻ , (1, 2, 3) ⁻⁺ }	Π_u

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

Inclusive Decays

- Dozens of XYZ states have been discovered (mass and decay rates measured) but physics still unknown.
- Several theoretical models for exotic states but no general consensus.
- Most of the exotic states discovered from decays to quarkonium. Decays might provide essential information on the structure of XYZ.
- Decay process of low-lying hybrids H_m :
 - ❖ Inclusive decays: $H_m \rightarrow X$
 - ❖ Semi-inclusive decays: $H_m \rightarrow Q_n + X$
- Both processes are characterized by energy gap ΔE .
 - ✓ $\Delta E \approx$ Energy difference of quark-antiquark pair in gluelump and singlet $\Rightarrow \Delta E \sim E_o + \Lambda_k - E_s$.
 - ✓ Assume hierarchy of scales: $mv \gg \Delta E \gg \Lambda_{QCD} \gg mv^2$
- Start with pNRQCD effective theory and obtain BOEFT by matching: Integrate out modes of scale $\sim \Delta$ and $\sim \Lambda_{QCD}$.

Inclusive Decays

- pNRQCD Lagrangian:

Weakly-coupled pNRQCD Lagrangian

$$\begin{aligned} L_{\text{pNRQCD}} = & \int d^3 R \left\{ \int d^3 r \left(\text{Tr} [S^\dagger (i\partial_0 - h_s) S + O^\dagger (iD_0 - h_o) O] \right. \right. \\ & + g \text{Tr} \left[S^\dagger \mathbf{r} \cdot \mathbf{E} O + O^\dagger \mathbf{r} \cdot \mathbf{E} S + \frac{1}{2} O^\dagger \mathbf{r} \cdot \{ \mathbf{E}, O \} \right] + \frac{g}{4m} \text{Tr} [O^\dagger \mathbf{L}_{Q\bar{Q}} \cdot [\mathbf{B}, O]] \\ & \left. \left. + \frac{gc_F}{m} \text{Tr} [S^\dagger (\mathbf{S}_1 - \mathbf{S}_2) \cdot \mathbf{B} O + O^\dagger (\mathbf{S}_1 - \mathbf{S}_2) \cdot \mathbf{B} S + O^\dagger \mathbf{S}_1 \cdot \mathbf{B} O - O^\dagger \mathbf{S}_2 O \cdot \mathbf{B}] - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} \right) \right\} \end{aligned}$$

- BOEFT:

BOEFT Hamiltonian

$$H_{\text{BOEFT}} = \int d^3 x \int d^3 R \text{Tr} \left[H^{i\dagger} \left(h_o \delta^{ij} + V_{soft}^{ij} \right) H^j \right]$$

Potential term in BOEFT

- Decays are computed from local imaginary terms in the BOEFT Lagrangian.
- Imaginary term in V_{soft}^{ij} from 1-loop diagram in pNRQCD and then matching to BOEFT .
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Inclusive Decays

- pNRQCD Lagrangian:

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

Weakly-coupled pNRQCD Lagrangian

$$L_{\text{pNRQCD}} = \int d^3 R \left\{ \int d^3 r \left(\text{Tr} [S^\dagger (i\partial_0 - h_s) S + O^\dagger (iD_0 - h_o) O] \right. \right.$$

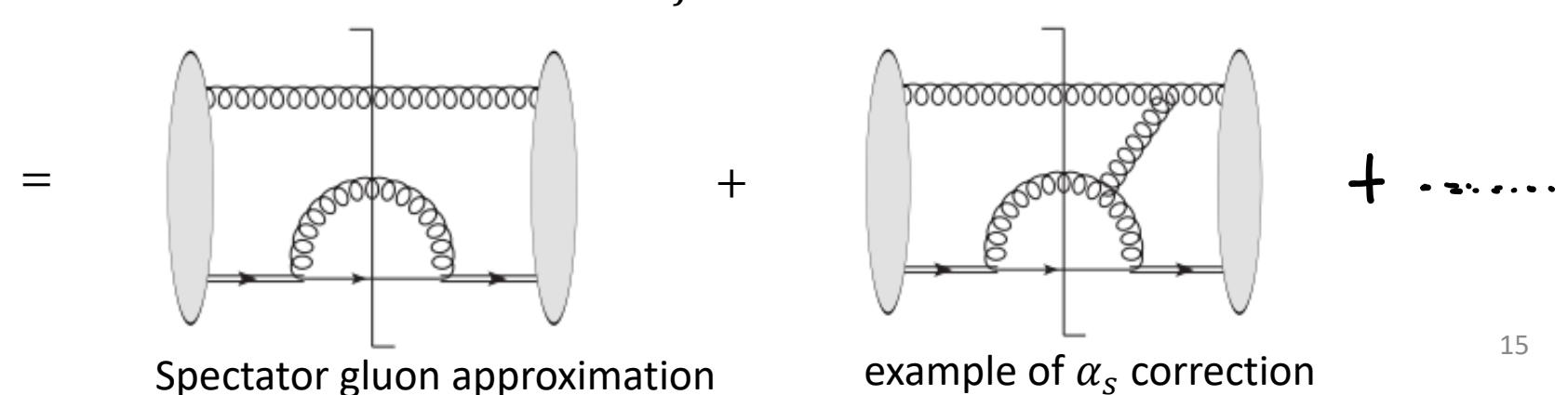
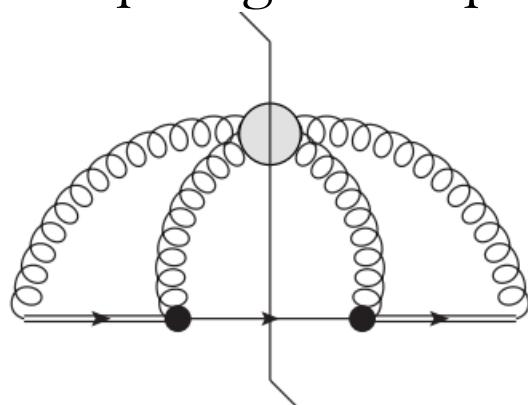
$$+ g \text{Tr} \left[S^\dagger \mathbf{r} \cdot \mathbf{E} O + O^\dagger \mathbf{r} \cdot \mathbf{E} S + \frac{1}{2} O^\dagger \mathbf{r} \cdot \{ \mathbf{E}, O \} \right] + \frac{g}{4m} \text{Tr} [O^\dagger \mathbf{L}_{Q\bar{Q}} \cdot [\mathbf{B}, O]]$$

$$\left. \left. + \frac{gc_F}{m} \text{Tr} [S^\dagger (S_1 - S_2) \cdot \mathbf{B} O + O^\dagger (S_1 - S_2) \cdot \mathbf{B} S + O^\dagger S_1 \cdot \mathbf{B} O - O^\dagger S_2 O \cdot \mathbf{B}] - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} \right\} \right)$$

- Spin preserving decays

- Spin flipping decays

- 1-loop diagram in pNRQCD contributing to $\text{Im } V_{\text{soft}}^{ij}$ in BOEFT:



Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

- Spin-conserving decay due to $\mathbf{r} \cdot \mathbf{E}$ term :

$$\begin{aligned} |S_H = 1\rangle &\longrightarrow |S_Q = 1\rangle \\ |S_H = 0\rangle &\longrightarrow |S_Q = 0\rangle \end{aligned}$$

$$\Gamma_{\text{Incl}} = \sum_{n'} \Gamma_{m,n'} + \int \frac{d^3 p_s}{(2\pi)^3} \Gamma_{m,p_s}$$

[red bracket under $\sum_{n'}$] bound singlet states
[purple bracket under \int] continuum singlet states

$$\Gamma_{m,q} = \frac{4\alpha_s T_F}{3N_c} \int \frac{d^3 l}{(2\pi)^3} \int \frac{d^3 l'}{(2\pi)^3} f_{m\mathbf{l}}^i g_{\mathbf{l}q}^k g_{\mathbf{l}'q}^{k\dagger} f_{m\mathbf{l}'}^{i\dagger} (\Lambda + E_{\mathbf{l}}^o/2 + E_{\mathbf{l}'}^o/2 - E_q^s)^3$$

$$q = (n', p_s)$$

Depends on several
Overlap functions:

$$\begin{aligned} f_{(m)\mathbf{l}}^i &\equiv \langle H_m | \Phi_{\mathbf{l}}^o \rangle = \int d^3 \mathbf{r} \Psi_{(m)}^{i\dagger}(\mathbf{r}) \Phi_{\mathbf{l}}^o(\mathbf{r}), \\ g_{\mathbf{l}q}^k &\equiv \langle \Phi_{\mathbf{l}}^o | r^k | \Phi_q^s \rangle = \int d^3 \mathbf{r} \Phi_{\mathbf{l}}^{o\dagger}(\mathbf{r}) r^k \Phi_q^s(\mathbf{r}), \end{aligned}$$

$\Psi_{(m)}^i$: Hybrid wf

$\Phi_{\mathbf{l}}^o$: Octet wf

Φ_q^s : Singlet wf

- ✓ Cubic factor $(\Lambda + E_{\mathbf{l}}^o/2 + E_{\mathbf{l}'}^o/2 - E_q^s)^3 \sim \Delta E^3$
- ✓ Including continuum states can account for decay to meson-meson thresholds.

- For singlet wf : $V_s = -\frac{4\alpha_s(mv)}{3}$, where $v \sim 1/\sqrt{3}$ for charm and $\sim 1/\sqrt{10}$ for bottom

Semi-inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

- Spin-conserving decay due to $\mathbf{r} \cdot \mathbf{E}$ term :

$$\Gamma(H_m \rightarrow Q_n) = \sum_{n'} |w_{nn'}|^2 \Gamma_{m,n'} + \int \frac{d^3 \mathbf{p}_s}{(2\pi)^3} |w_{np_s}|^2 \Gamma_{m,p_s}$$

bound singlet states
continuum singlet states

$$\Gamma_{m,q} = \frac{4\alpha_s T_F}{3N_c} \int \frac{d^3 \mathbf{l}}{(2\pi)^3} \int \frac{d^3 \mathbf{l}'}{(2\pi)^3} f_{m \mathbf{l}}^i g_{\mathbf{l} q}^k g_{\mathbf{l}' q}^{k\dagger} f_{m \mathbf{l}'}^{i\dagger} (\Lambda + E_{\mathbf{l}}^o/2 + E_{\mathbf{l}'}^o/2 - E_q^s)^3$$

$$q = (n', p_s)$$

Depends on several Overlap functions:

$$f_{(m) \mathbf{l}}^i \equiv \langle H_m | \Phi_{\mathbf{l}}^o \rangle = \int d^3 \mathbf{r} \Psi_{(m)}^{i\dagger}(\mathbf{r}) \Phi_{\mathbf{l}}^o(\mathbf{r}),$$

$$g_{\mathbf{l} q}^k \equiv \langle \Phi_{\mathbf{l}}^o | r^k | \Phi_q^s \rangle = \int d^3 \mathbf{r} \Phi_{\mathbf{l}}^{o\dagger}(\mathbf{r}) r^k \Phi_q^s(\mathbf{r}),$$

$$w_{qn} = \int d^3 \mathbf{r} \Phi_q^{s\dagger}(\mathbf{r}) \boxed{\Phi_{(n)}^Q(\mathbf{r})}$$

Quarkonium wf

- ✓ Significant overlap between quarkonium and continuum singlet states except for 1s quarkonium.

Decay rate different from R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017) and J. Castellà, E. Passemar, Phys. Rev. D104, 034019 (2021).

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

- Spin-flipping decay due to $\mathbf{S} \cdot \mathbf{B}$ term:

$$\begin{aligned} |S_H = 1\rangle &\longrightarrow |S_Q = 0\rangle \\ |S_H = 0\rangle &\longrightarrow |S_Q = 1\rangle \end{aligned}$$

$$\Gamma_{\text{Incl}} = \sum_{n'} \Gamma_{m,n'} + \int \frac{d^3 \mathbf{p}_s}{(2\pi)^3} \Gamma_{m,p_s}$$

[red bracket under sum]
[purple bracket under integral]
bound singlet states **continuum singlet states**

$$\Gamma_{m,q} = \frac{4\alpha_s T_F}{3N_c m_Q^2} \int \frac{d^3 \mathbf{l}}{(2\pi)^3} \int \frac{d^3 \mathbf{l}'}{(2\pi)^3} f_{m \mathbf{l}}^i g_{\mathbf{l} q}^k g_{\mathbf{l}' q}^{k\dagger} f_{m \mathbf{l}'}^{i\dagger} (\Lambda + E_{\mathbf{l}}^o/2 + E_{\mathbf{l}'}^o/2 - E_q^s)^3 \quad q = (n', p_s)$$

Depends on several
Overlap functions:

$$\begin{aligned} f_{(m) \mathbf{l}}^i &\equiv \langle H_m | \Phi_{\mathbf{l}}^o \rangle = \int d^3 \mathbf{r} \Psi_{(m)}^{i\dagger}(\mathbf{r}) \Phi_{\mathbf{l}}^o(\mathbf{r}), \\ g_{\mathbf{l} q}^k &\equiv \langle \Phi_{\mathbf{l}}^o | (S_1^k - S_2^k) | \Phi_{n'}^s \rangle = \left[\int d^3 \mathbf{r} \Phi_{\mathbf{l}}^{o\dagger}(\mathbf{r}) \Phi_{n'}^s(\mathbf{r}) \right] \langle \chi_o | (S_1^k - S_2^k) | \chi_s \rangle \end{aligned}$$

$|\chi_{o,s}\rangle$: Spin wf of octet and singlet

- $Q_m \rightarrow Q_n + X$ spin-flipping decays: Decay rate suppressed by additional $(\mathbf{r} \cdot \mathbf{E})^2 \sim v^2$ vertex factor.

Difficulties with continuum singlet states

- Dipole matrix element:

W. Gordon, Ann. Phys. (Leipzig) 2, 1031 (1929)

A. Maquet, Phys. Rev. A 15, 1088 (1977)

$$g_{\mathbf{k}_o \mathbf{p}_s}^k \equiv \langle \Phi_{\mathbf{k}_o}^o | \mathbf{r} | \Phi_{\mathbf{p}_s}^s \rangle = \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \left[\int r^2 A_l^*(k_o, r) r B_{l'}(p_s, r) dr \right] \left[d\Omega P_l(\hat{\mathbf{k}}_o \cdot \hat{\mathbf{r}}) \hat{r} P_{l'}(\hat{\mathbf{p}}_s \cdot \hat{\mathbf{r}}) \right]$$

Continuum radial wf for Coulomb octet and singlet

After integrating over $d\Omega$: $l' = l + 1$ or $l' = l - 1$

- ✓ Radial matrix element:

$$R_{l,l+1}(k_o, p_s) = C_{l,l+1}(k_o, p_s)$$

$$R_{l,l-1}(k_o, p_s) = C_{l,l-1}(k_o, p_s)$$

$$\begin{aligned} & \mathcal{J}_{l+2+i\eta_s, l+1+i\eta_o, 2l+4, 2}^{-2ip_s, -2ik_o, 1} \\ & \mathcal{J}_{l+1+i\eta_o, l+i\eta_s, 2l+2, 2}^{-2ik_o, -2ip_s, 1} \end{aligned}$$

$$\eta_o = m_Q \alpha_s / 12 k_o$$

$$\eta_s = -4 m_Q \alpha_s / 6 p_s$$

Smooth function of octet and singlet momentum k_o and p_s

Singular function:
“Diagonal Singularity” for $k_o \rightarrow p_s$

Madajczyk, Trippenbach, J. Phys. A: Math. Gen. 22 2369 (1989)

Veniard, Piraux, Phys. Rev. A 41, 4019 (1989)

Difficulties with continuum singlet states

$$\begin{aligned} \mathcal{J}_{l+2+i\eta_s, l+1+i\eta_o, 2l+4, 2}^{-2ip_s, -2ik_o, 1} &= -\frac{(2l+3)!e^{-\frac{\pi m_Q}{4}\left(\frac{\alpha_s}{6k_o} + \frac{4\alpha_s}{3p_s}\right)} e^{-\frac{\pi m_Q}{4}\left(-\frac{\alpha_s}{6k_o} + \frac{4\alpha_s}{3p_s}\right)\text{sgn}(p_s-k_o)}}{4(p_s-k_o)^{2l+4} p_s^2} \left| \frac{p_s - k_o}{k_o + p_s} \right|^{i\frac{m_Q}{2}\left(\frac{-\alpha_s}{6k_o} + \frac{4\alpha_s}{3p_s}\right)} \left({}_2F_1 \left[l+2+i\eta_s, l+1+i\eta_o, 2l+2, -\frac{4p_s k_o}{(p_s - k_o)^2} \right] \left(2ip_s + \frac{3m_Q \alpha_s}{2} \right) \right. \\ &\quad \left. + 3m_Q \alpha_s \left(\frac{p_s - k_o}{k_o + p_s} \right) {}_2F_1 \left[l+1+i\eta_s, l+1+i\eta_o, 2l+2, -\frac{4p_s k_o}{(p_s - k_o)^2} \right] + {}_2F_1 \left[l+i\eta_s, l+1+i\eta_o, 2l+2, -\frac{4p_s k_o}{(p_s - k_o)^2} \right] \left(-2ip_s + \frac{3m_Q \alpha_s}{2} \right) \left(\frac{p_s - k_o}{k_o + p_s} \right)^2 \right) \end{aligned}$$

$$\begin{aligned} \mathcal{J}_{l+1+i\eta_o, l+i\eta_s, 2l+2, 2}^{-2ik_o, -2ip_s, 1} &= \frac{(2l+1)!e^{-\frac{\pi m_Q}{4}\left(\frac{\alpha_s}{6k_o} + \frac{4\alpha_s}{3p_s}\right)} e^{-\frac{\pi m_Q}{4}\left(-\frac{\alpha_s}{6k_o} + \frac{4\alpha_s}{3p_s}\right)\text{sgn}(p_s-k_o)}}{4(p_s-k_o)^{2l+2} k_o^2} \left| \frac{p_s - k_o}{k_o + p_s} \right|^{i\frac{m_Q}{2}\left(\frac{-\alpha_s}{6k_o} + \frac{4\alpha_s}{3p_s}\right)} \left({}_2F_1 \left[l+1+i\eta_o, l+i\eta_s, 2l, -\frac{4p_s k_o}{(p_s - k_o)^2} \right] \left(-2ik_o + \frac{3m_Q \alpha_s}{2} \right) \right. \\ &\quad \left. - 3m_Q \alpha_s \left(\frac{p_s - k_o}{k_o + p_s} \right) {}_2F_1 \left[l+i\eta_o, l+i\eta_s, 2l, -\frac{4p_s k_o}{(p_s - k_o)^2} \right] + {}_2F_1 \left[l-1+i\eta_o, l+i\eta_s, 2l, -\frac{4p_s k_o}{(p_s - k_o)^2} \right] \left(2ik_o + \frac{3m_Q \alpha_s}{2} \right) \left(\frac{p_s - k_o}{k_o + p_s} \right)^2 \right) \end{aligned}$$

“Diagonal Singularity” for $k_o \rightarrow p_s$: Singular Gauss hypergeometric ${}_2F_1$ function

Preliminary Results

- Inclusive Rate:

Decays not allowed in
**R. Oncala, J. Soto,
 Phys. Rev. D96,
 014004 (2017)**

$H_m [J^{PC}]$ (Mass)	$\Gamma_{r.E}$ (MeV)	$\Gamma_{S.B}$ (MeV)	$\Gamma_{Incl} = \Gamma_{r.E} + \Gamma_{S.B}$ (MeV)
charmonium hybrid decay			
$H_1 [1^{--}]$ (4155)	$27^{+13}_{-6} {}^{+18}_{-12}$	$8^{+4}_{-2} {}^{+3}_{-3}$	$35^{+14}_{-6} {}^{+18}_{-12}$
$H'_1 [1^{--}]$ (4507)	$45^{+19}_{-10} {}^{+24}_{-18}$	$9^{+4}_{-2} {}^{+4}_{-3}$	$54^{+19}_{-10} {}^{+24}_{-18}$
$H_2 [1^{++}]$ (4286)	$4^{+2}_{-1} {}^{+1}_{-1}$	$19^{+9}_{-4} {}^{+10}_{-7}$	$23^{+9}_{-4} {}^{+10}_{-7}$
$H'_2 [1^{++}]$ (4667)	$60^{+22}_{-12} {}^{+40}_{-28}$	$11^{+4}_{-2} {}^{+7}_{-5}$	$71^{+22}_{-12} {}^{+41}_{-28}$
$H_3 [0^{++}]$ (4590)	$15^{+5}_{-3} {}^{+4}_{-3}$	$12^{+5}_{-3} {}^{+6}_{-5}$	$27^{+7}_{-4} {}^{+7}_{-6}$
$H'_3 [0^{++}]$ (5054)	$47^{+16}_{-9} {}^{+23}_{-17}$	$11^{+4}_{-2} {}^{+6}_{-4}$	$58^{+16}_{-9} {}^{+24}_{-17}$
$H_4 [2^{++}]$ (4367)	$4^{+2}_{-1} {}^{+1}_{-1}$	$17^{+7}_{-4} {}^{+9}_{-6}$	$21^{+7}_{-4} {}^{+9}_{-6}$
bottomonium hybrid decay			
$H_1 [1^{--}]$ (10786)	$63^{+34}_{-16} {}^{+36}_{-26}$	$12^{+6}_{-3} {}^{+5}_{-4}$	$75^{+35}_{-16} {}^{+36}_{-26}$
$H'_1 [1^{--}]$ (10976)	$45^{+21}_{-11} {}^{+22}_{-16}$	$8^{+4}_{-2} {}^{+3}_{-3}$	$53^{+21}_{-11} {}^{+22}_{-16}$
$H''_1 [1^{--}]$ (11172)	$16^{+7}_{-3} {}^{+8}_{-6}$	$2^{+1}_{-0} {}^{+1}_{-1}$	$18^{+7}_{-3} {}^{+8}_{-6}$
$H_2 [1^{++}]$ (10846)	$21^{+11}_{-5} {}^{+12}_{-9}$	$7^{+4}_{-2} {}^{+4}_{-3}$	$28^{+12}_{-5} {}^{+13}_{-9}$
$H'_2 [1^{++}]$ (11060)	$59^{+27}_{-13} {}^{+27}_{-20}$	$1^{+0.42}_{-0.21} {}^{+0.59}_{-0.42}$	$60^{+27}_{-13} {}^{+27}_{-20}$
$H_3 [0^{++}]$ (11065)	$13^{+6}_{-3} {}^{+10}_{-7}$	$5^{+2}_{-1} {}^{+3}_{-2}$	$18^{+6}_{-3} {}^{+10}_{-6}$
$H'_3 [0^{++}]$ (11352)	$48^{+19}_{-10} {}^{+23}_{-18}$	$1.42^{+0.56}_{-0.30} {}^{+0.86}_{-0.62}$	$49^{+19}_{-10} {}^{+23}_{-18}$
$H''_3 [0^{++}]$ (11616)	$19^{+7}_{-4} {}^{+9}_{-7}$	$1.20^{+0.43}_{-0.24} {}^{+0.63}_{-0.46}$	$20^{+7}_{-4} {}^{+9}_{-7}$

. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

$$m_c^{RS} = 1.477(40) \text{ GeV}$$

$$m_b^{RS} = 4.863(55) \text{ GeV}$$

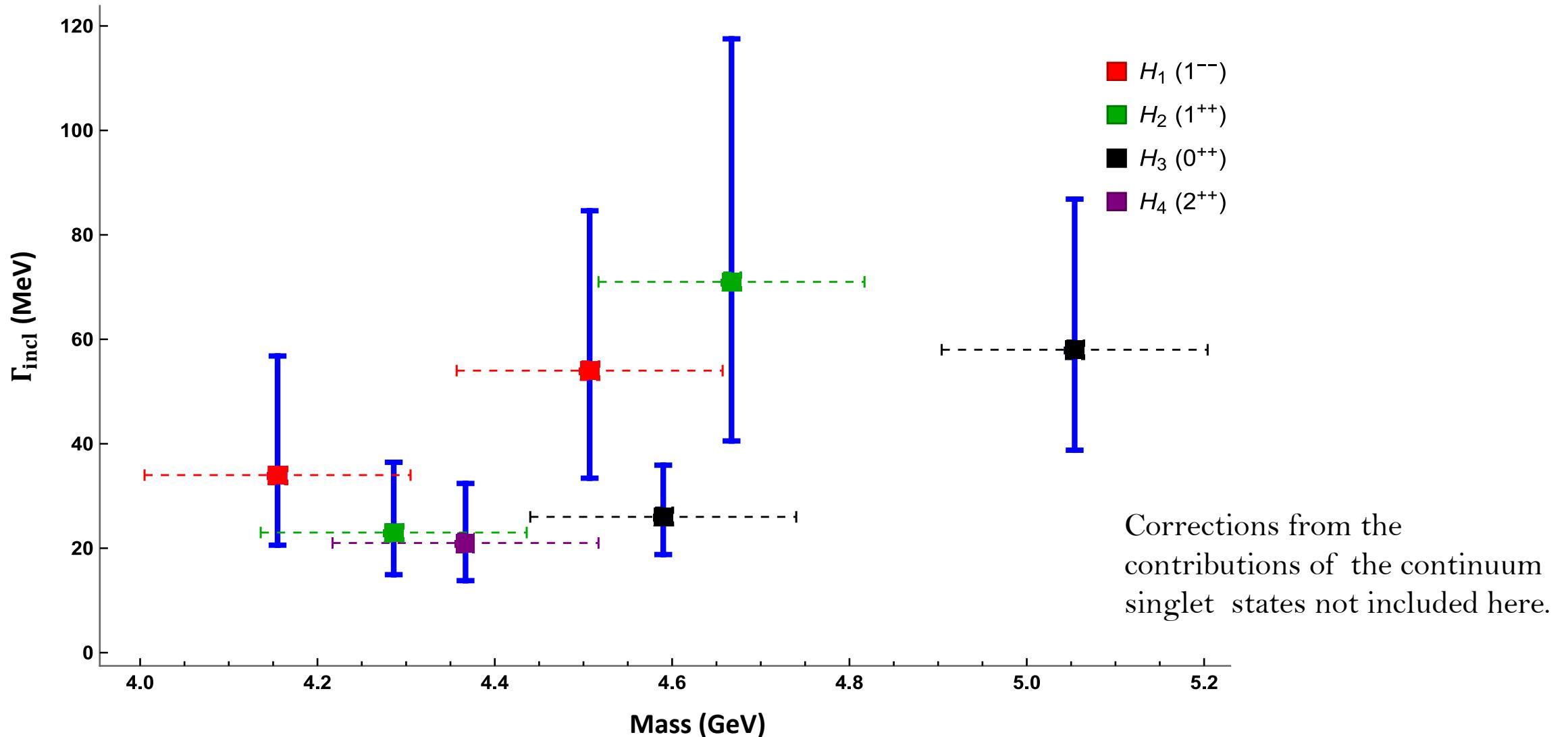
Error bars from higher order corrections in α_s + Error bar from gluclump mass (± 0.15 GeV).

The hybrid states with a prime and double prime denotes the first and the second excited state

Corrections from the contributions of the continuum singlet states not included here.

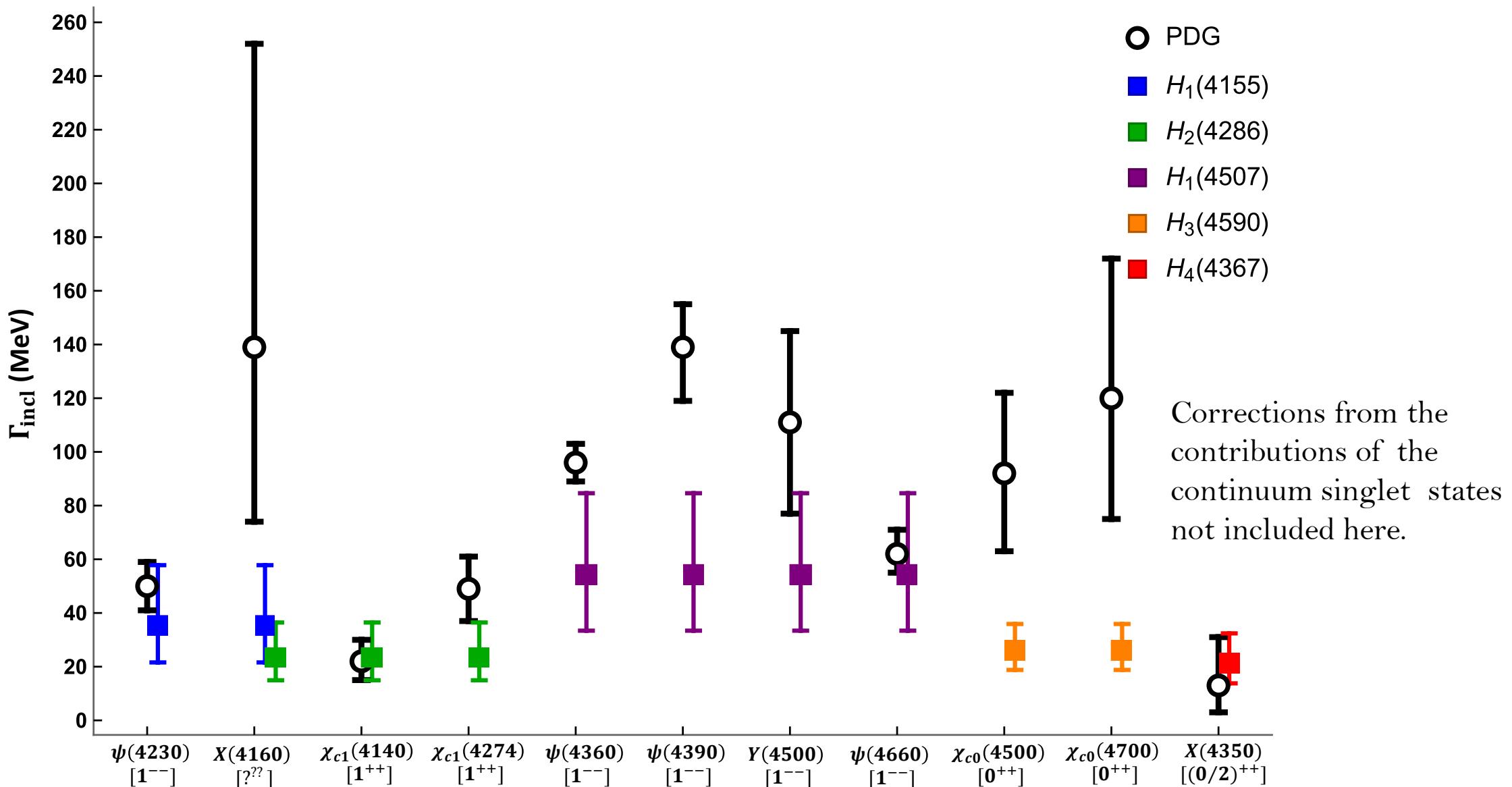
Preliminary Results

- Inclusive Rate for Charmonium Hybrids:



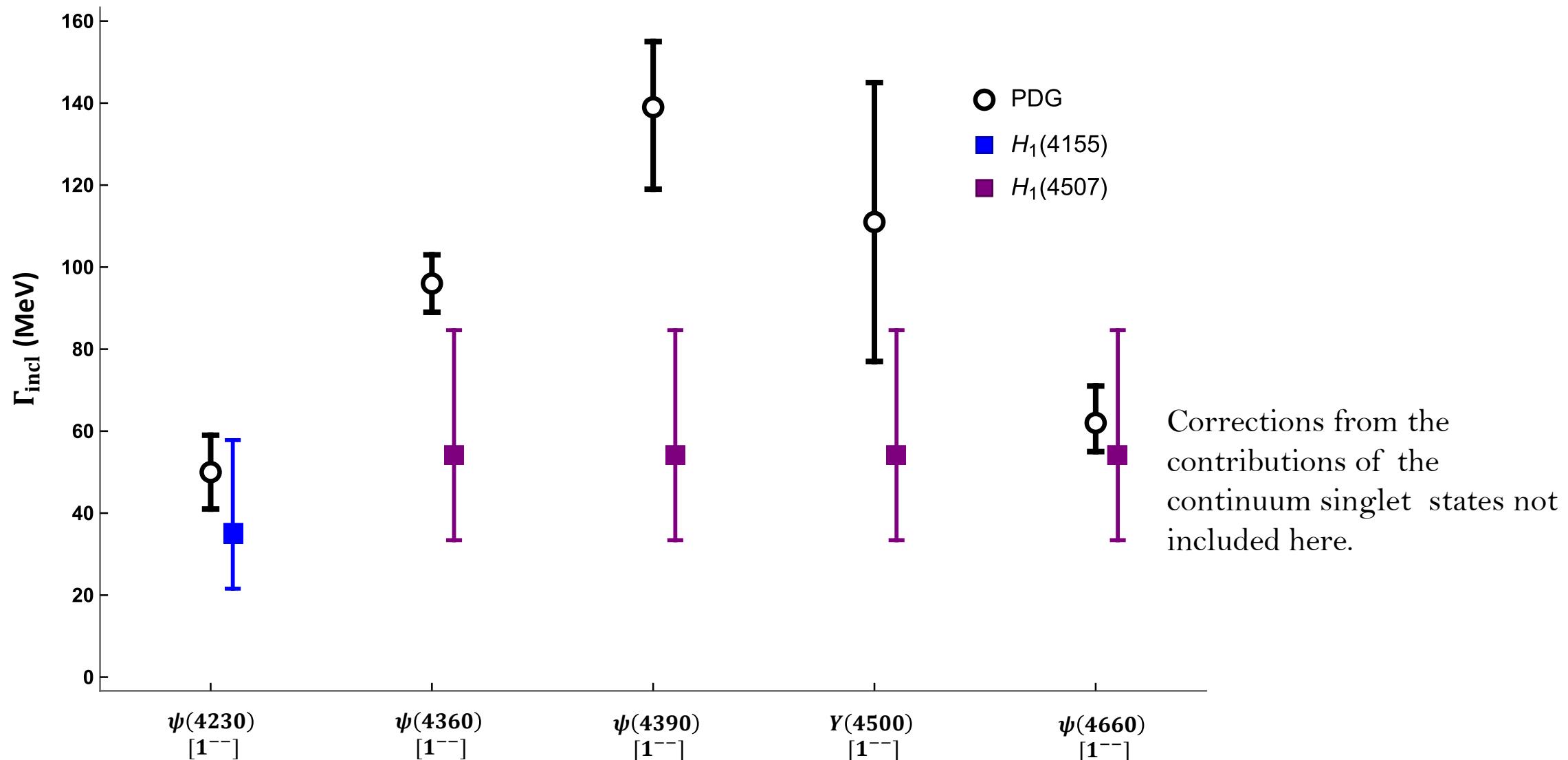
Preliminary Results

- Comparison with PDG states:



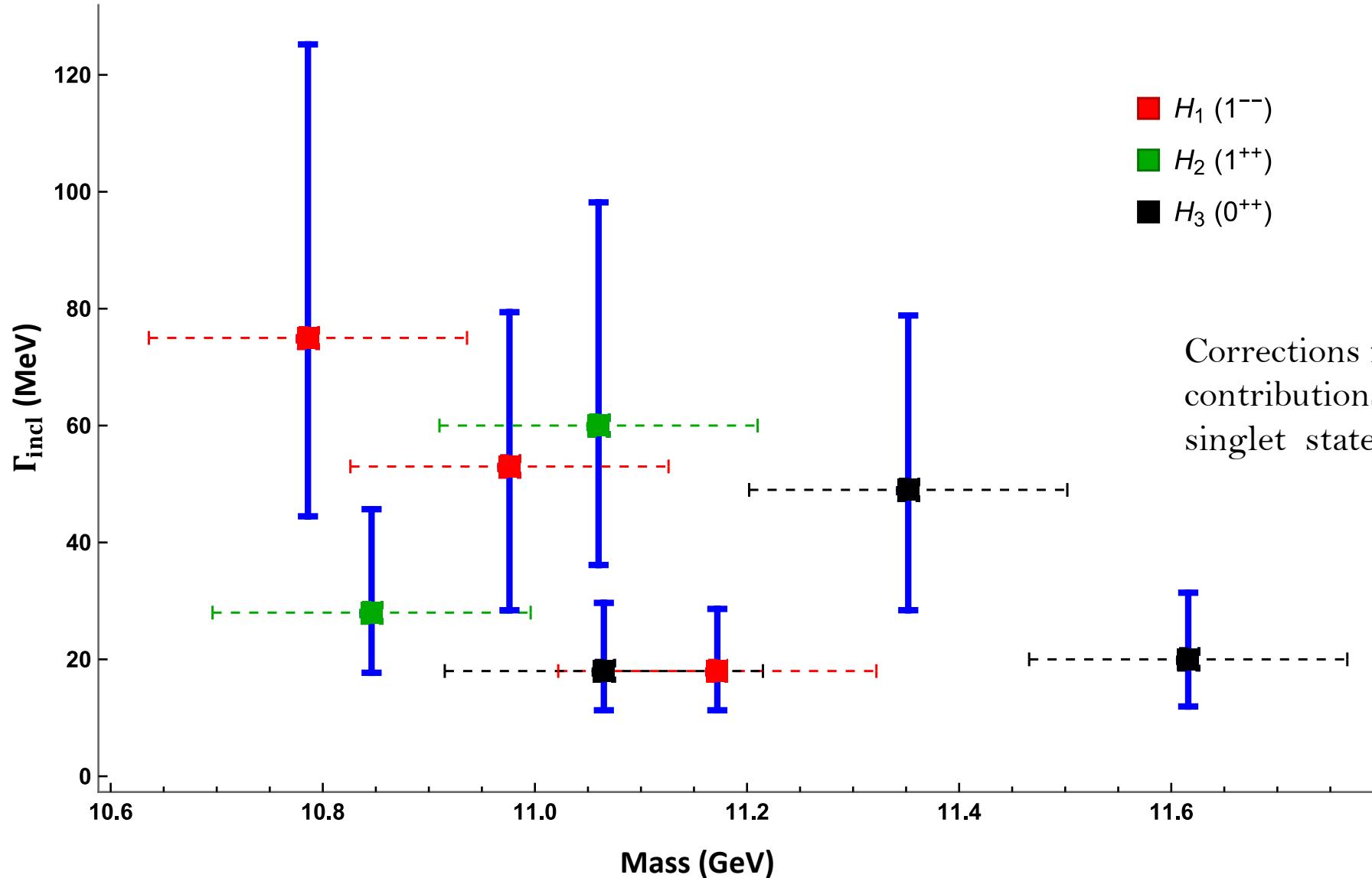
Preliminary Results

- Comparison for Y-states:



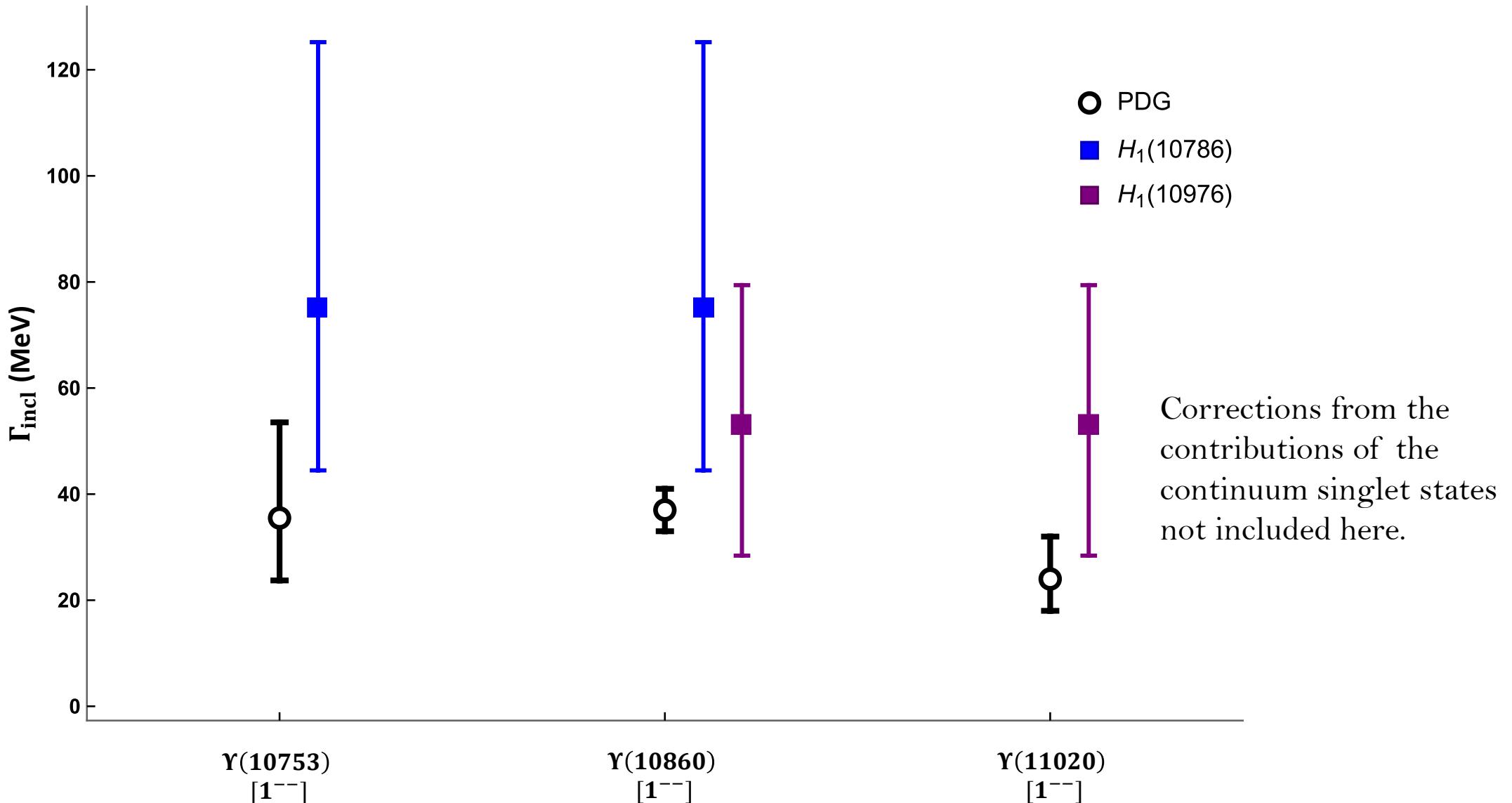
Preliminary Results

- Inclusive Rate for Bottomonium Hybrids:



Preliminary Results

- Comparison with PDG states:



Summary/Outlook

- BOEFT provides a model-independent & systematic way to study heavy quark hybrids (exotic) and decays.
- General formula for the H_m inclusive decay based on overlap functions:

Inclusive decay rate for $H_m \rightarrow Q_n + X$

$$\Gamma(H_m \rightarrow Q_n + X) = \frac{4\alpha_s T_F}{3N_c} \sum_{n'} |h_{nn'}|^2 \sum_{q,q'} \int dE \int dE' f_{mq}^i(E) g_{qn}^j(E) \\ \times g_{q'n}^{j\dagger}(E') f_{mq'}^{i\dagger}(E') (\Lambda + E/2 + E'/2 - E_n^s)^3$$

- Computed preliminary results on decay rates for spin-flipping and spin preserving decays.



- Future and ongoing work includes:
 - Computing contributions to the inclusive rate from the continuum singlet states.
 - Quantifying errors in the decay rates & comparing with the PDG data for observed exotic states.
 - Include effect of mixing with excited quarkonia $Q' \rightarrow Q + X$.
 - Extending this analysis to study Quarkonium tetraquarks.

Thank you!!

Backup Slides

Quarkonium hybrids: Spectrum

- Results for Hybrids from Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

multiplet	J^{PC}	$c\bar{c}$				$b\bar{c}$				$b\bar{b}$			
		m_H	$\langle 1/r \rangle$	E_{kin}	P_{II}	m_H	$\langle 1/r \rangle$	E_{kin}	P_{II}	m_H	$\langle 1/r \rangle$	E_{kin}	P_{II}
H_1	$\{1^{--}, (0, 1, 2)^{+-}\}$	4.15	0.42	0.16	0.82	7.48	0.46	0.13	0.83	10.79	0.53	0.09	0.86
		4.51	0.34	0.34	0.87	7.76	0.38	0.27	0.87	10.98	0.47	0.19	0.87
H_2	$\{1^{++}, (0, 1, 2)^{+-}\}$	4.28	0.28	0.24	1.00	7.58	0.31	0.19	1.00	10.84	0.37	0.13	1.00
		4.67	0.25	0.42	1.00	7.89	0.28	0.34	1.00	11.06	0.34	0.23	1.00
H_3	$\{0^{++}, 1^{+-}\}$	4.59	0.32	0.32	0.00	7.85	0.37	0.27	0.00	11.06	0.46	0.19	0.00
H_4	$\{2^{++}, (1, 2, 3)^{+-}\}$	4.37	0.28	0.27	0.83	7.65	0.31	0.22	0.84	10.90	0.37	0.15	0.87
H_5	$\{2^{--}, (1, 2, 3)^{+-}\}$	4.48	0.23	0.33	1.00	7.73	0.25	0.27	1.00	10.95	0.30	0.18	1.00
H_6	$\{3^{--}, (2, 3, 4)^{+-}\}$	4.57	0.22	0.37	0.85	7.82	0.25	0.30	0.87	11.01	0.30	0.20	0.89
H_7	$\{3^{++}, (2, 3, 4)^{+-}\}$	4.67	0.19	0.43	1.00	7.89	0.22	0.35	1.00	11.05	0.26	0.24	1.00

$$m_c^{RS} = 1.477(40) \text{ GeV}$$

$$m_b^{RS} = 4.863(55) \text{ GeV}$$

Other notation of hybrid states

	l	$J^{PC}\{s = 0, s = 1\}$	$E_n^{(0)}$
$N(s/d)_1$	1	$\{1^{--}, (0, 1, 2)^{+-}\}$	Σ_u^-, Π_u
Np_1	1	$\{1^{++}, (0, 1, 2)^{+-}\}$	Π_u
Np_0	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
$N(p/f)_2$	2	$\{2^{++}, (1, 2, 3)^{+-}\}$	Σ_u^-, Π_u
Nd_2	2	$\{2^{--}, (1, 2, 3)^{+-}\}$	Π_u

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

R. Oncala, J. Soto, Phys. Rev. D96 (2017)

$$H_{\text{BOEFT}} = \int d^3x \int d^3R \text{Tr} \left[H^{i\dagger} \left(h_o \delta^{ij} + V_{soft}^{ij} \right) H^j \right]$$

$$V_{soft}^{ij} = \Lambda + b^{ij} r^2 + \dots$$

Λ = gluelump mass ($= 0.87(15)$ GeV for lowest lying $\kappa = 1^{+-}$ gluelump)

For two insertions of the $\mathbf{r} \cdot \mathbf{E}$ vertex, the contribution to the two-point function is

$$\begin{aligned} & I_{ij}^{(2)}(\mathbf{r}, \mathbf{R}, \mathbf{r}', \mathbf{R}') \\ &= - \lim_{T \rightarrow \infty} g^2 \frac{T_F}{N_c} \int_{-T/2}^{T/2} dt \int_{-T/2}^t dt' e^{-ih_o(T/2-t)} r^k e^{-ih_s(t-t')} r^l e^{-ih_o(t'+T/2)} \\ & \quad \times \langle 0 | G^{ib}(T/2) \phi^{ab}(T/2, t) E^{kb}(t) E^{lc}(t') \phi^{cd}(t', -T/2) G^{jd}(-T/2) | 0 \rangle \mathbb{I} \delta^3(\mathbf{r} - \mathbf{r}') \delta^3(\mathbf{R} - \mathbf{R}') \end{aligned}$$

Decay Rate: Calculation

Details.

To separate the scales Δ and Λ_{QCD} , write $\mathbf{E} = \mathbf{E}_h + \mathbf{E}_s$, $\mathbf{E}_h \sim \Delta$, $\mathbf{E}_s \sim \Lambda_{\text{QCD}}$. Replace \mathbf{E} by \mathbf{E}_h to get the leading contribution.

$$\begin{aligned} & \langle 0 | G^{ib}(T/2) \phi^{ab}(T/2, t) E_h^{kb}(t) E_h^{lc}(t') \phi^{cd}(t', -T/2) G^{jd}(-T/2) | 0 \rangle \\ &= \langle 0 | G^{ib}(T/2) \phi^{ab}(T/2, t) \phi^{cd}(t', -T/2) G^{jd}(-T/2) | 0 \rangle \langle 0 | E_h^{kb}(t) E_h^{lc}(t') | 0 \rangle \\ &= \langle 0 | G^{ib}(T/2) \phi^{ab}(T/2, t) \phi^{bd}(t', -T/2) G^{jd}(-T/2) | 0 \rangle \frac{\delta^{kl}}{3} \int \frac{d^3 k}{(2\pi)^3} |\mathbf{k}| e^{-i|\mathbf{k}|(t-t')} \\ &\approx \frac{\delta^{kl}}{3} e^{i\Lambda(t-t')} \langle 0 | G^{ib}(T/2) \phi^{ab}(T/2, -T/2) G^{jb}(-T/2) | 0 \rangle \int \frac{d^3 k}{(2\pi)^3} |\mathbf{k}| e^{-i|\mathbf{k}|(t-t')} \\ &= \delta^{ij} \frac{\delta^{kl}}{3} e^{i\Lambda(t-t'-T)} \int \frac{d^3 k}{(2\pi)^3} |\mathbf{k}| e^{-i|\mathbf{k}|(t-t')} . \end{aligned}$$

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Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)

- Spin-conserving:

$$\Gamma_{\text{Incl}} = \text{Re} \frac{2g^2}{3} \frac{T_F}{N_c} \int d^3\mathbf{r} \int_0^\infty dt \Psi_{(m)}^{i\dagger}(\mathbf{r}) \left[e^{i\Lambda t} e^{ih_o t/2} r^k e^{-ih_s t} r^k e^{ih_o t/2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} |\mathbf{k}| e^{-i|\mathbf{k}|t} \right] \Psi_{(m)}^i(\mathbf{r}).$$

Using complete set of
octet and singlet states

$$\boxed{\Gamma_{\text{Incl}} = \sum_{n'} \Gamma_{m,n'} + \int \frac{d^3\mathbf{p}_s}{(2\pi)^3} \Gamma_{m,p_s}}$$

$$\Gamma_{m,q} = \frac{4\alpha_s T_F}{3N_c} \int \frac{d^3\mathbf{l}}{(2\pi)^3} \int \frac{d^3\mathbf{l}'}{(2\pi)^3} f_{m\mathbf{l}}^i g_{\mathbf{l}q}^k g_{\mathbf{l}'q}^{k\dagger} f_{m\mathbf{l}'}^{i\dagger} (\Lambda + E_{\mathbf{l}}^o/2 + E_{\mathbf{l}'}^o/2 - E_q^s)^3$$

$q = (n', p_s)$

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo
(in progress)

Spin-preserving inclusive decay rate for $H_m \rightarrow Q_n + X$

$$\begin{aligned} \Gamma(H_m \rightarrow Q_n + X) &= \frac{4\alpha_s T_F}{3N_c} \sum_{n'} |h_{nn'}|^2 \sum_{q,q'} \int dE \int dE' f_{mq}^i(E) g_{qn}^j(E) \\ &\times g_{q'n}^{j\dagger}(E') f_{mq'}^{i\dagger}(E') (\Lambda + E/2 + E'/2 - E_n^s)^3 \end{aligned}$$

Assumption:

$f_{mq}^i(E) \neq 0$ only for $E_m \approx E + \Lambda$

$h_{nn'} \approx 1$ and $E_m^Q \approx E_m^s$ (replace singlet with quarkonium)

Spin-preserving inclusive decay rate for $H_m \rightarrow Q_n + X$

$$\Gamma(H_m \rightarrow Q_n + X) = \frac{4\alpha_s T_F}{3N_c} (E_m - E_n^Q)^3 T^{ij} (T^{ij})^*$$

$$T^{ij} \equiv \int d^3r \Psi_m^{i\dagger}(\mathbf{r}) r^j \Phi_n^Q(\mathbf{r})$$

- Above result looks similar to the one in R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017). In general has **tensor structure T^{ij}** that agrees with J. Castellà, E. Passemar, arXiv:2104.03975.

Inclusive Decays

N. Brambilla, W.K. Lai, AM, A. Vairo
(in progress)

Spin-preserving inclusive decay rate for $H_m \rightarrow Q_n + X$

$$\Gamma(H_m \rightarrow Q_n + X) = \frac{4\alpha_s T_F}{3N_c} (E_m - E_n^Q)^3 T^{ij} (T^{ij})^*$$

$$T^{ij} \equiv \int d^3r \Psi_m^{i\dagger}(\mathbf{r}) r^j \Phi_n^Q(\mathbf{r})$$

- R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017): only **diagonal elements** T^{ii} are considered

Inclusive decay rate for $H_m \rightarrow Q_n + X$ computed by Oncala and Soto

$$\Gamma^{\text{Oncala}}(H_m \rightarrow Q_n + X) = \frac{4\alpha_s T_F}{3N_c} (E_m - E_n^Q)^3 T^{ii} (T^{jj})^*$$

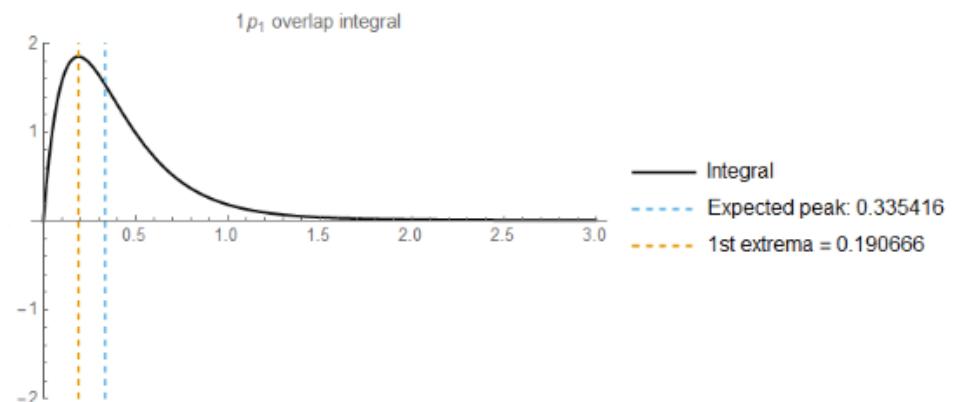
- ~~T^{ii} leads to selection rule: Hybrids such as Np_1 (H_2) where $L=J$ don't decay to quarkonium.~~
- \mathbf{T}^{ij} : allows for the decay of Np_1 (H_2) hybrid decays.

Inclusive Decays

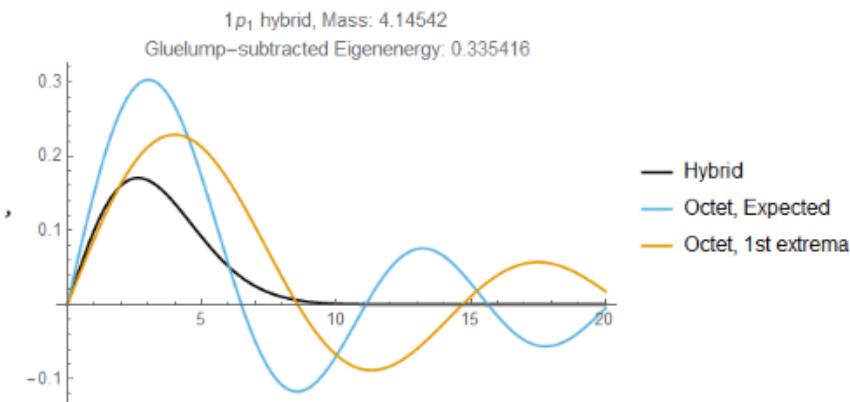
It is interesting to see how $f_{mq}^i(E) = \left[\int d^3r \Psi_m^i(\mathbf{r}) \Phi_{E,q}^o(\mathbf{r}) \right]$ looks like as a function of E :

H_2 -multiplet, $l = 1, J^{PC} = [1^{++}, (0, 1, 2)^{+-}]$
 $H_2(4145)$:

N. Brambilla, W.K. Lai, AM, A. Vairo (in progress)



Radial integral of $f_{mq}^i(E)$ vs E (GeV)



Radial hybrid wave function vs r (GeV⁻¹)

- The actual peak is slightly off (at a lower E) from the expected peak at $E = E_m - \Lambda$.
- The peak is broad, with width ~ 1 GeV. The assumption that $f_{mq}^i(E)$ is nonzero only when $E_m \approx E + \Lambda$ is not true.

Singlet-Quarkonium overlap

2) Overlap of Coulomb singlet bound states & Quarkonium

$$w_{nn'} = \left\langle \Phi_n^{q\bar{q}} \mid \Phi_{n'}^s \right\rangle$$

↓
 Quarkonium wf → Coulomb Singlet
 bound state wf.

3) Overlap of continuum singlet & Quarkonium

$$w_n = \int \frac{d^3k}{(2\pi)^3} \left| \left\langle \Phi_n^{q\bar{q}} \mid \Phi_k^s \right\rangle \right|^2$$

→ Continuum
singlet states

Charm

n	$\sum_n w_{nn'} ^2$	w_n
1s	0.73	0.26
2s	0.26	0.72
1p	0.18	0.76
2p	0.20	0.73

Bottom.

n	$\sum_n w_{nn'} ^2$	w_n
1s	0.90	0.10
2s	0.42	0.55
1p	0.45	0.51
2p	0.37	0.59

Exotic States

State (PDG)	State (Former)	M (MeV)	Γ (MeV)	J^{PC}	Decay modes
χ_{c1} (4140)	$Y(4140)$	4146.8 ± 2.4	22^{+8}_{-7}	1^{++}	$\phi J/\psi$
X (4160)		4156^{+29}_{-35}	139^{+113}_{-65}	$?^{?+}$	$D^* \bar{D}^*$
ψ (4230)	$Y(4230)$	4220 ± 15	50 ± 9	1^{--}	$\pi^+ \pi^- J/\psi, \omega \chi_{c0}(1P),$
	$Y(4260)$				$\pi^+ \pi^- h_c(1P)$
χ_{c1} (4274)	$Y(4274)$	4274^{+8}_{-6}	49 ± 12	1^{++}	$\phi J/\psi$
X (4350)		$4350.6^{+4.7}_{-5.1}$	13^{+18}_{-10}	$(0/2)^{++}$	$\phi J/\psi$
ψ (4360)	$Y(4360)$	4368 ± 13	96 ± 7	1^{--}	$\pi^+ \pi^- J/\psi,$
	$Y(4320)$				$\pi^+ \pi^- \psi(2S)$
ψ (4390)	$Y(4390)$	4390 ± 6	139^{+16}_{-20}	1^{--}	$\eta J/\psi, \pi^+ \pi^- h_c(1P)$
χ_{c0} (4500)	$X(4500)$	4506^{+16}_{-19}	92^{+30}_{-29}	0^{++}	$\phi J/\psi$
$Y(4500)^a$		4484.7 ± 27.5	111 ± 34	1^{--}	
ψ (4660)	$Y(4660)$	4630 ± 6	62^{+9}_{-7}	1^{--}	$\pi^+ \pi^- \psi(2S), \Lambda_c^+ \bar{\Lambda}_c^-,$
	$X(4630)$				$D_s^+ D_{s1}(2536)$
χ_{c0} (4700)	$X(4700)$	4704^{+17}_{-26}	120^{+52}_{-45}	0^{++}	$\phi J/\psi$
Υ (10753)		$10752.7^{+5.9}_{-6.0}$	36^{+18}_{-12}	1^{--}	$\pi\pi\Upsilon(1S, 2S, 3S)$
Υ (10860)	$\Upsilon(5S)$	$10885.2^{+2.6}_{-1.6}$	37 ± 4	1^{--}	$\pi\pi\Upsilon(1S, 2S, 3S),$
					$\pi^+ \pi^- h_b(1P, 2P),$
					$\eta\Upsilon(1S, 2S), \pi^+ \pi^- \Upsilon(1D)$
					(see PDG listings)
Υ (11020)	$\Upsilon(6S)$	11000 ± 4	24^{+8}_{-6}	1^{--}	$\pi\pi\Upsilon(1S, 2S, 3S),$
					$\pi^+ \pi^- h_b(1P, 2P),$
					(see PDG listings)

Neutral meson states above the open-flavor thresholds which are potential candidates for hybrids

Table adapted from PDG 2021