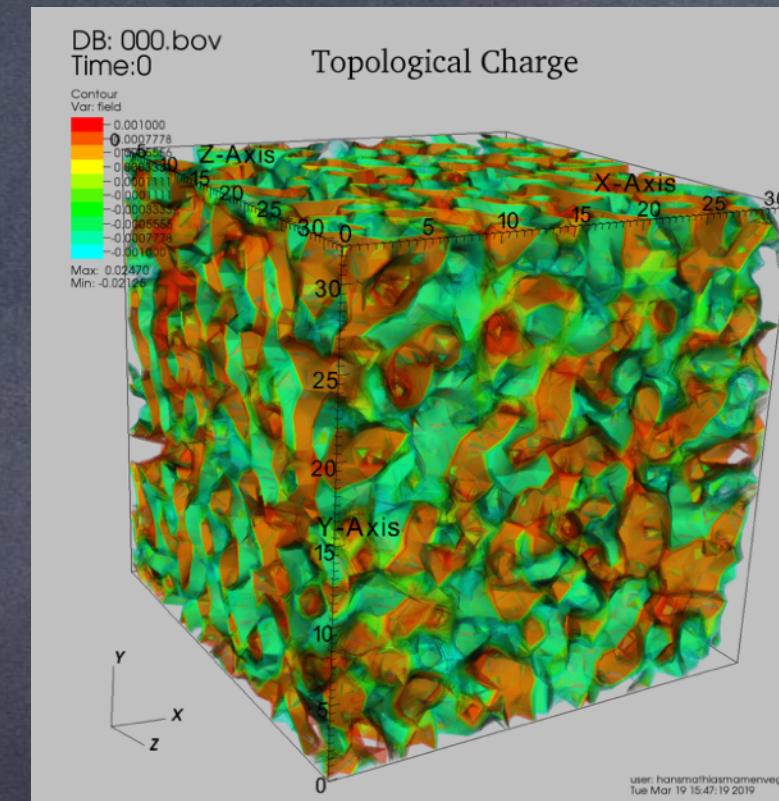


# Gradient Flow: perturbative and non-perturbative renormalization

Andrea Shindler



Quark Confinement  
08.01.2022



MICHIGAN STATE  
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# Some numbers and a disclaimer

- ~ 200 papers with “Gradient Flow” in the title

- “Properties and uses of the Wilson flow in lattice QCD” – M. Lüscher – 845 citations

## Applications:

- Scale setting
- Renormalization
  - Running coupling, finite volume scheme
- Electroweak Hamiltonian – 4 quark operators ( $B_K$ )
- CP-violating effective operators → EDM
- QCD at finite temperature – Equation of state
- Polyakov loops and static potential
- Heavy quark diffusion coefficients
- Topological susceptibility – theta term
- Smearing of gauge configurations

Narayanan, Neuberger: 2006

Lüscher 2010-2013

Lüscher, Weisz 2011

Suzuki – FlowQCD: 2012

Fritzsch, Ramos: 2012

BMW

Del Debbio, Patella, Rago: 2013

Carosso, A. Hasenfratz, Neil, Rebbi, Witzel: 2018 –

ALPHA Coll.

Artz, Harlander, Lange, Neumann, Prausa: 2019

Taniguchi, Kanaya, Kitazawa, Suzuki: 2016

Brambilla, Leino, Mayer-Steudte, Petreczky: 2022

# Outline

- ⦿ Motivation
- ⦿ Gradient Flow for gauge fields
- ⦿ Applications:
  - ⦿ topological charge, strong coupling, scale setting
- ⦿ Gradient flow for fermions
- ⦿ Applications:
  - ⦿ renormalization local operators, quark-chromo electric dipole moment

# Motivation

## Gradient flow

- Reduces short distance fluctuations → noise reduction
- Provides regularization independent and free of UV divergences definition of  $Q$
- Provides an extra dimensional scale in the theory: scale setting
- Provides a natural definition of the strong coupling
- Provides a natural way to define renormalized matrix elements with simplified renormalization → power divergences, complicated mixing patterns
- Provides an interesting tool into the study of RG flows and the definition of non-perturbative renormalization scheme

# Gradient flow

$x_\mu \quad t = \text{flow-time} \quad [t] = -2 \quad A_\mu(x) = A_\mu^a(x) T^a \rightarrow \text{gluon fields}$

$$\partial_t B_\mu(x, t) = D_\nu G_{\nu\mu}(x, t)$$

$$B_\mu(x, t)|_{t=0} = A_\mu(x)$$

$$D_\nu = \partial_\nu + [B_\nu(x, t), \cdot]$$

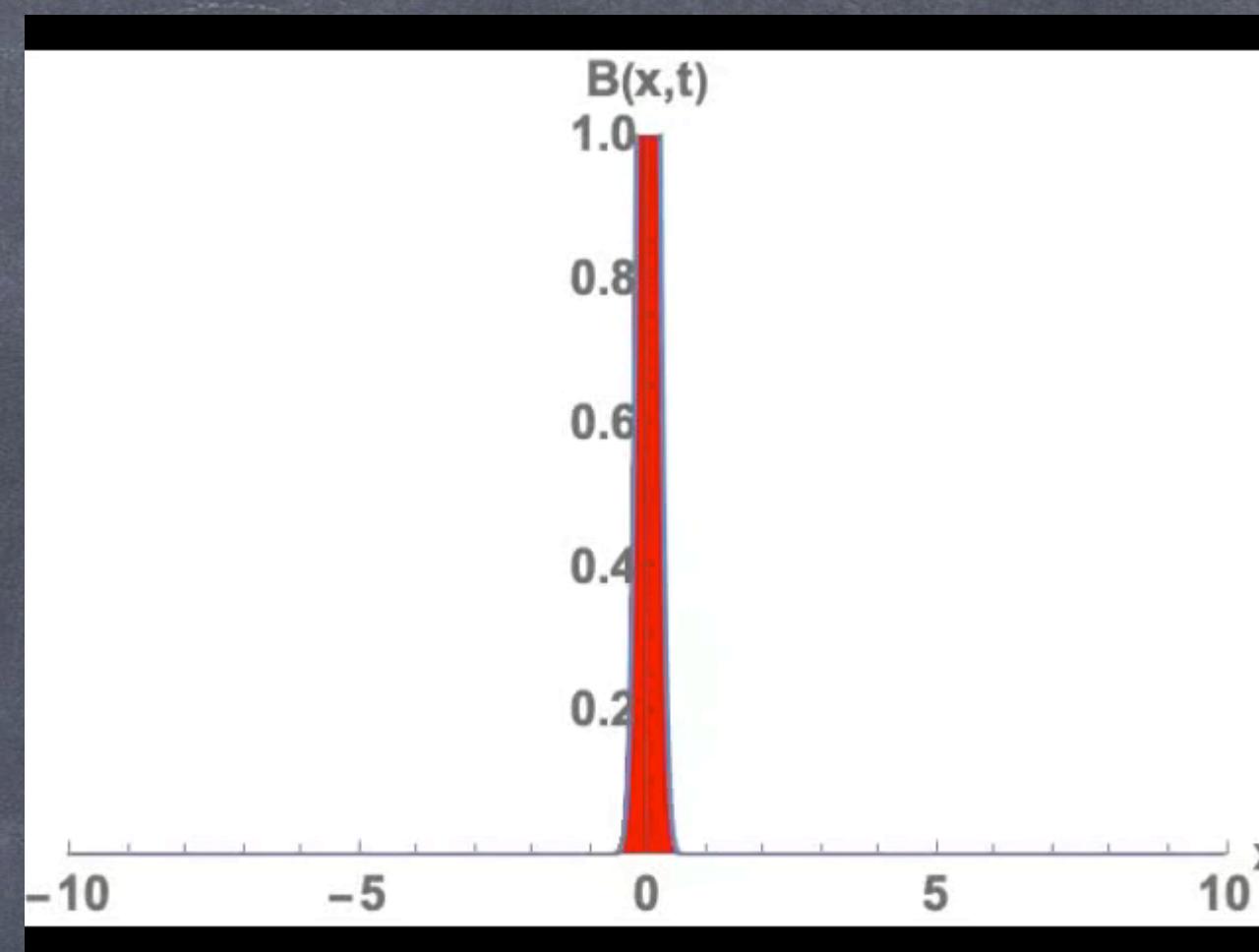
$$G_{\mu\nu}(x, t) = \partial_\mu B_\nu(x, t) - \partial_\nu B_\mu(x, t) + [B_\mu, B_\nu]$$

# Gradient flow and large momenta

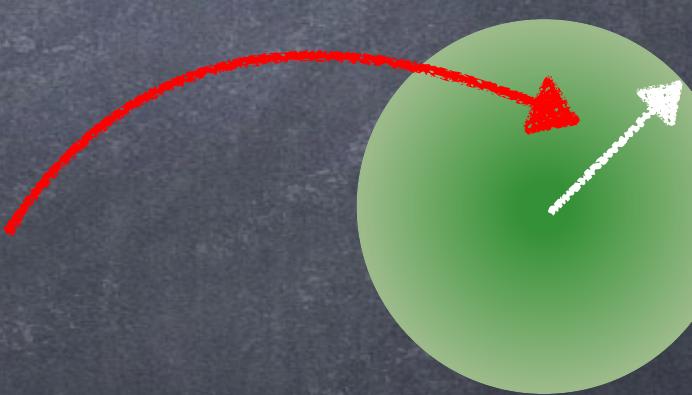
$$\partial_t B_\mu = \partial_\nu \partial_\nu B_\mu$$

$$B_\mu(x, t) = \int d^4y \ K(x - y; t) A_\mu(y)$$

$$K(x; t) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} e^{-tp^2} = \frac{e^{-x^2/4t}}{(4\pi t)^2}$$



- ⦿ Gaussian damping at large momenta
- ⦿ Smoothing at short distance over a range  $\sqrt{8t}$



$$B_\mu(x, t) \quad t > 0 \quad \text{finite}$$

No additional renormalization

Lüscher, Weisz: 2011

# Gradient flow and topological charge

Lüscher: 2010

$$Q(t) = \int d^4x \ q(x, t) \quad q(x, t) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr} \{ G_{\mu\nu}(x, t) G_{\rho\sigma}(x, t) \}$$

$$S[A] \geq \frac{8\pi^2}{g_0^2} |Q[A]|$$

$$\delta B_\mu(x, t) = \partial_t B_\mu \delta t$$



Bianchi identities

$$\partial_t q(x, t) = \partial_\mu \mathcal{K}_\mu(x, t)$$

$$\mathcal{K}_\mu(x, t) = \frac{1}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr} [G_{\nu\rho}(x, t) D_\alpha G_{\alpha\sigma}(x, t)]$$

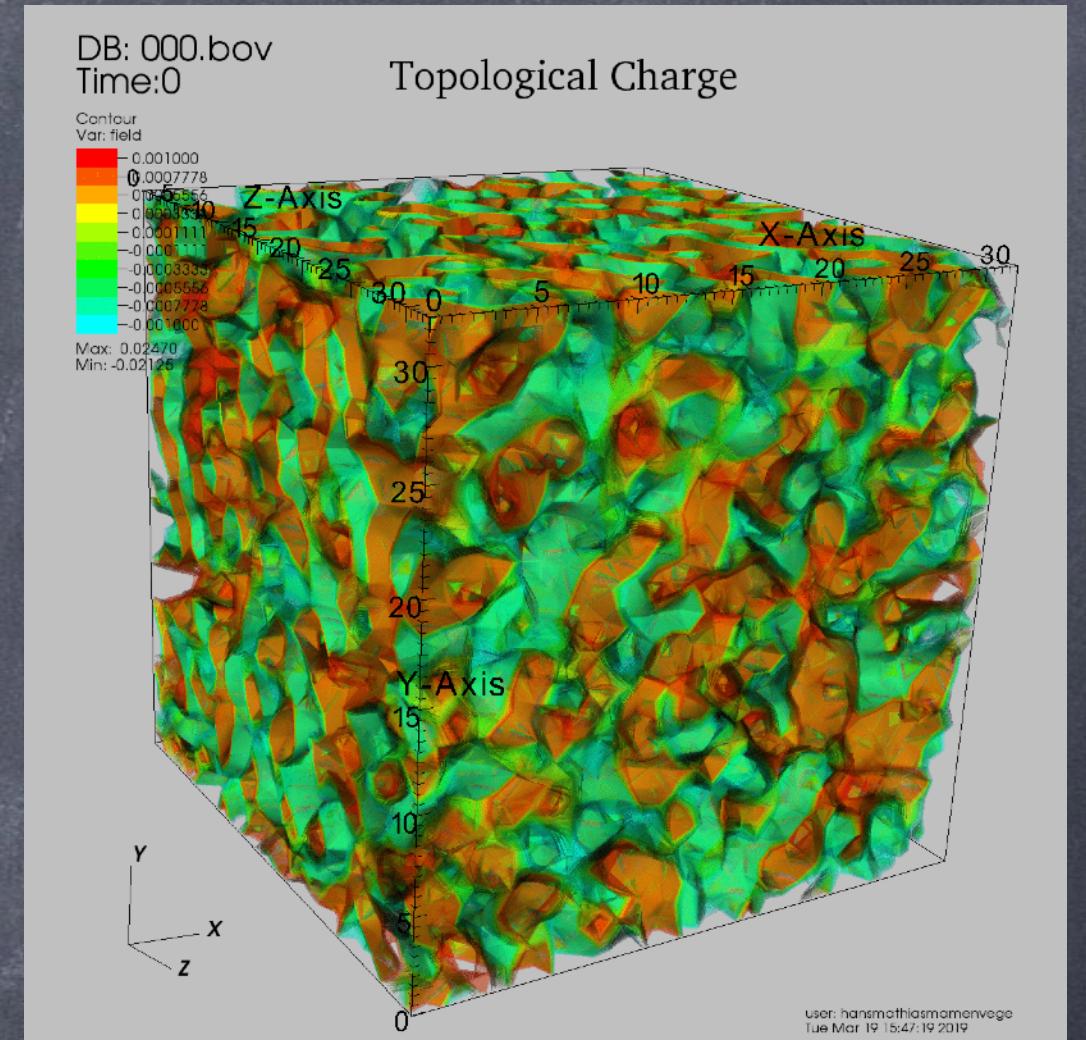
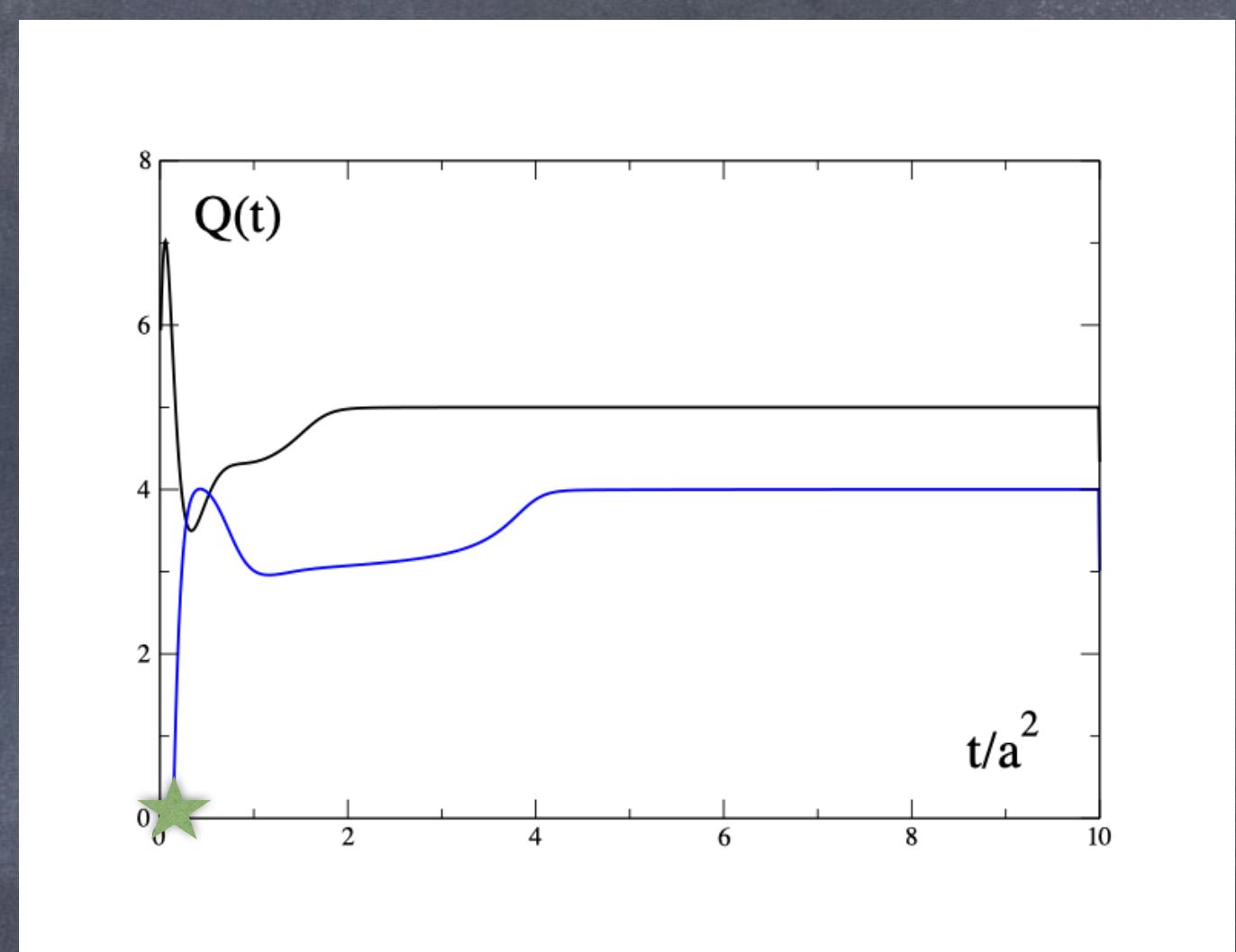
$$\langle \cdots [Q(t)]_R \cdots \rangle = \langle \cdots Q(t) \cdots \rangle$$

$$\partial_t Q(t) = 0 \quad Q = \int d^4x \ q(x, t)$$

Equivalent to fermionic definition

Polyakov: 1987, Lüscher: 2010

Ce', Consonni, Engel, Giusti: 2015 Lüscher: 2021



Pederiva, Vege: 2018  
LatViz

# EDM from $\theta$ -term

Collaborators:

J. DeVries

J. Kim

T. Luu

E. Mereghetti

C. Monahan

G. Pederiva

M. Rizik

P. Stoffer

J. Dragos

H. M. Vege

1409.2735

1507.02343

1711.04730

1809.03487

1810.05637

1810.10301

1902.03254

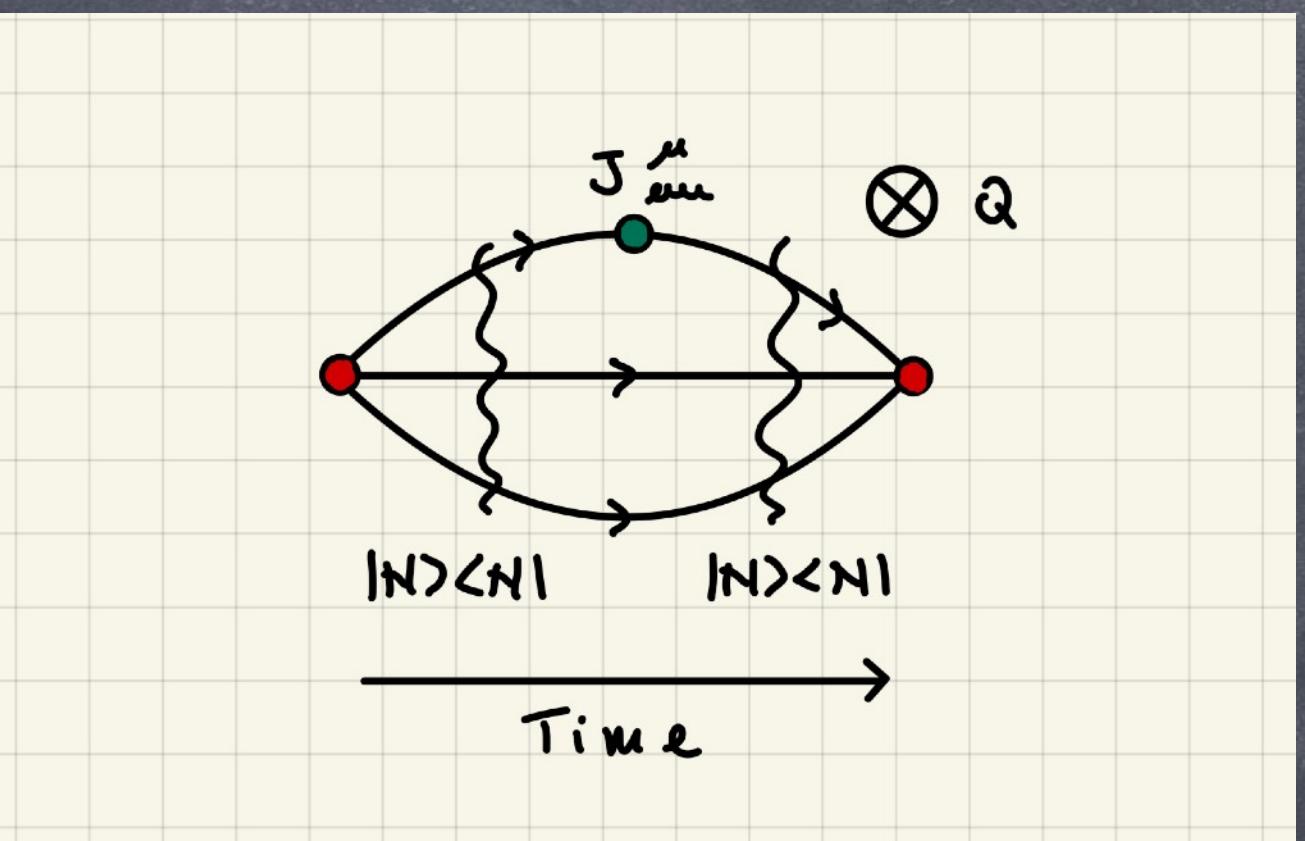
2005.04199

2106.07633

2111.1149

$$\langle N^\theta(\underline{p}', s') | J_\mu^{\text{em}} | N^\theta(\underline{p}, s) \rangle = \bar{u}_N(\underline{p}', s') \Gamma_\mu^{\bar{\theta}}(q^2) u_N(\underline{p}, s)$$

$$G_N^\theta J_\mu N = \langle N(y_0, p_2) J_\mu^\mu(x_0, q) N^\dagger(0, p_1) \rangle_\theta$$



$$e^{-S} \simeq e^{-S_{\text{QCD}}} [1 + i\theta Q]$$

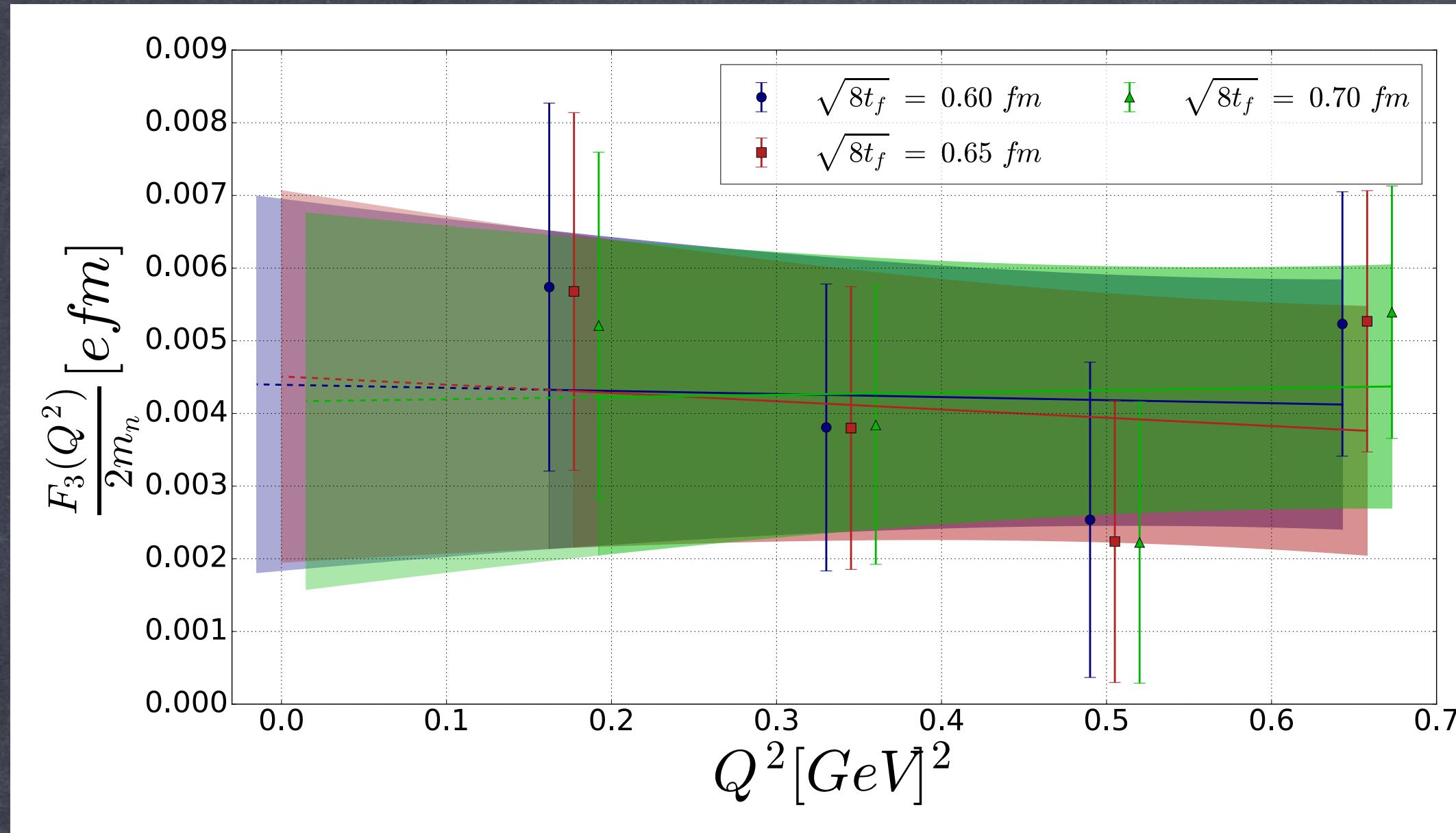
$$\langle \mathcal{O} \rangle_{\bar{\theta}} \simeq \langle \mathcal{O} \rangle_{\bar{\theta}=0} + i\bar{\theta} \langle \mathcal{O} | Q \rangle_{\bar{\theta}=0} + O(\bar{\theta}^2)$$

$$Q = \int d^4x \, q(x)$$

Problem: definition of  $Q$  on the lattice

# CP-odd form factor

Dragos, Luu, A.S.,  
de Vries, Yousif: 2019

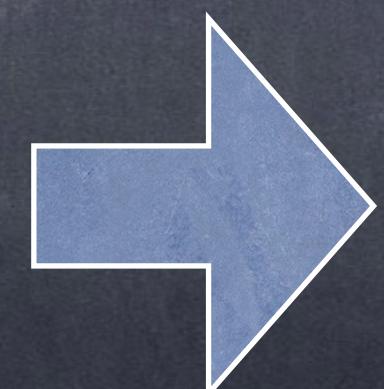


$$\frac{F_3^{P/N}(Q^2)}{2M_N} = d_{P/N} + S_{P/N}Q^2 + H_{P/N}(Q^2)$$

$$\frac{d_P}{d_N} < 0 \quad \frac{S_P}{S_N} < 0$$

Otnad, Kubis, Meißner, Gut: 2010  
Mereghetti, de Vries, Hockings,  
Maekawa, van Kolck: 2011

$$d_n^{\text{phys}} = -0.00152(71) \bar{\theta} e \text{ fm}$$



$$|\bar{\theta}| < 1.98 \times 10^{-10} (90\% \text{CL})$$

$$\bar{g}_0^{\bar{\theta}} = -1.28(64) \cdot 10^{-2} \bar{\theta}$$

$$\bar{g}_0^{\bar{\theta}} = -1.47(23) \cdot 10^{-2} \bar{\theta}$$

Crewther et al.: 1980  
de Vries et al.: 2015

Ab-initio determination of  $\bar{g}_0^{\bar{\theta}}$

# Strong coupling

Lüscher: 2010

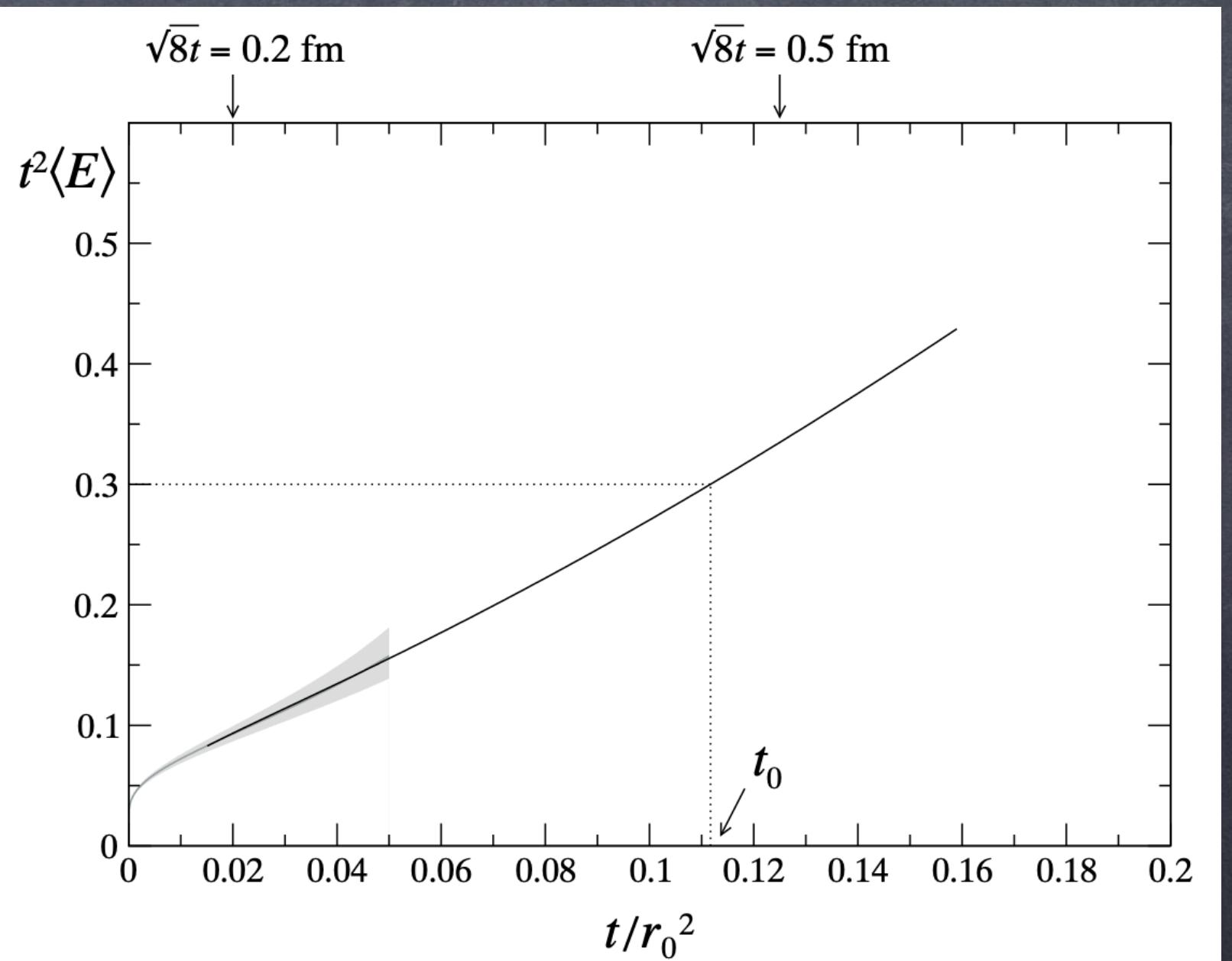
$$E(t) = \frac{1}{4} G_{\mu\nu}^a(t) G_{\mu\nu}^a(t) \quad \langle E \rangle = \frac{1}{2} g_0^2 \frac{N_c^2 - 1}{(8\pi t)^{D/2}} (D - 1) [1 + c_1 g_0^2 + O(g_0^4)] \quad \bar{g}^2(\mu) = 48\pi^2 t^2 \langle E(t) \rangle \Big|_{8t\mu^2=1}$$

$$c_1 = \frac{1}{16\pi^2} (4\pi)^\epsilon (8t)^\epsilon \left[ N_c \left( \frac{11}{3} \frac{1}{\epsilon} + \frac{52}{9} - 3 \ln 3 \right) - N_f \left( \frac{2}{3} \frac{1}{\epsilon} + \frac{4}{9} - \frac{4}{3} \ln 2 \right) + O(\epsilon) \right]$$

$$\langle E \rangle = \frac{3}{4\pi t^2} \alpha(\mu) [1 + k_1 \alpha(\mu) + O(\alpha^2)] \quad N_c = 3$$

$$k_1 = \frac{1}{4\pi} \left[ (11L + \frac{52}{3} - 9 \ln 3) - N_f \left( \frac{2}{3}L + \frac{4}{9} - \frac{4}{3} \ln 2 \right) \right]$$

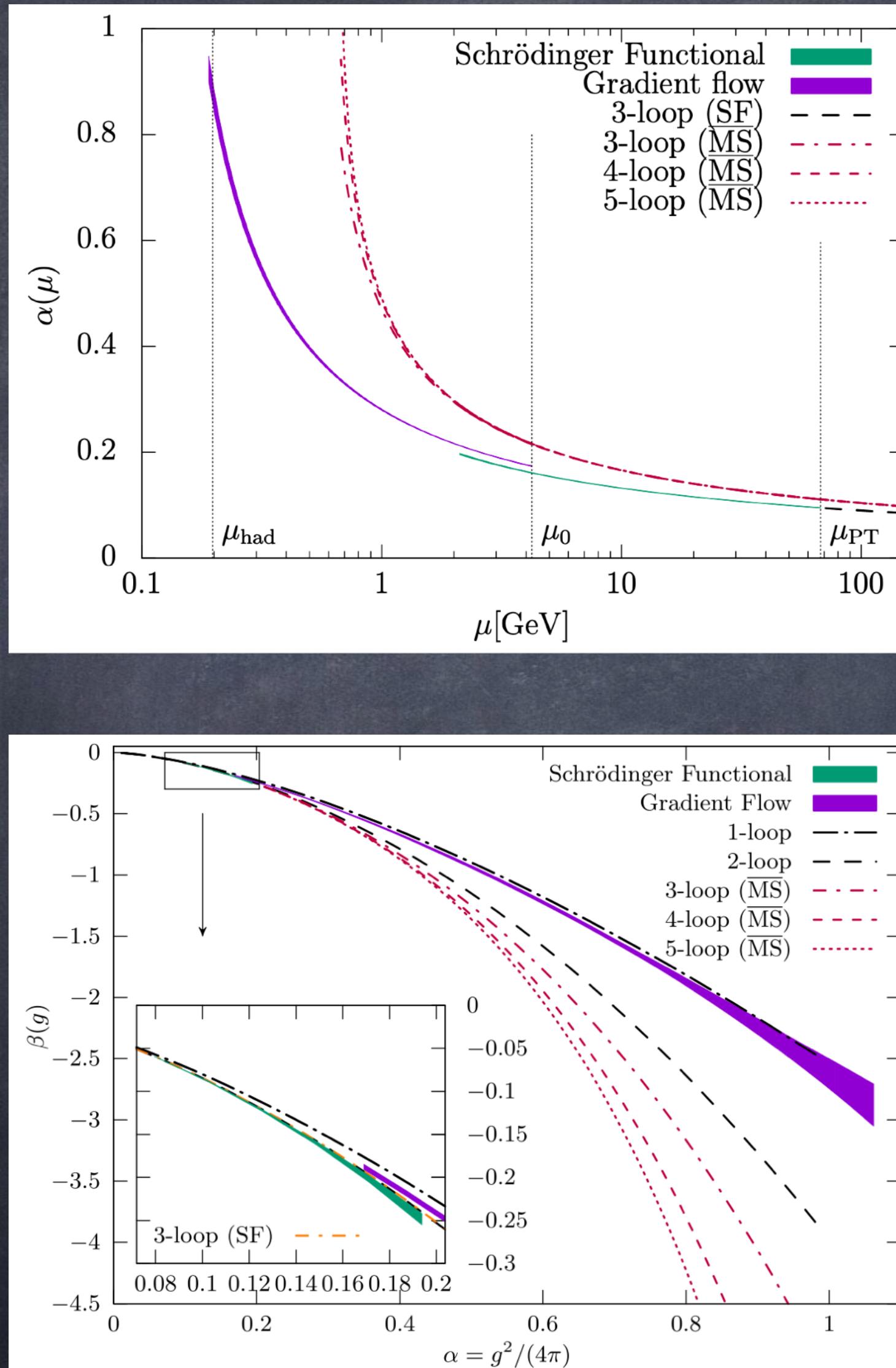
$$L = \ln(8\mu^2 t) + \gamma_E$$



Similar calculation for Static potential

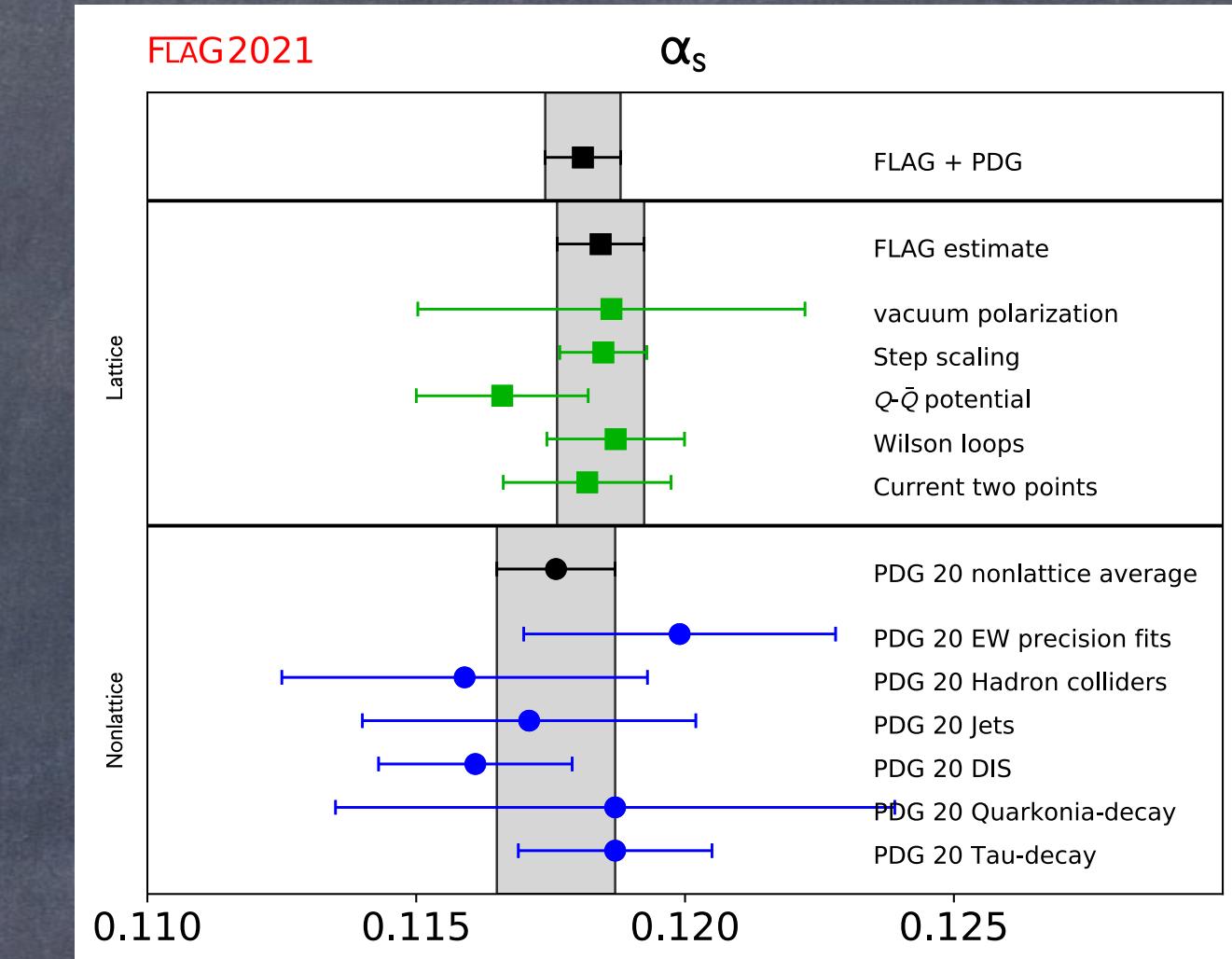
Brambilla, Chung, Vairo, Wang: 2021  
Leino, Brambilla, Mayer-Steudte, Vairo: 2021

# Strong coupling



$$\bar{g}_{\text{GF}}^2(L) = [\mathcal{N}^{-1}(c, T/L, x_4/T) \cdot t^2 \langle E(t, x_4) \rangle]_{t=c^2 L^2/8}$$

Fritzsch, Ramos: 2012



FLAG: 2021

With periodic boundary conditions +  
Infinite volume limit

Fodor, Holland, Kuti, Nogradi, Wong: 2012

A. Hasenfratz, Witzel: 2019

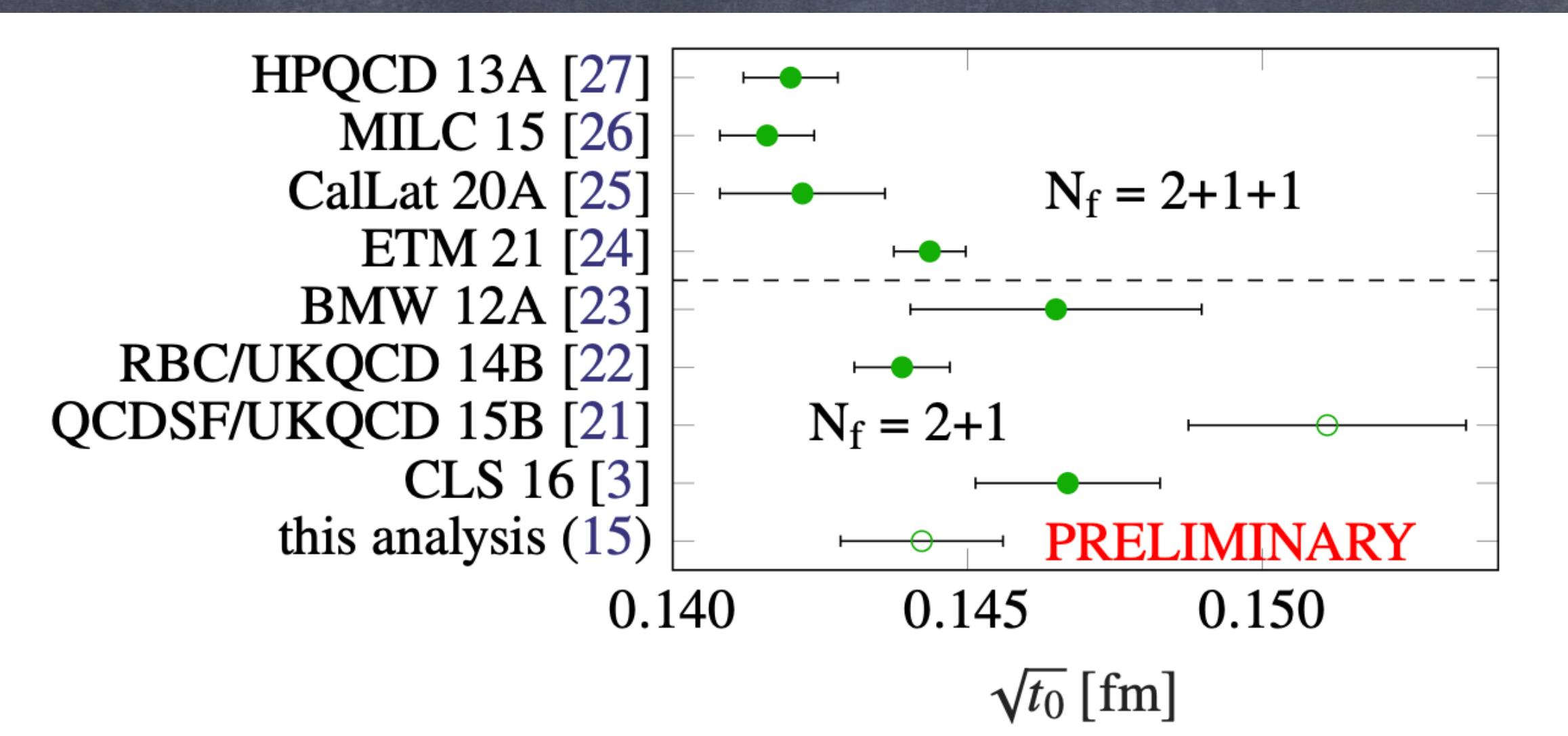
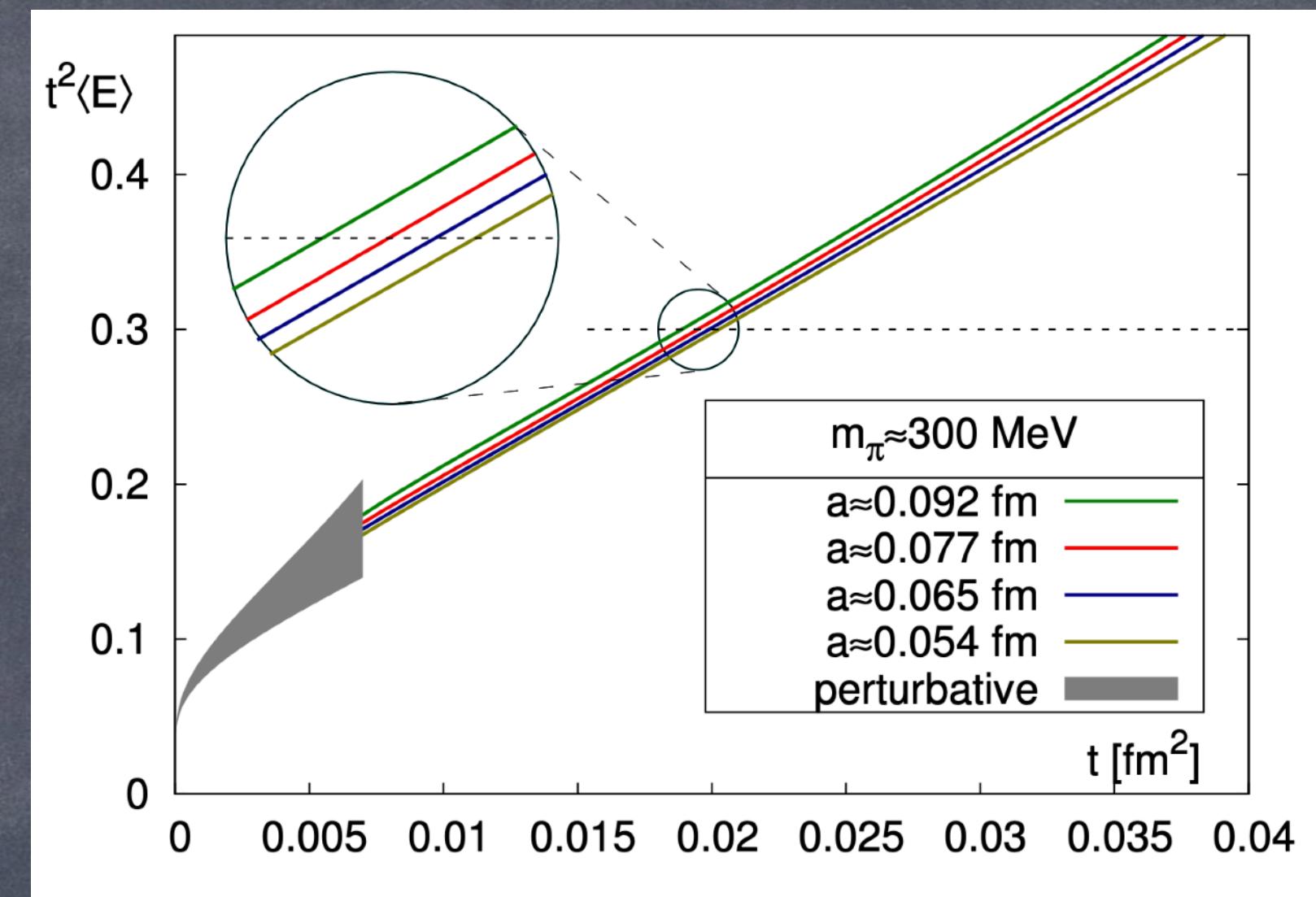
# Fixing the scale

Lüscher: 2010

$$t^2 \langle E(t) \rangle \Big|_{t=t_0} = 0.3$$

BMW Coll.: 2012

$$t \frac{d}{dt} [t^2 \langle E(t) \rangle] \Big|_{t=w_0^2} = 0.3$$



BMW Coll.: 2012

CLS: 2021

# Gradient flow for fermions

Lüscher: 2013

$$\partial_t \chi(x, t) = \Delta \chi(x, t) \quad \partial_t \bar{\chi}(x, t) = \bar{\chi}(x, t) \overleftarrow{\Delta}$$

$$\chi(x, t=0) = \psi(x)$$

$$\bar{\chi}(x, t=0) = \bar{\psi}(x)$$

$$x_\mu = (x_0, \mathbf{x}) \quad t = \text{flow - time} \quad [t] = -2$$

$$\Delta = D_{\mu,t} D_{\mu,t} \quad D_{\mu,t} = \partial_\mu + B_{t,\mu}$$

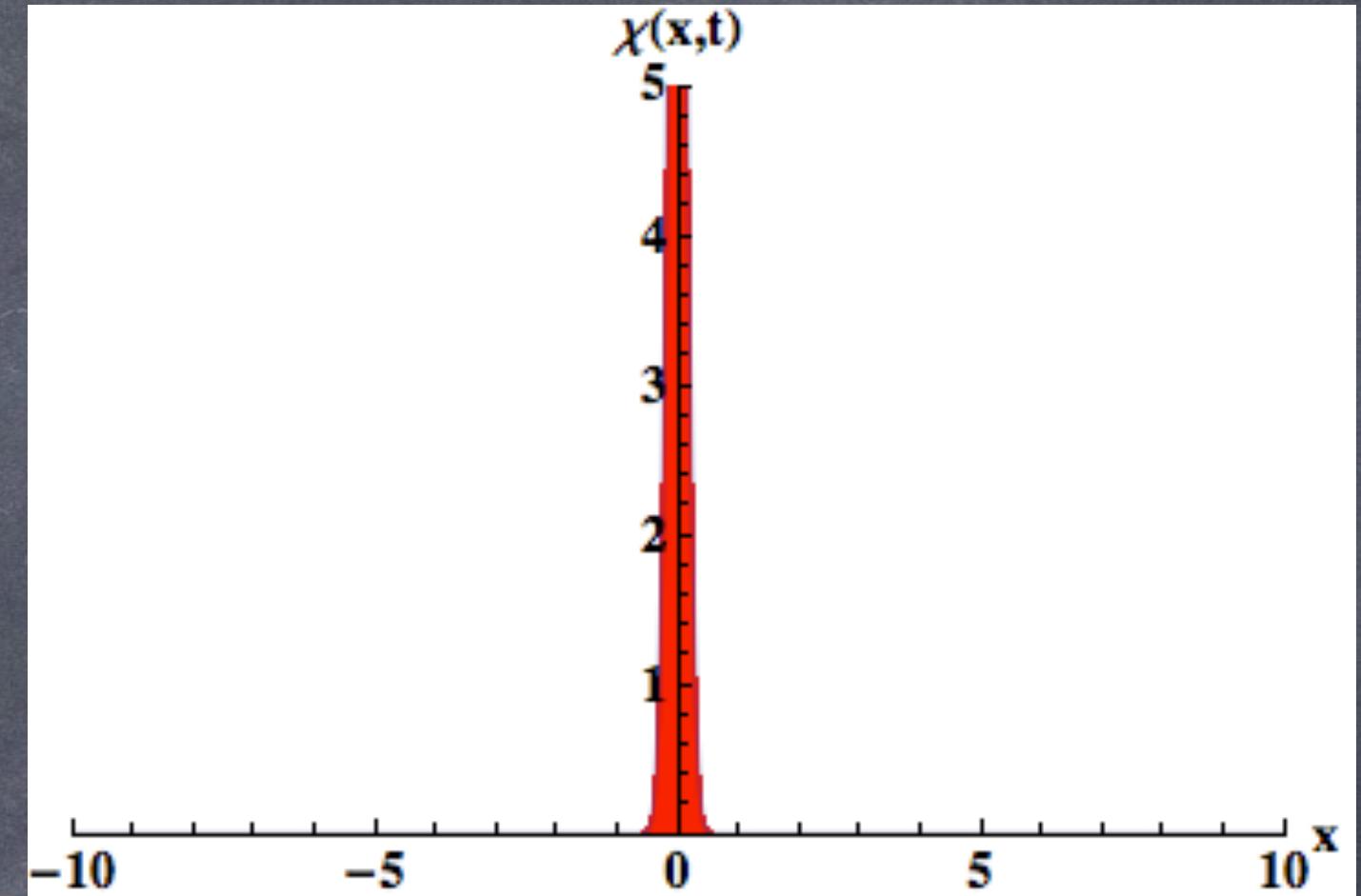
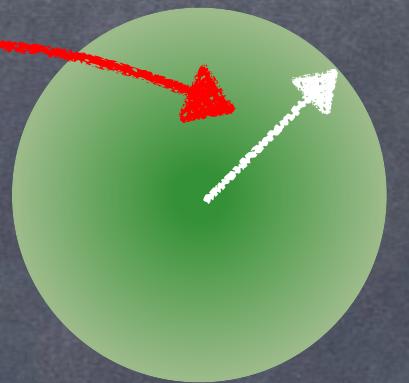
# Gradient flow

Lüscher: 2013

$$\chi(x, t) = \int d^4y K(x - y, t)\psi(y)$$

$$K(t, x) = \frac{e^{-\frac{x^2}{4t}}}{(4\pi t)^2}$$

- Smooth over a range  $\sqrt{8t}$
- Gaussian damping at large momenta



$$\chi_R(x, t) = Z_\chi^{1/2} \chi(x, t) \quad \mathcal{O}(x, t) = \bar{\chi}(x, t) \Gamma(x, t) \chi(x, t) \quad \mathcal{O}_R = Z_\chi \mathcal{O}$$

$$\Sigma_t = \langle \bar{\chi}(x, t) \chi(x, t) \rangle \quad \Sigma_{t,R} = Z_\chi \Sigma_t$$

Lüscher: 2010, 2013  
Lüscher, Weisz: 2011

No additive divergences

All fermion operators renormalize multiplicatively with same factor

# 4+1 Local field theory

$$S = S_G + S_{G,\text{fl}} + S_{F,\text{QCD}} + S_{F,\text{fl}}$$

$$S_{F,\text{fl}} = \int_0^\infty dt \int d^4x \left[ \bar{\lambda}(t, x) (\partial_t - \Delta) \chi(t, x) + \bar{\chi}(t, x) \left( \overleftarrow{\partial}_t - \overleftarrow{\Delta} \right) \lambda(t, x) \right]$$

- ⦿ Wick contractions

Lüscher, Weisz: 2011

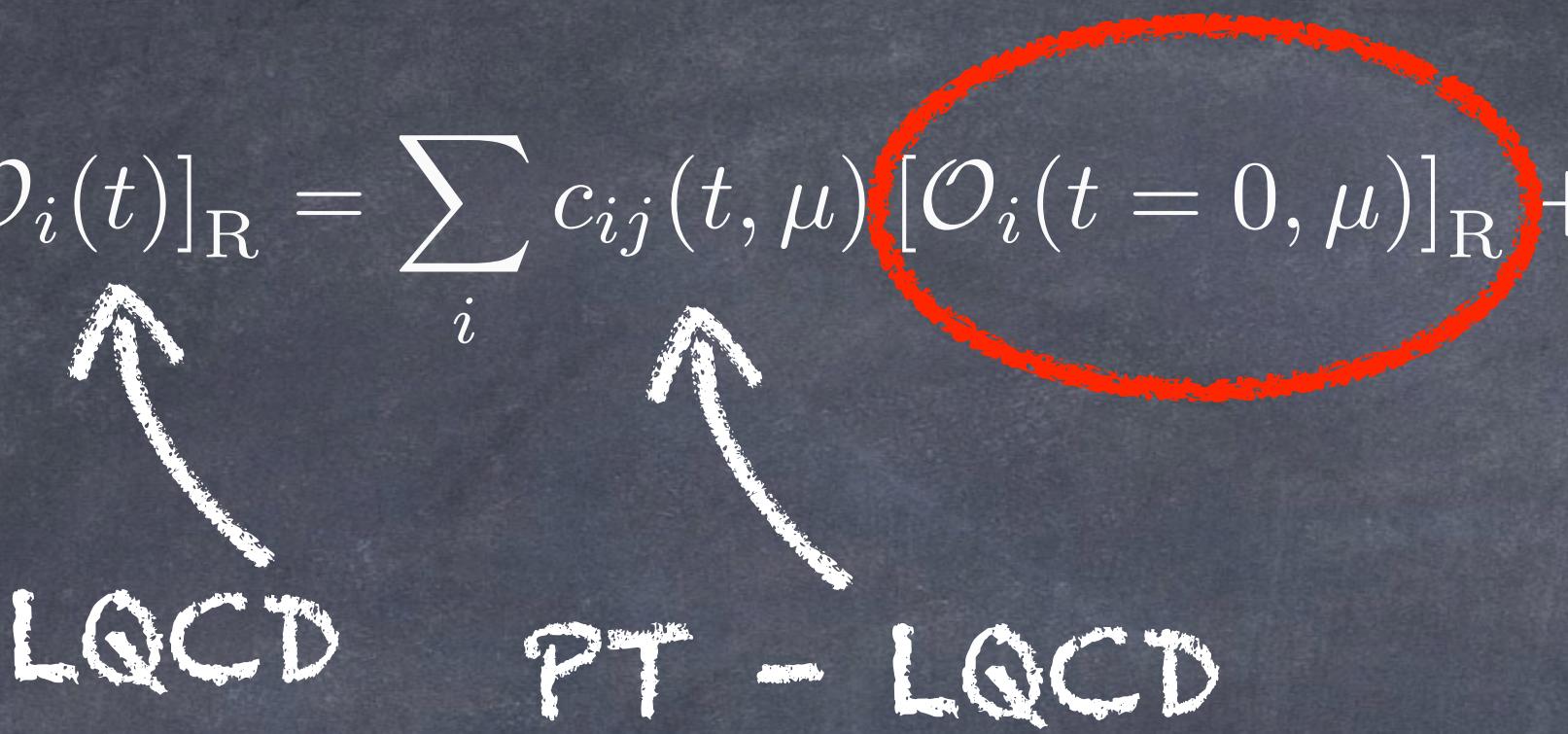
- ⦿ Renormalization. All order proof for gauge sector

Lüscher: 2013  
A.S.:2013

- ⦿ Chiral symmetry and Ward identities

# Short flow-time expansion

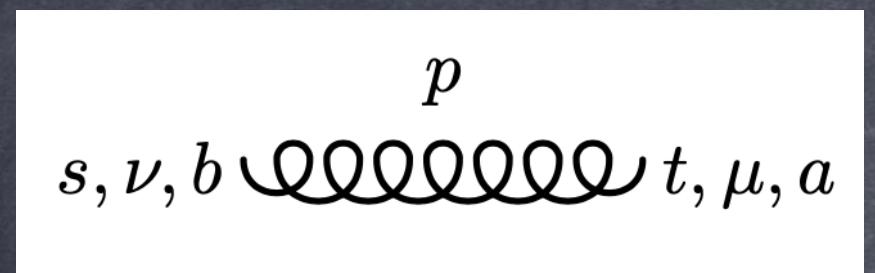
$$[\mathcal{O}_i(t)]_R = \sum_i c_{ij}(t, \mu) [\mathcal{O}_i(t=0, \mu)]_R + O(t)$$
$$c_{ij}(t, \mu) = \delta_{ij} + \frac{\alpha_s(\mu)}{4\pi} c_{ij}^{(1)}(t, \mu) + O(\alpha_s^2)$$

  
LQCD      PT - LQCD

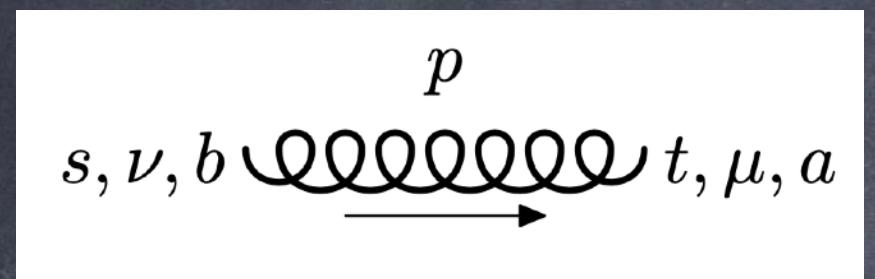
- ⦿ Calculation of matrix elements with flowed fields
  - ⦿ Easy renormalization (no power divergences)
- ⦿ Calculation of Wilson coefficients
  - ⦿ Insert OPE in off-shell amputated 1PI Green's functions
- ⦿ Determination of the physical renormalized matrix element at zero flow-time

# Feynman rules

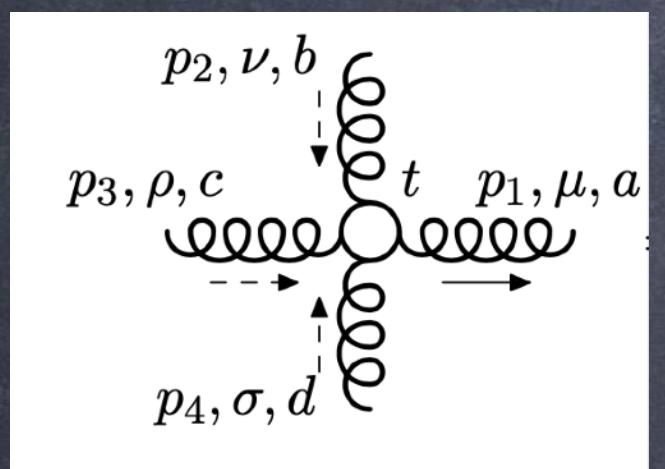
Rizik, Monahan, A.S.: 2020  
 Mereghetti, Monahan, Rizik, A.S.,  
 Stoffer : 2021



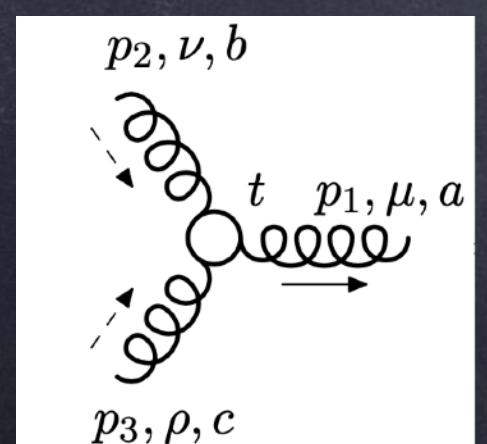
$$\left\langle \tilde{B}_\mu^a(p, t) \tilde{B}_\nu^b(-p, s) \right\rangle = \tilde{D}_{\mu\nu}^{ab}(p, s+t) = g_0^2 \delta^{ab} \frac{1}{p^2} \left[ \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) e^{-(s+t)p^2} + \xi \frac{p_\mu p_\nu}{p^2} e^{-(s+t)p^2} \right]$$



$$= \delta^{ab} \theta(t-s) \frac{1}{p^2} \left[ (\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-(t-s)p^2} + p_\mu p_\nu e^{-(t-s)p^2} \right]$$



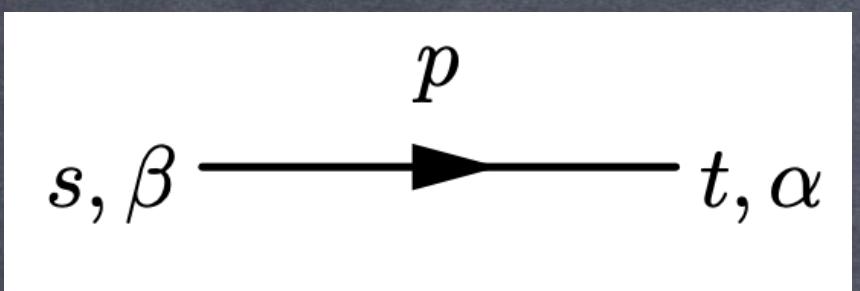
$$= -i f^{abc} \int_0^\infty dt \left( \delta_{\nu\rho} (p_2 - p_3)_\mu + 2\delta_{\mu\rho} p_{3\nu} - 2\delta_{\mu\nu} p_{2\rho} \right)$$



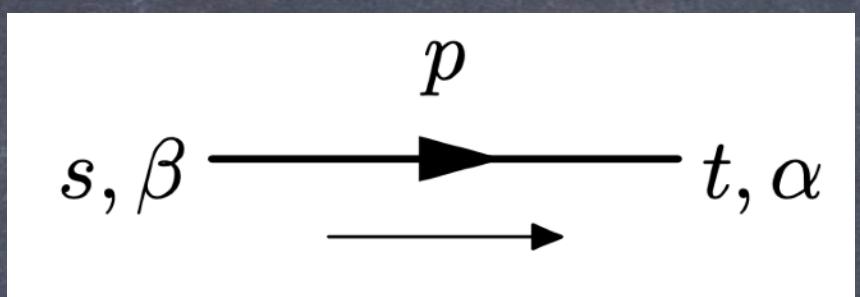
$$= - \int_0^\infty dt \left( f^{abe} f^{cde} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\rho\nu}) + f^{ace} f^{bde} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) + f^{ade} f^{bce} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\rho} \delta_{\nu\sigma}) \right)$$

# Feynman rules

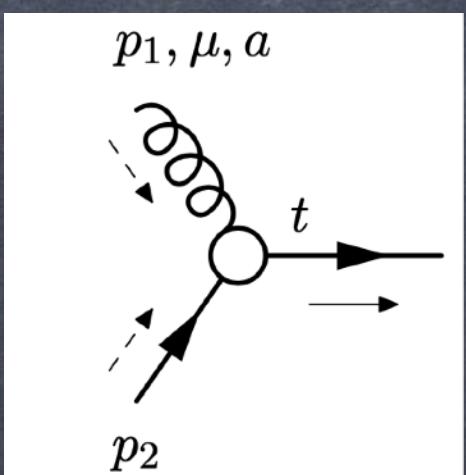
Rizik, Monahan, A.S.: 2020  
 Mereghetti, Monahan, Rizik, A.S.,  
 Stoffer : 2021



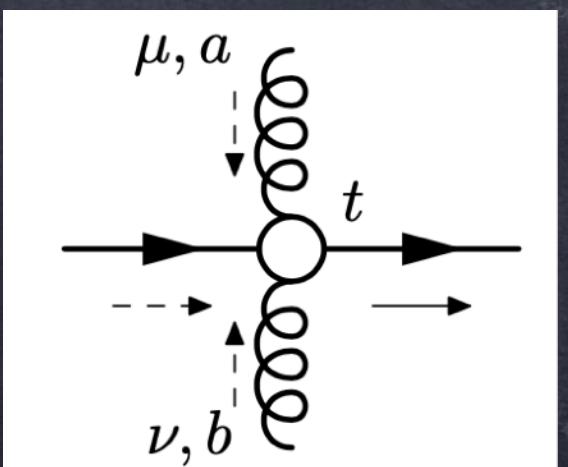
$$\left\langle \tilde{\chi}(p, t) \tilde{\bar{\chi}}(p, s) \right\rangle = \tilde{S}(p, s + t) = \frac{-i\cancel{p} + m}{p^2 + m^2} e^{-(s+t)p^2}$$



$$= \delta^{\alpha\beta} \theta(t - s) e^{-(t-s)p^2}$$



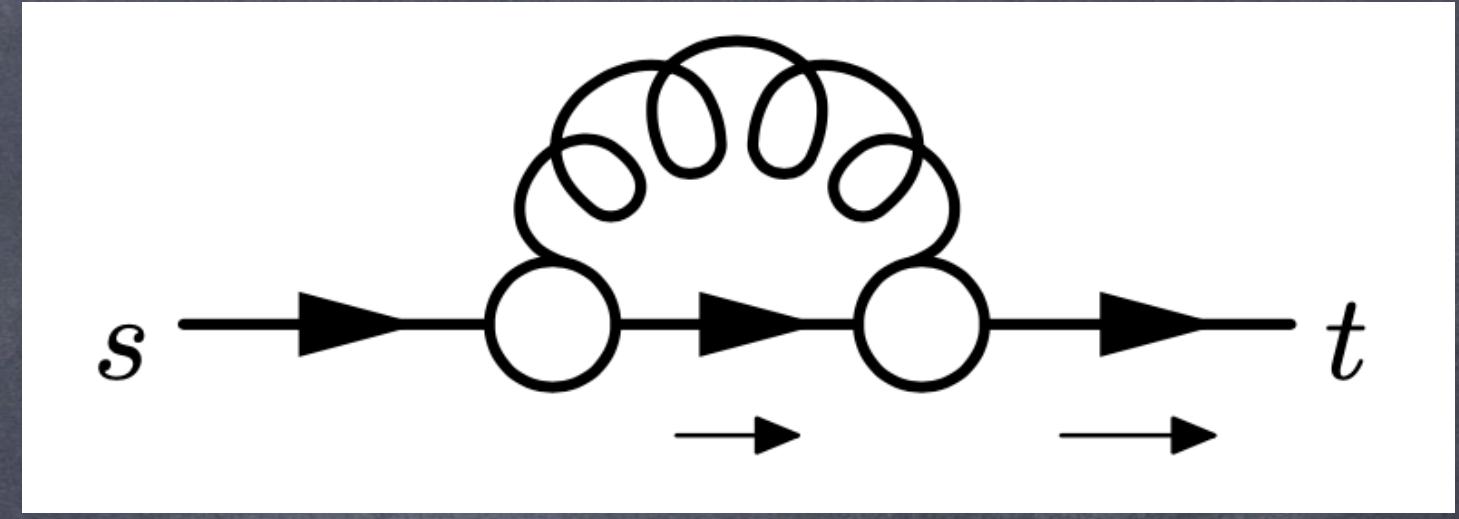
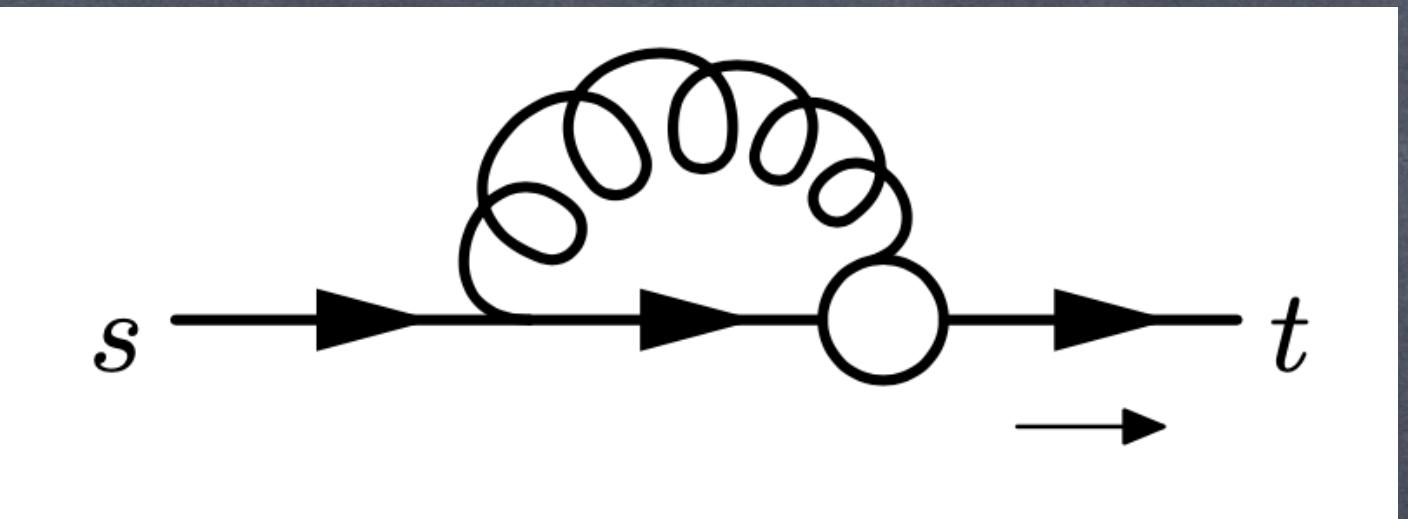
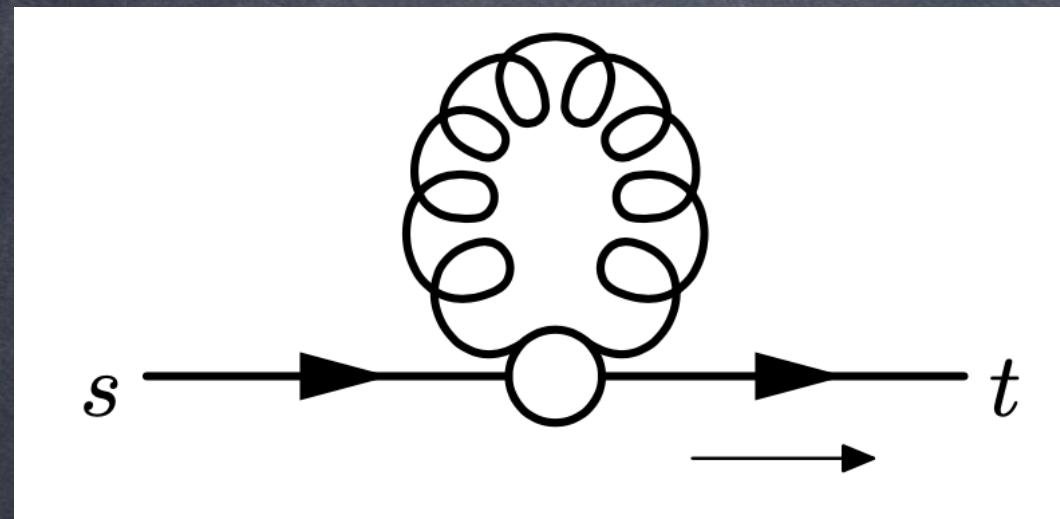
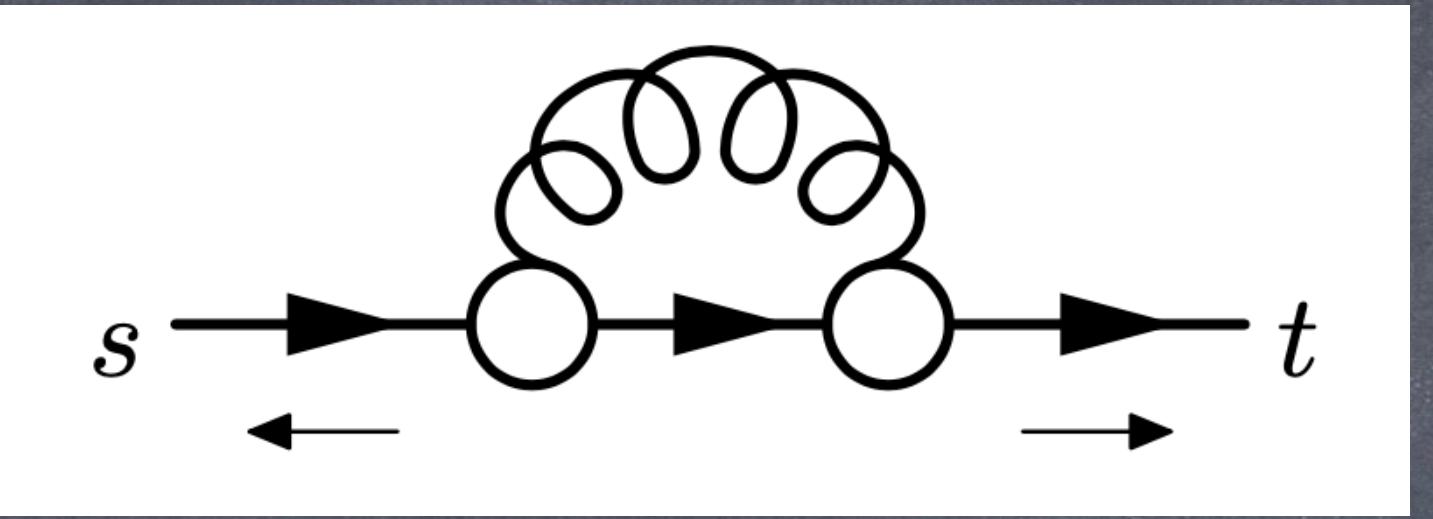
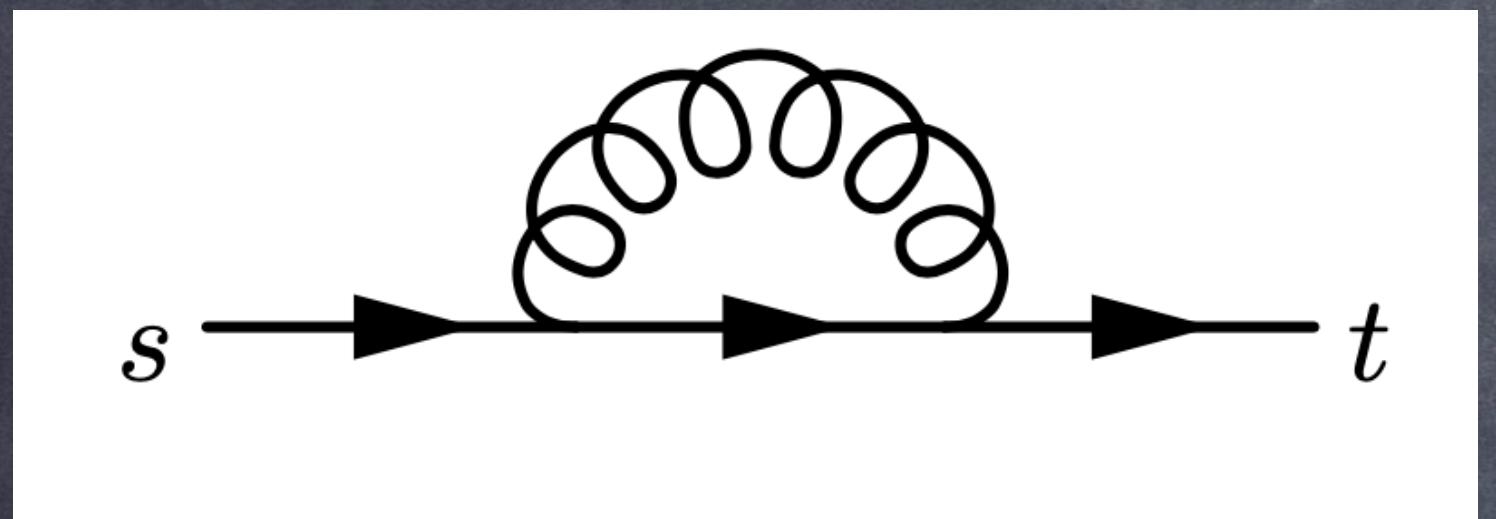
$$= -it^a \int_0^\infty dt 2p_{2\mu}$$



$$= \delta_{\mu\nu} \{t^a, t^b\} \int_0^\infty dt$$

# Sample calculation

Rizik, Monahan, A.S.: 2020



$$\tilde{S}(p, t, s) = \tilde{S}^{(0)}(p, t, s) \left\{ 1 - \frac{g_0^2}{(4\pi)^2} C_F \left[ \frac{3}{\epsilon} + \log(8\pi\mu^2 t) + \log(8\pi\mu^2 s) + \log\left(\frac{4\pi\mu^2}{p^2}\right) - \gamma_E + 1 \right] \right\} + O(s, t, g_0^4)$$

$$\chi_R(x, t) = Z_\chi^{1/2} \chi(x, t) \quad S_R(p, t, s) = Z_\chi S(p, t, s)$$

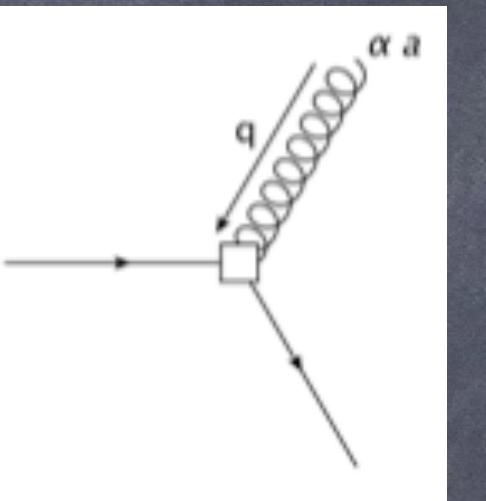
$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2} C_F \frac{3}{\epsilon}$$

# Quark-chromo EDM

$$[\mathcal{O}_i(t)]_R = \sum_i c_{ij}(t, \mu) [\mathcal{O}_i(t=0, \mu)]_R + O(t)$$

$$\mathcal{O}_{\text{CE}}(x) = \sum_{f=u,d,s,\dots} \bar{\psi}_f(x) \gamma_5 \sigma_{\mu\nu} G_{\mu\nu}^a T^a \psi_f(x)$$

$$P(x) = \sum_{f=u,d,s,\dots} \bar{\psi}_f(x) \gamma_5 \psi_f(x)$$



Bhattacharya, Cirigliano,  
Gupta, Mereghetti, Yoon: 2015

$$[\mathcal{O}_{\text{CE}}]_R = Z_{\text{CE}} \left[ \mathcal{O}_{\text{CE}} - \frac{C}{a^2} P \right] + \dots$$

## RI-MOM Off-shell

$$\frac{1}{a} \quad d=4 \rightarrow 2 \text{ operators} + 3 \text{ } O(m)$$

$$\log a \quad d=5 \rightarrow 3 \text{ operators} + (7 + 5) \text{ } O(m,m^2) + 4 \text{ "nuisance"}$$

Power divergences need to be subtracted non-perturbatively

Maiani, Martinelli, Sachrajda: 1992

# Quark-Chromo EDM

Rizik, Monahan, A.S.: 2020  
Mereghetti, Monahan, Rizik, A.S.,  
Stoffer : 2021

$$\mathcal{O}_{CE}(x, t) = \bar{\chi}(x, t)\tilde{\sigma}_{\mu\nu}G_{\mu\nu}(x, t)\chi(x, t) \quad \tilde{\sigma}_{\mu\nu}^{\text{HV}} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\sigma_{\alpha\beta} \quad \tilde{\sigma}_{\mu\nu}^{\text{NDR}} = \sigma_{\mu\nu}\gamma_5$$

$$\begin{aligned} \mathcal{O}_{CE}^R(x; t) &= c_P(t, \mu)\mathcal{O}_P^{\text{MS}}(x; \mu) + c_{m\theta}(t, \mu)\mathcal{O}_{m\theta}^{\text{MS}}(x; \mu) + c_E(t, \mu)\mathcal{O}_E^{\text{MS}}(x; \mu) \\ &\quad + c_{CE}(t, \mu)\mathcal{O}_{CE}^{\text{MS}}(x; \mu) + c_{m^2 P}(t, \mu)\mathcal{O}_{m^2 P}^{\text{MS}}(x; \mu) + \dots \end{aligned}$$

$$\mathcal{O}_P(x) = \bar{\psi}(x)\gamma_5\psi(x) \quad \mathcal{O}_{m^2 P}(x) = m^2\bar{\psi}(x)\gamma_5\psi(x)$$

$$\mathcal{O}_{m\theta}(x) = m\text{tr}[G_{\mu\nu}\tilde{G}_{\mu\nu}] \quad \mathcal{O}_E(x) = \bar{\psi}(x)\tilde{\sigma}_{\mu\nu}F_{\mu\nu}(x)\psi(x)$$

# Quark-Chromo EDM

Rizik, Monahan, A.S.: 2020

Mereghetti, Monahan, Rizik, A.S., Stoffer : 2021

$$Z_\chi^{-n/2} \left\langle (\psi)^{n_\psi} (\bar{\psi})^{n_{\bar{\psi}}} (A_\mu)^{n_A} \mathcal{O}_i(t) \right\rangle^{\text{amp}} = c_{ij}(t) (Z_{jk}^{\text{MS}})^{-1} \left\langle (\psi)^{n_\psi} (\bar{\psi})^{n_{\bar{\psi}}} (A_\mu)^{n_A} \mathcal{O}_k \right\rangle^{\text{amp}}$$

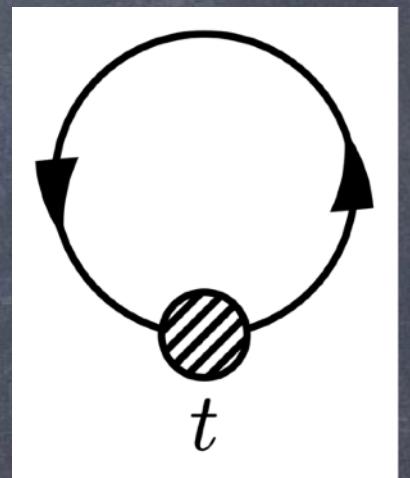
$$c_{ij}(t, \mu) = \delta_{ij} + \frac{\alpha_s(\mu)}{4\pi} c_{ij}^{(1)}(t, \mu) + O(\alpha_s^2)$$

$$\left\langle \overset{\circ}{\chi}(x; t) \overset{\leftrightarrow}{D} \overset{\circ}{\chi}(x; t) \right\rangle = -\frac{2N_c N_f}{(4\pi)^2 t^2}$$

$$\chi_R(x; t) = (8\pi t)^{\varepsilon/2} \zeta_\chi^{1/2} \overset{\circ}{\chi}(x; t)$$

$$\bar{\chi}_R(x; t) = (8\pi t)^{\varepsilon/2} \zeta_\chi^{1/2} \overset{\circ}{\chi}(x; t)$$

$$\zeta_\chi = 1 - \frac{\alpha_s C_F}{4\pi} (3 \log(8\pi\mu^2 t) - \log(432)) + O(\alpha_s^2)$$



Makino, Suzuki: 2014

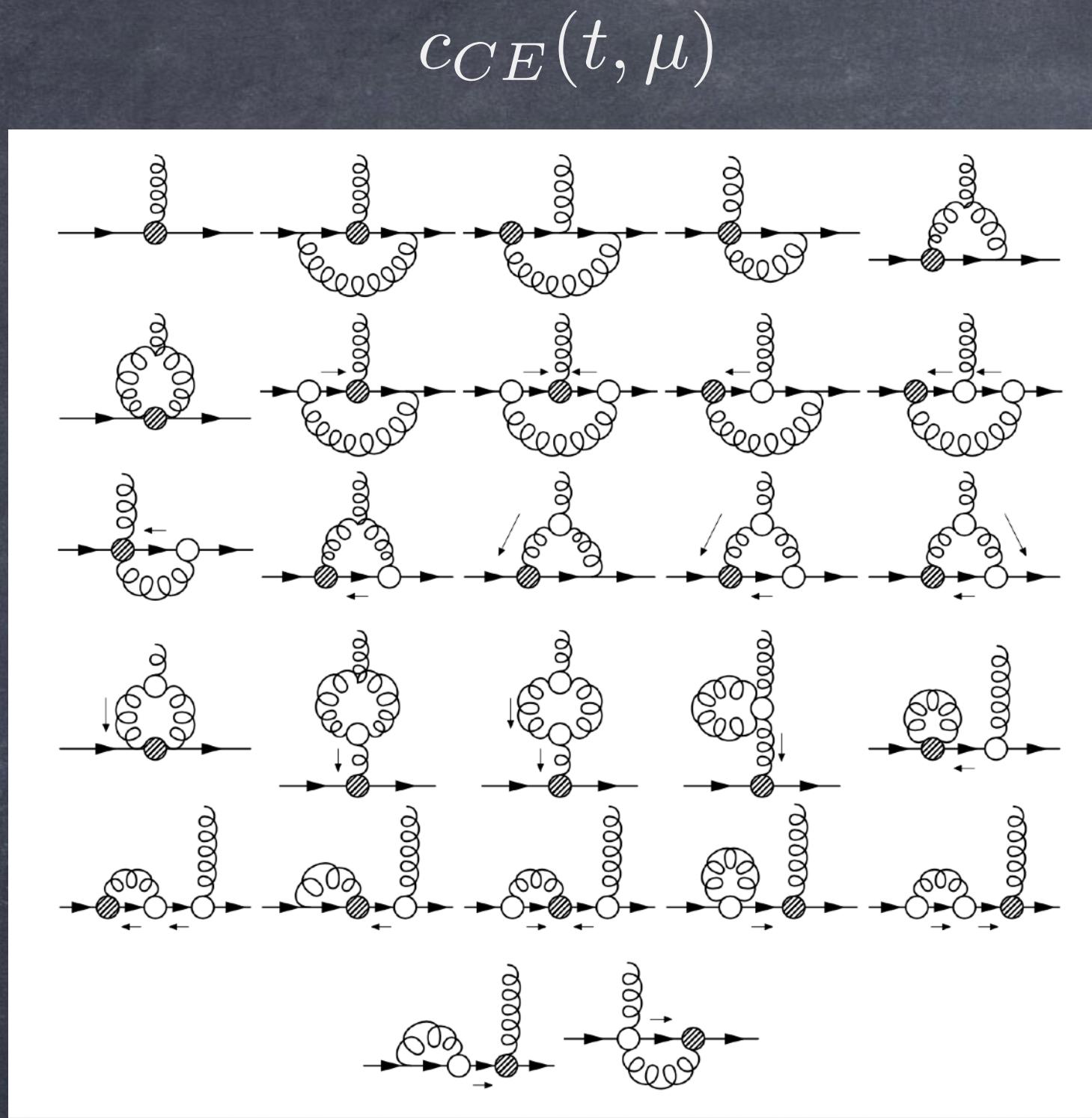
2-loops

Harlander, Kluth, Lange :2018

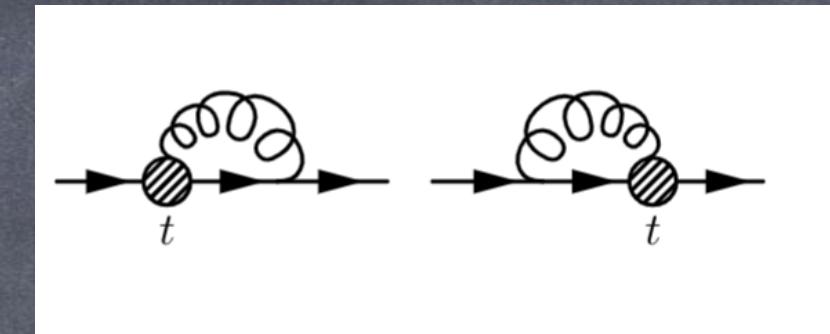
Artz, Harlander, Lange,  
Neumann, Prausa: 2019

# Quark-Chromo EDM

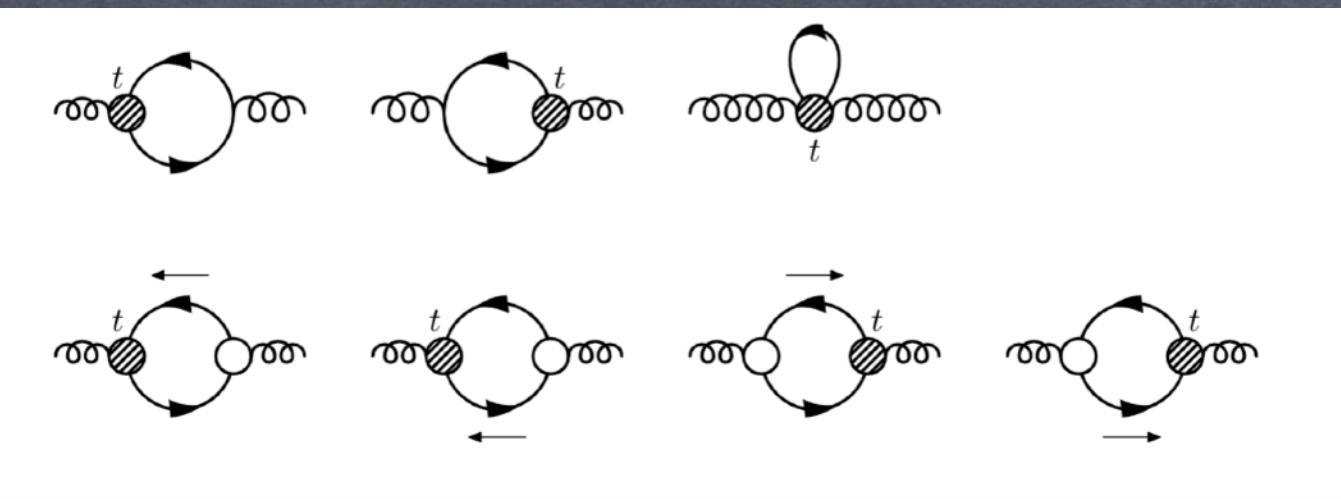
Rizik, Monahan, A.S.: 2020  
 Mereghetti, Monahan, Rizik, A.S.,  
 Stoffer : 2021



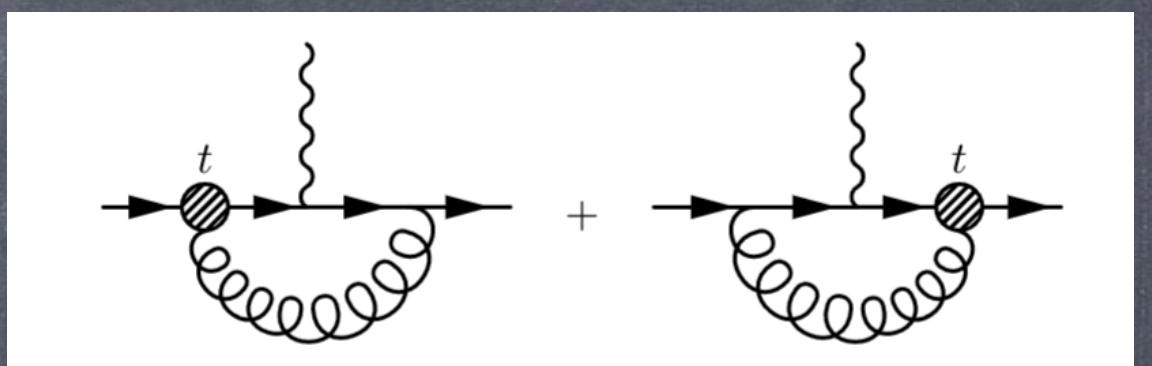
$c_P(t, \mu)$      $c_{m^2 P}(t, \mu)$



$c_{m\theta}(t, \mu)$



$c_E(t, \mu)$



- ⦿ Expand integrands of loop integrals in all scales excluding  $t$ 
  - ⦿ Analytic structure altered  $\rightarrow$  distortion of IR structure
  - ⦿ in matching equation the IR modification drops out in the difference
  - ⦿ Expanding loop integrals in the RHS vanish in DR  $\rightarrow$  UV and IR are identical
  - ⦿ The LHS is UV-finite, beside the renormalization of the bare parameters and flowed fermion fields
  - ⦿ The IR singularities on the LHS exactly match the UV MS counterterms

# Quark-Chromo EDM

Mereghetti, Monahan, Rizik, A.S.,  
Stoffer : 2021

$$\mathcal{O}_{CE}^R(x; t) = \bar{\chi}(x; t) \tilde{\sigma}_{\mu\nu} G_{\mu\nu}(x; t) \dot{\chi}(x; t)$$

$$\begin{aligned} \mathcal{O}_{CE}^R(x; t) &= c_P(t, \mu) \mathcal{O}_P^{\text{MS}}(x; \mu) + c_{m\theta}(t, \mu) \mathcal{O}_{m\theta}^{\text{MS}}(x; \mu) + c_E(t, \mu) \mathcal{O}_E^{\text{MS}}(x; \mu) \\ &\quad + c_{CE}(t, \mu) \mathcal{O}_{CE}^{\text{MS}}(x; \mu) + c_{mP^2}(t, \mu) \mathcal{O}_P^{\text{MS}}(x; \mu) + O(t) \end{aligned}$$

$$\begin{aligned} c_{CE}(t, \mu) &= \zeta_\chi^{-1} + \frac{\alpha_s}{4\pi} \left[ 2(C_F - C_A) \log(8\pi\mu^2 t) - \frac{1}{2} \left( (4 + 5\delta_{\text{HV}})C_A + (3 - 4\delta_{\text{HV}})C_F \right) \right] \\ &= 1 + \frac{\alpha_s}{4\pi} \left[ (5C_F - 2C_A) \log(8\pi\mu^2 t) \right. \\ &\quad \left. - \frac{1}{2} \left( (4 + 5\delta_{\text{HV}})C_A + (3 - 4\delta_{\text{HV}})C_F \right) - \log(432)C_F \right] \end{aligned}$$

$$c_P(t, \mu) = \frac{\alpha_s C_F}{4\pi} \frac{6i}{t} \quad c_E(t, \mu) = \frac{\alpha_s C_F}{4\pi} (4 \log(8\pi\mu^2 t) + 3 + 2\delta_{\text{HV}}) \quad c_{m^2 P}(t, \mu) = \frac{\alpha_s C_F}{4\pi} i \left( 12 \log(8\pi\mu^2 t) + \frac{1}{2} (33 - 16\delta_{\text{HV}}) \right)$$

# Scale dependence matching coefficients

$$\bar{\mu}_0 = 3 \text{ GeV} \rightarrow \mu_0 = 1.13 \text{ GeV}$$

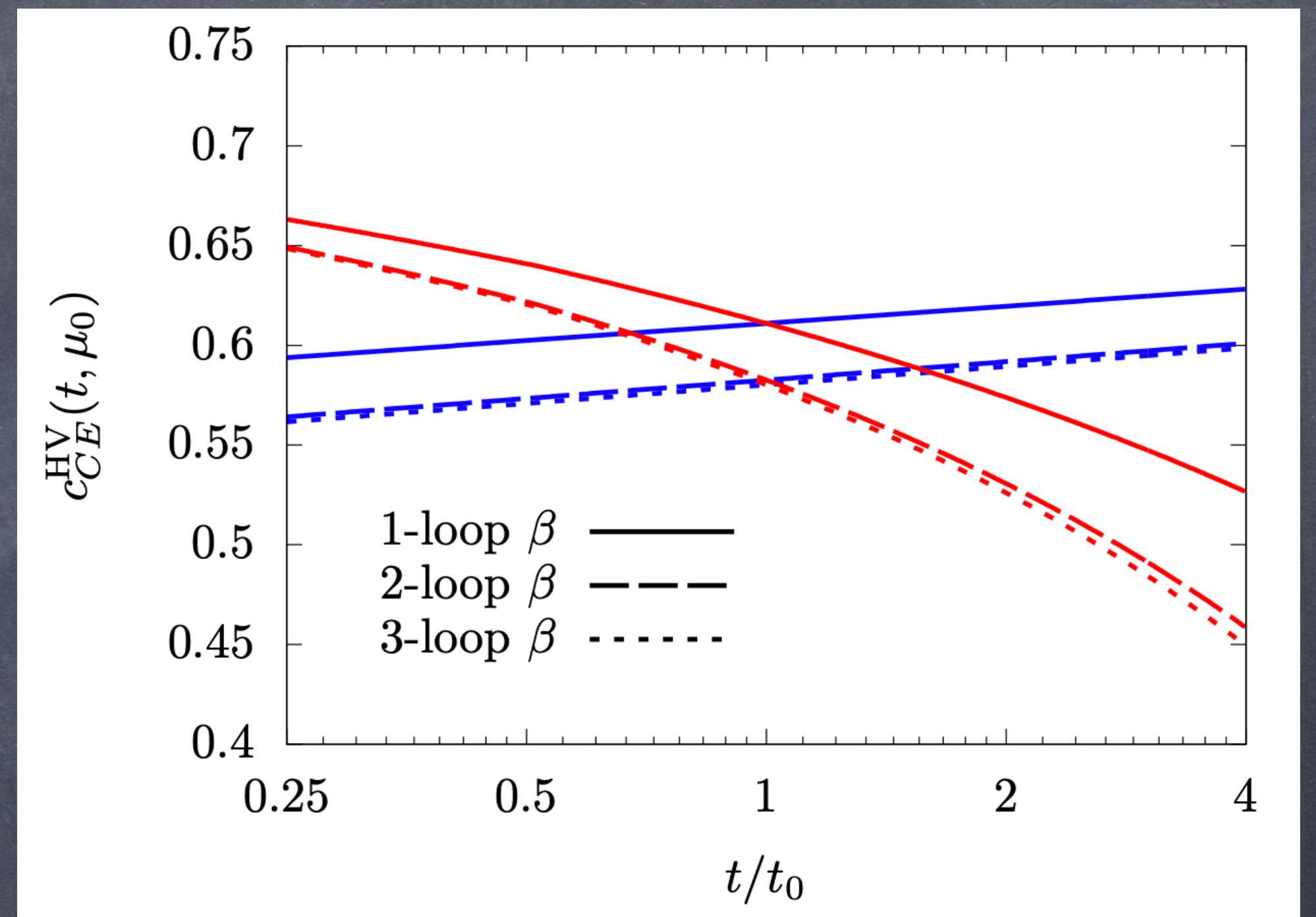
$$t_0 = \frac{1}{8\pi\mu_0^2}$$

Red - Blue =

$$A_1 \alpha_s^2(\mu_0^2) \log^2(8\pi t \mu_0^2) + A_2 \alpha_s^2(\mu_0^2) \log(8\pi t \mu_0^2) + O(\alpha_s^3)$$

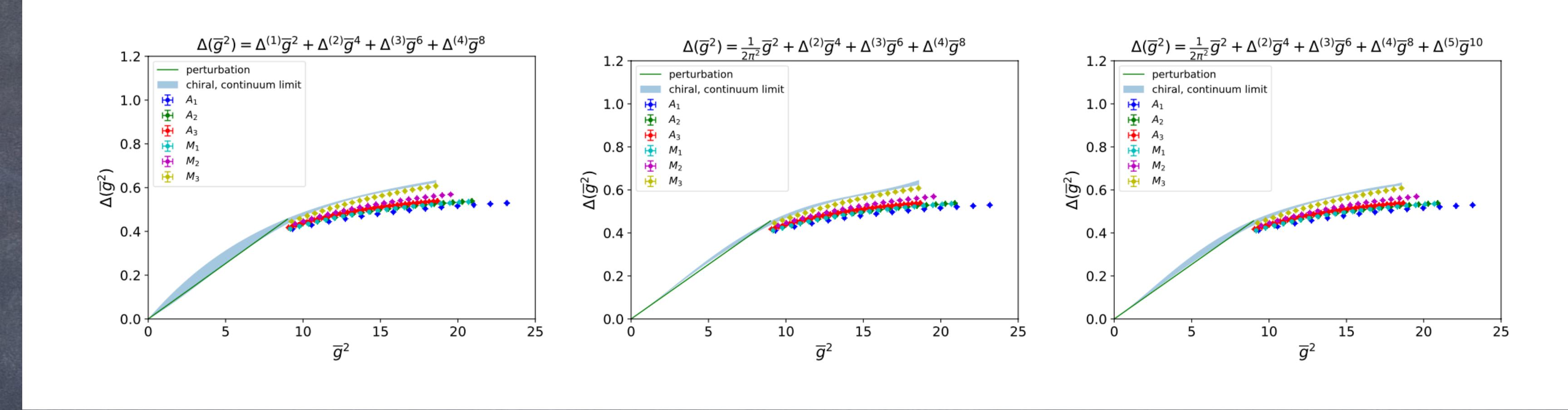
$$t \in [t_0/4, 4t_0]$$

10%-20% uncertainties from PT at 1-loop



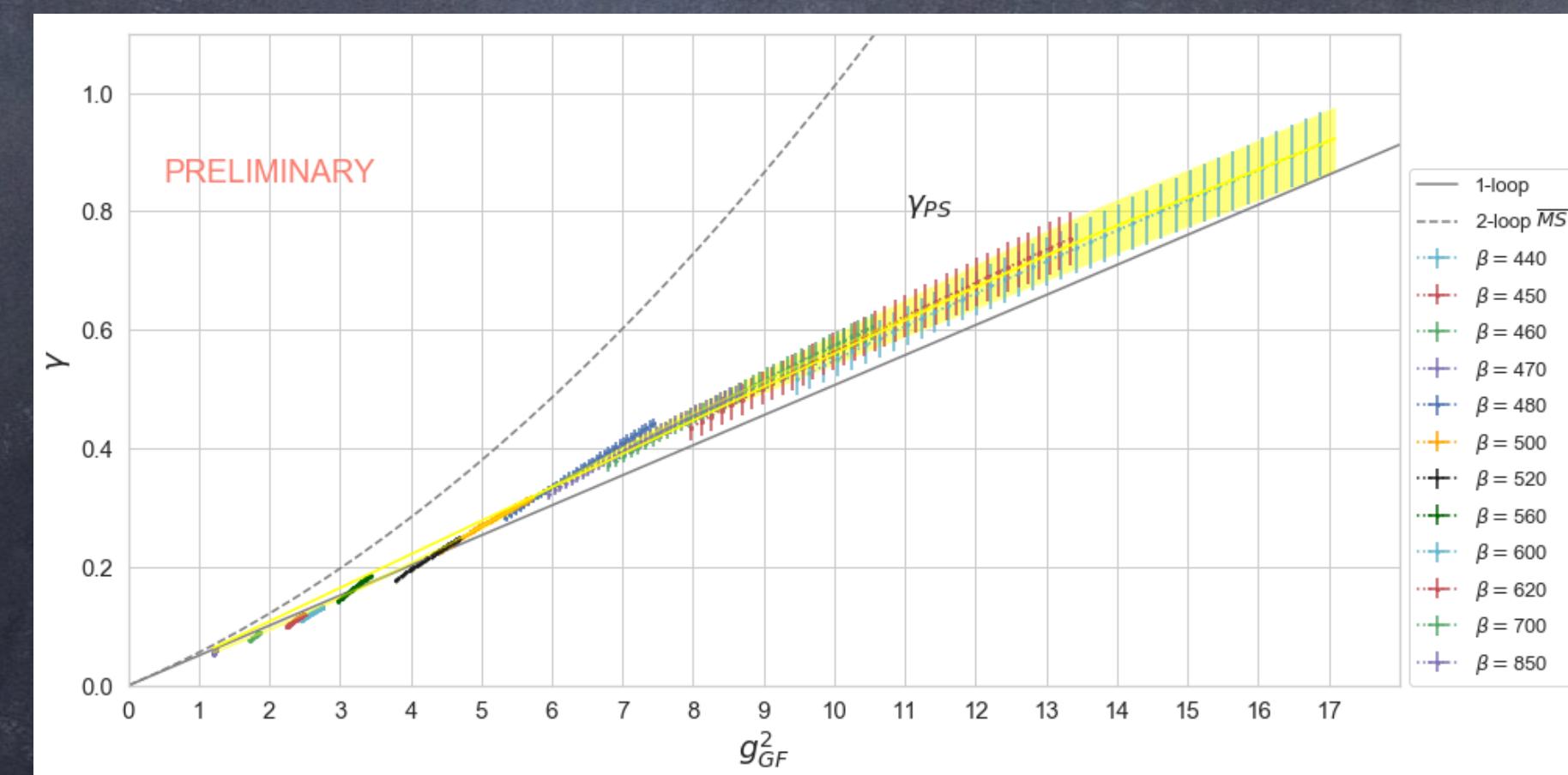
# Non-perturbative applications

Non-perturbative  
determination  
of power divergences



Kim, Luu, Rizik, A.S.:2020

Non-perturbative  
renormalization  
scheme



$$\gamma_{GF} = -2t \frac{d \log R_{\mathcal{O}}(a, x_4, t)}{dt}$$

$$Z_{\mathcal{O}}^{GF}(a, \mu) \frac{R_{\mathcal{O}}(a, x_4, t=0)}{R_{\mathcal{O}}(a, x_4, t)} \Big|_{t=c/\mu^2 x_4 \gg \sqrt{t/c}} = \text{t.l.}$$

Hasenfratz, Monahan, Rizik,  
A.S., Witzel: in progress

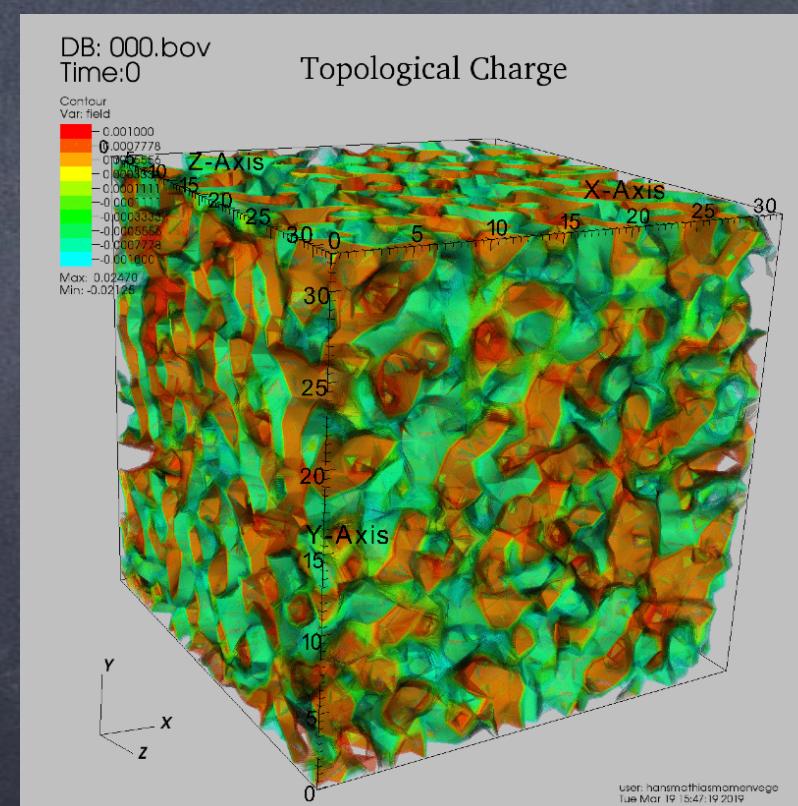
# Phenomenological applications

- ⦿ CP-violating operators → EDM
- ⦿ Electroweak Hamiltonian Harlander and Lange: 2022
- ⦿ qCEDM + qCMDM → heavy SM and BSM particles on flavor observables:
  - ⦿ CP-conserving long distance contributions to Kaon mixing
  - ⦿ direct CP-violation on hyperon decays
  - ⦿  $\Delta I=1/2$  and  $K \rightarrow \pi \pi$
  - ⦿ CP-violating part of the  $K \rightarrow 3\pi$
  - ⦿ Neutral B-mesons: masses and decay rates differences

# Conclusive remarks

- ⦿ The Gradient Flow is a an important tool for renormalization
- ⦿ Several calculations show the potential of the GF
- ⦿ Results need to be matched with  $t=0$ 
  - ⦿ Topological quantities  $\rightarrow t$ -independent
  - ⦿ Ward Identities  $\rightarrow$  chiral condensate
  - ⦿ Short flow time expansion: strong coupling, strange content, qCEDM
- ⦿ Non-perturbative determination of power divergences  
 $\rightarrow \varepsilon'/\varepsilon$ , Effective electro-weak Hamiltonian, higher moments PDF

# Takk!



# Backup slides

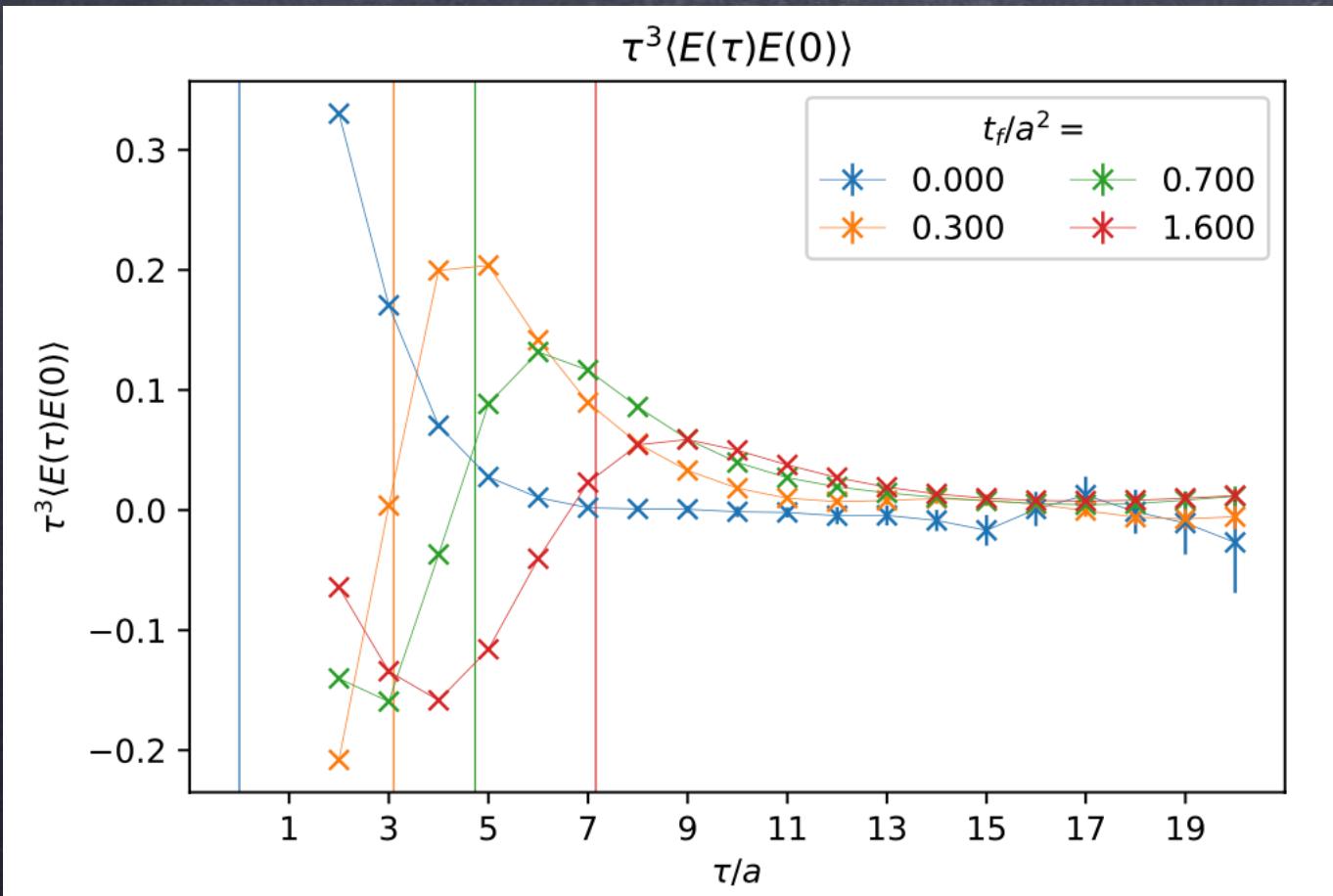
# Inclusive annihilation P-wave spin-triplet quarkonium

$$\Gamma_{\chi_{QJ}} = \frac{3N_c}{2\pi} |R'(0)|^2 \frac{32}{M^4} \left[ \Im f_1(^3P_J)(\Lambda) + \Im f_8(^3S_1) \frac{2T_F}{9N_c} \mathcal{E}_3(\Lambda) \right]$$

$$\mathcal{E}_3 = \frac{T_F}{N_c} \int dx_4 x_4^3 \langle 0 | g E_i^a(x_4, 0) \Phi_{ab}(t, 0) g E_i^b(0, 0) | 0 \rangle$$

Requires renormalization

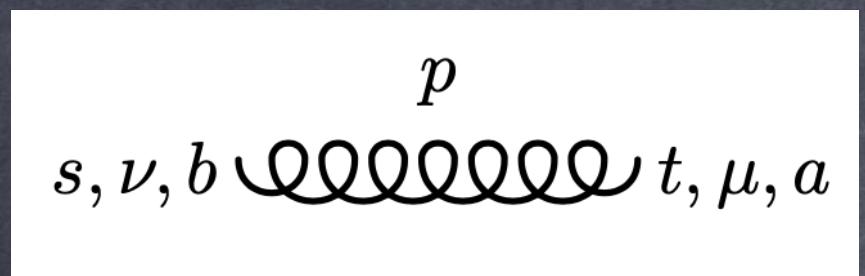
$$\mathcal{E}_3 = \frac{6\alpha_s C_F}{\pi} \left[ \frac{1}{\epsilon} + \log(2t\bar{\mu}) - \frac{2}{3} + \gamma_E \right]$$



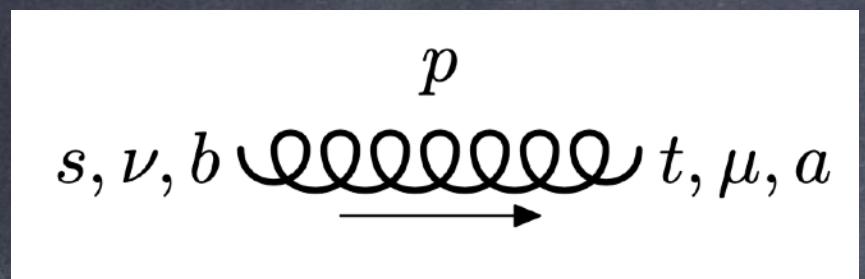
Brambilla, Chung, Leino,  
Mayer-Stedte, Wang: in progress

# Feynman rules

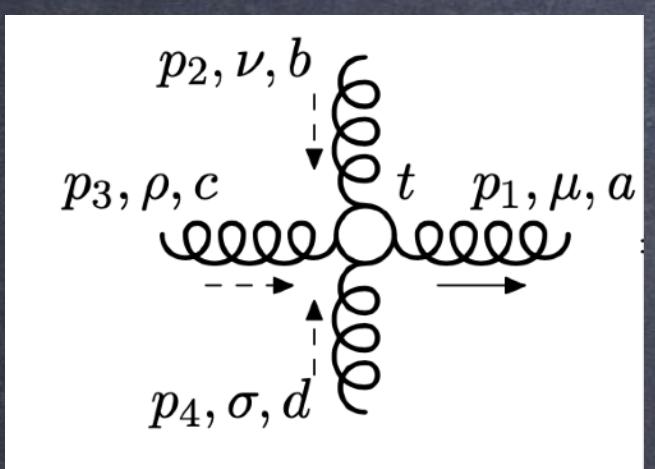
Rizik, Monahan, A.S.: 2020  
 Mereghetti, Monahan, Rizik, A.S.,  
 Stoffer : 2021



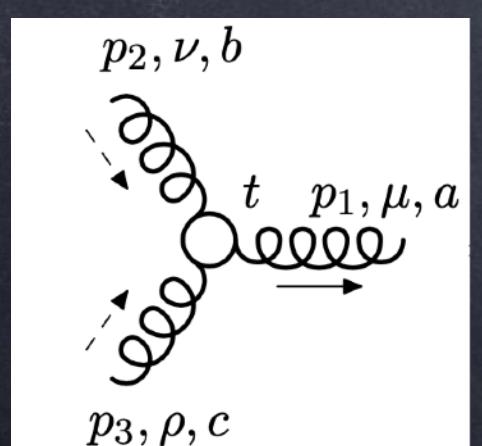
$$\left\langle \tilde{B}_\mu^a(p, t) \tilde{B}_\nu^b(-p, s) \right\rangle = \tilde{D}_{\mu\nu}^{ab}(p, s+t) = g_0^2 \delta^{ab} \frac{1}{p^2} \left[ \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) e^{-(s+t)p^2} + \xi \frac{p_\mu p_\nu}{p^2} e^{-\alpha_0(s+t)p^2} \right]$$



$$= \delta^{ab} \theta(t-s) \frac{1}{p^2} \left[ (\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-(t-s)p^2} + p_\mu p_\nu e^{-\alpha_0(t-s)p^2} \right]$$



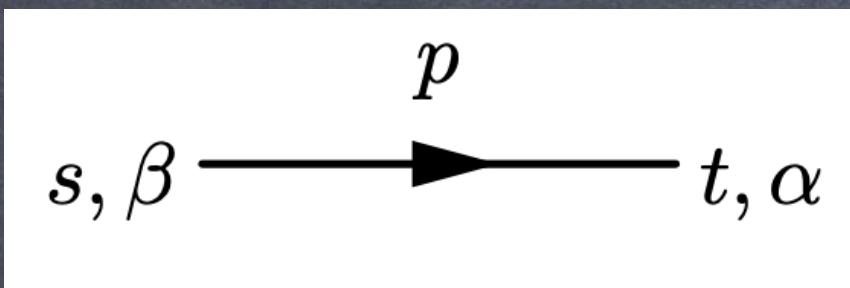
$$= -i f^{abc} \int_0^\infty dt \left( \delta_{\nu\rho} (p_2 - p_3)_\mu + 2\delta_{\mu\rho} p_{3\nu} - 2\delta_{\mu\nu} p_{2\rho} + (\alpha_0 - 1)(\delta_{\mu\nu} p_{3\rho} - \delta_{\mu\rho} p_{2\nu}) \right)$$



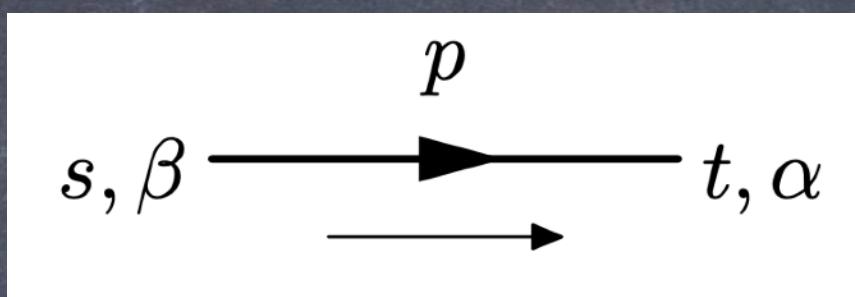
$$= - \int_0^\infty dt \left( f^{abe} f^{cde} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\rho\nu}) + f^{ace} f^{bde} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) + f^{ade} f^{bce} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\rho} \delta_{\nu\sigma}) \right)$$

# Feynman rules

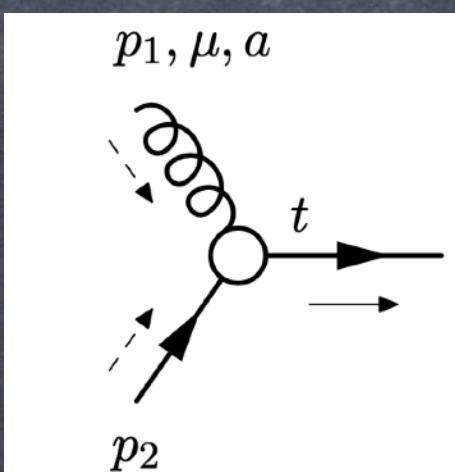
Rizik, Monahan, A.S.: 2020  
 Mereghetti, Monahan, Rizik, A.S.,  
 Stoffer : 2021



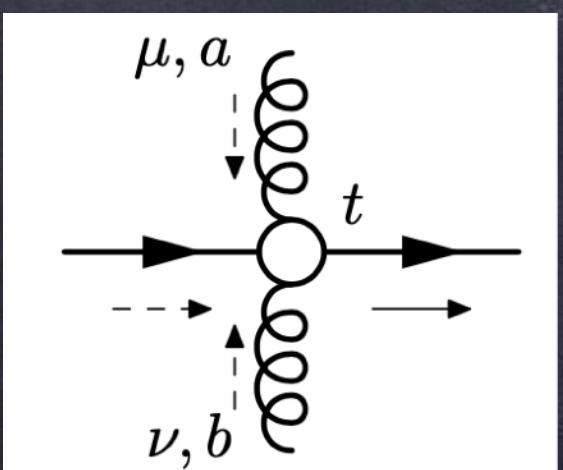
$$\left\langle \tilde{\chi}(p, t) \tilde{\bar{\chi}}(p, s) \right\rangle = \tilde{S}(p, s + t) = \frac{-i\cancel{p} + m}{p^2 + m^2} e^{-(s+t)p^2}$$



$$= \delta^{\alpha\beta} \theta(t - s) e^{-(t-s)p^2}$$



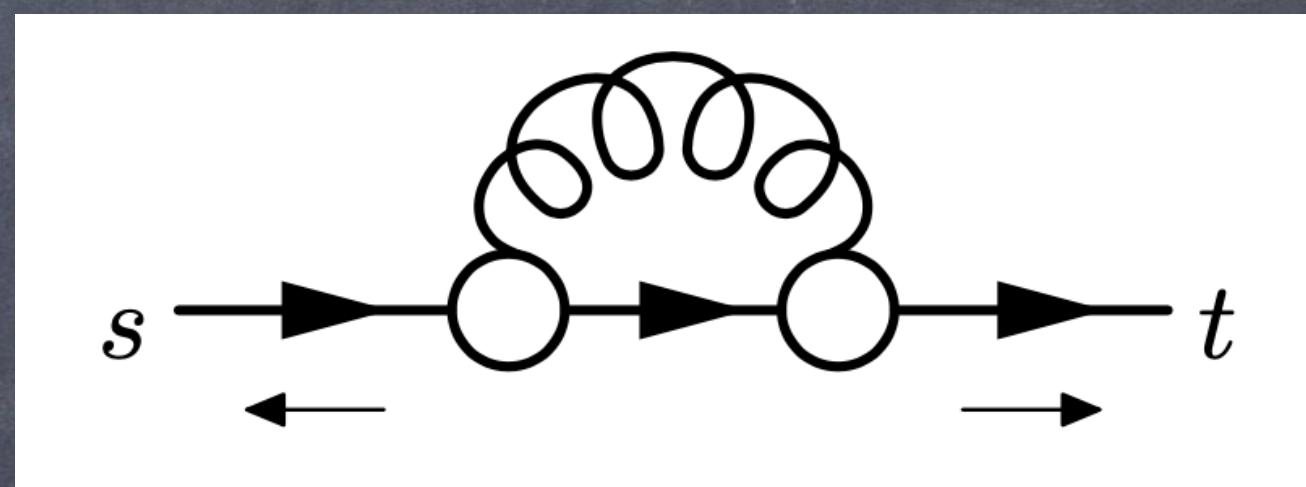
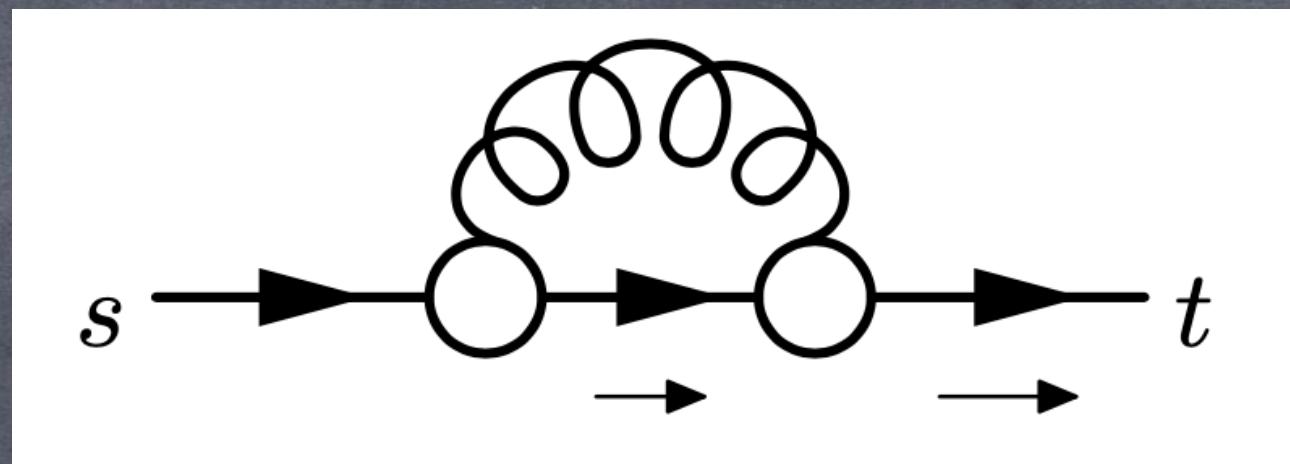
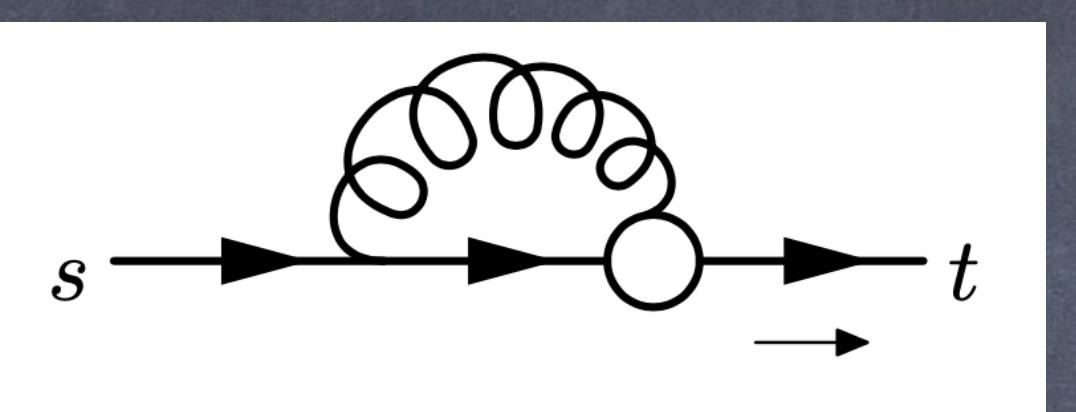
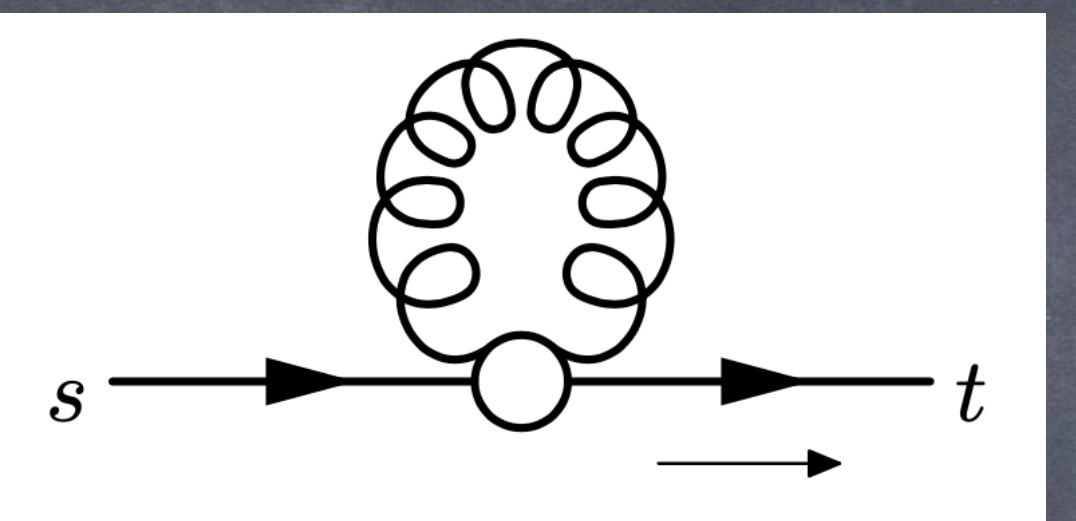
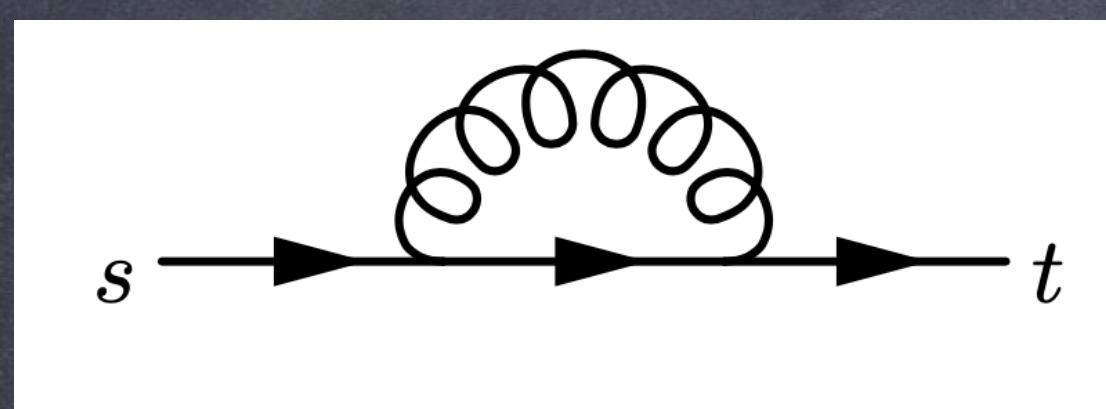
$$= -it^a \int_0^\infty dt \left( (1 - \alpha_0) p_{1\mu} + 2 p_{2\mu} \right)$$



$$= \delta_{\mu\nu} \{t^a, t^b\} \int_0^\infty dt$$

# Sample calculation

Rizik, Monahan, A.S.: 2020



$$\tilde{S}(p, t, s) = \tilde{S}^{(0)}(p, t, s) \left\{ 1 - \frac{g_0^2}{(4\pi)^2} C_F \left[ \frac{3}{\epsilon} + \log(8\pi\mu^2 t) + \log(8\pi\mu^2 s) + \log\left(\frac{4\pi\mu^2}{p^2}\right) - \gamma_E + 1 \right] \right\} + O(s, t, g_0^4)$$

$$\chi_R(x, t) = Z_\chi^{1/2} \chi(x, t) \quad S_R(p, t, s) = Z_\chi S(p, t, s)$$

$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2} C_F \frac{3}{\epsilon}$$

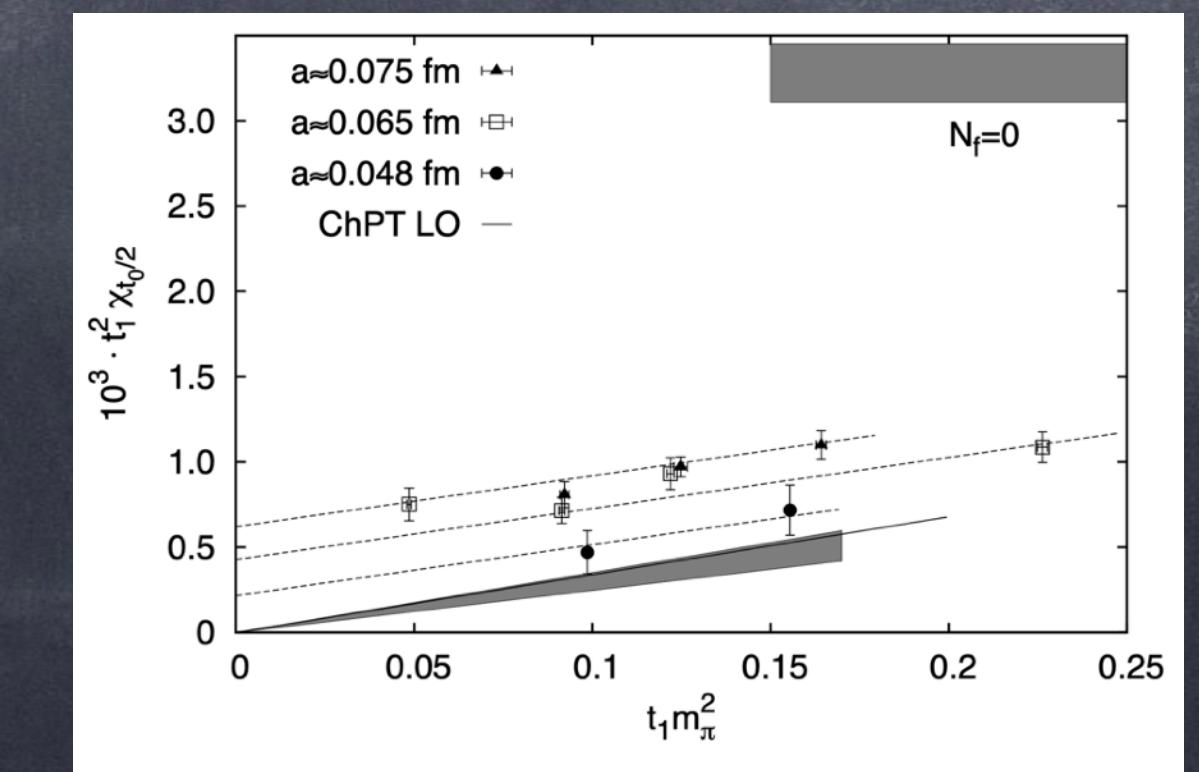
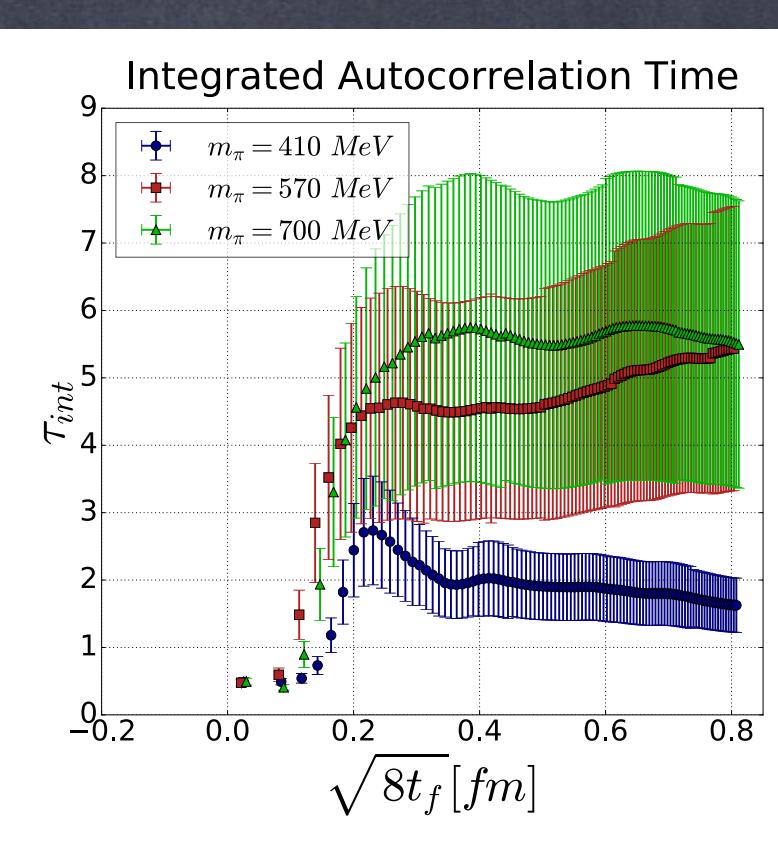
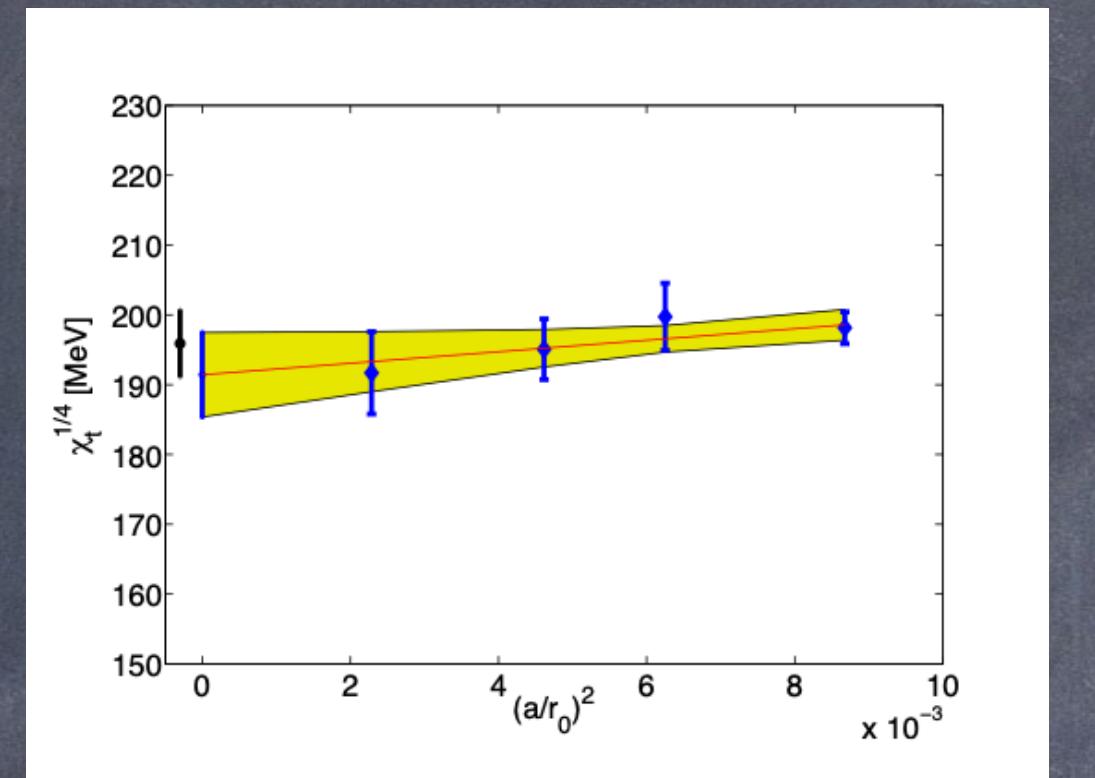
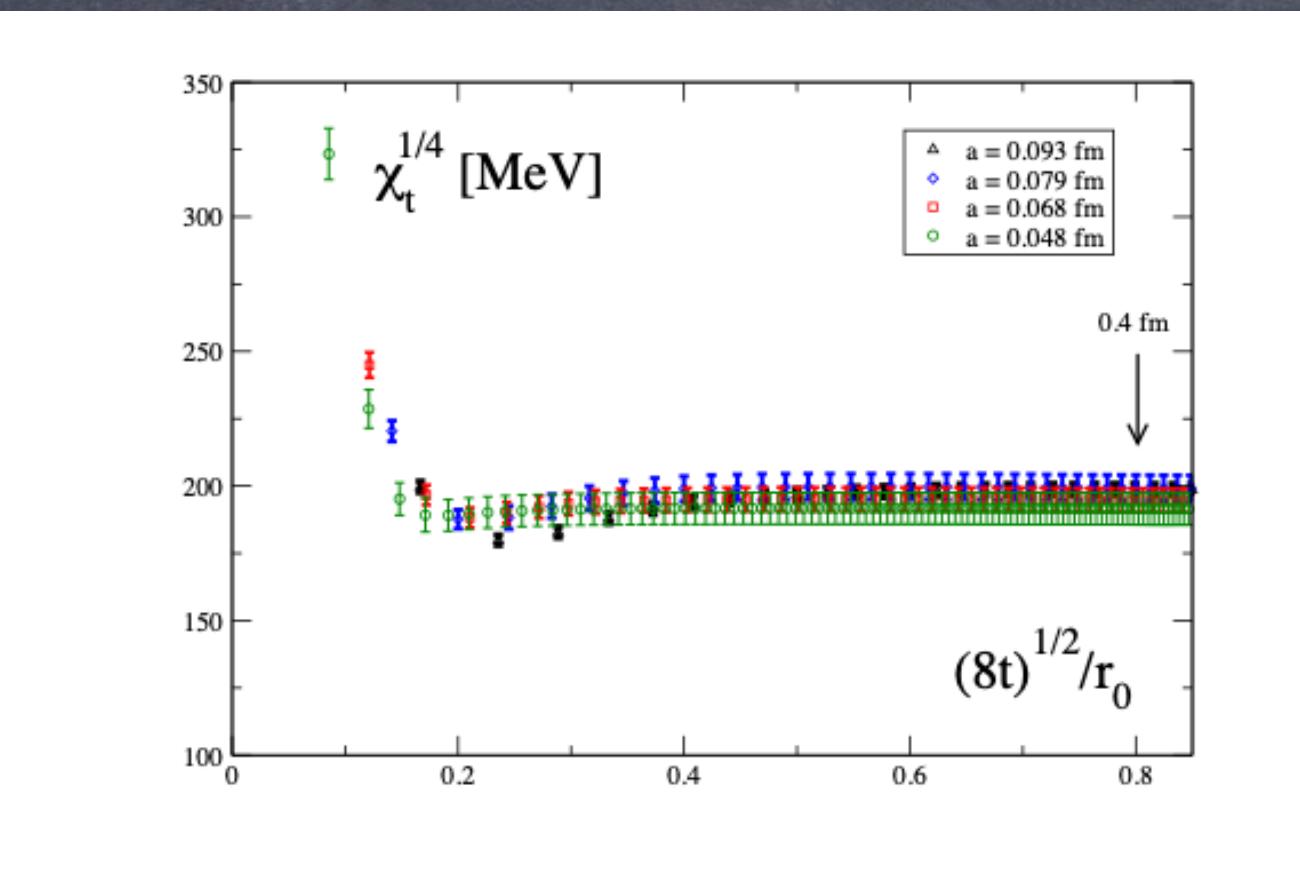
# Topological susceptibility

$$q(x, t) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr} \{ G_{\mu\nu}(x, t) G_{\rho\sigma}(x, t) \} \quad Q(t) = \int d^4x \ q(x, t)$$

A.S., de Vries, Luu: 2014,  
2015

$$\chi_t^{1/4} = 191(7) \text{ MeV}$$

$$\chi_t = \frac{1}{V} \int d^4x \ d^4y \ \langle q(x, t_f) q(y, t_f) \rangle$$



Bruno, Schaefer, Sommer:  
2014

# The role of lattice QCD

$$d_N = M_N^\theta \bar{\theta} + \left(\frac{v}{\Lambda}\right)^2 \sum_i M_N^{(i)} \tilde{d}_i \quad \langle N | J_\mu \mathcal{O}_{CP} | N \rangle \rightarrow d \underline{E} \cdot \underline{S}$$

$M_N^\theta$   Hadronic matrix element topological charge

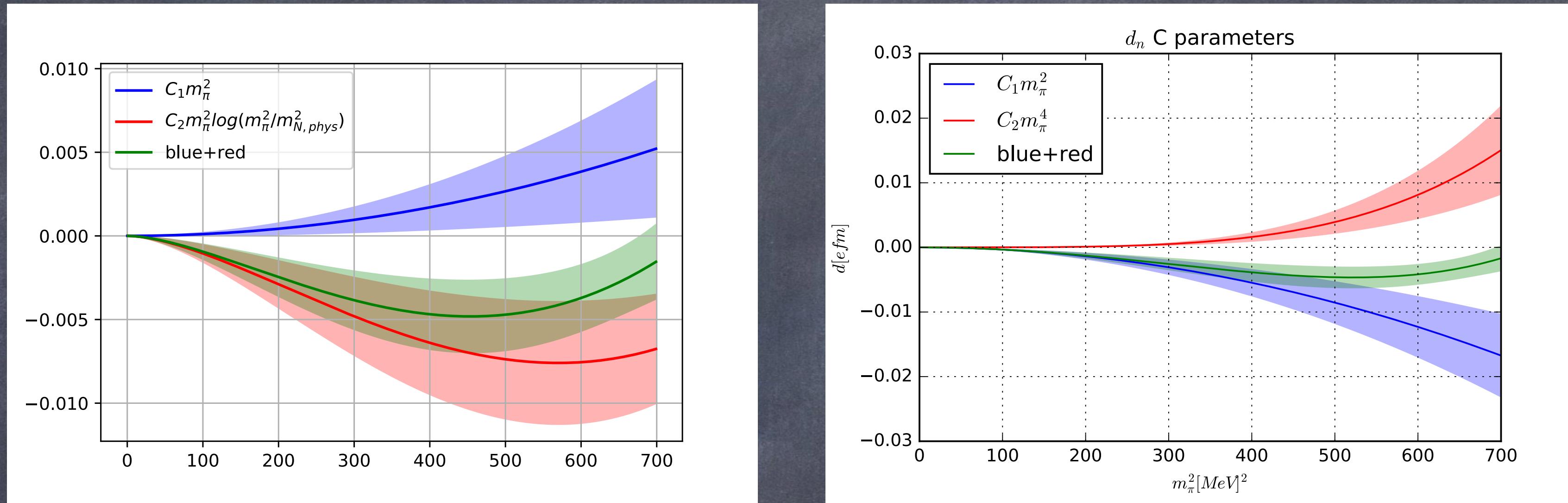
$M_N^{(i)}$   Hadronic matrix element CP odd operators

The role of LQCD is to provide the renormalized matrix elements

Only in this way it is  
possible to interpret  
experimental results and  
disentangle all CP violating  
sources

Shintani et al.: 2005  
Berruto, Blum, Orginos, Soni 2006  
A.S., Luu, de Vries: 2014-2015  
Guo, Meißner, et al. : 2010-  
Alexandrou et al. (ETMC): 2015-2020  
Abramczyk et al. : 2017-  
Dragos, Kim, Luu, Monahan, Rizik, A.S., de Vries,  
Yousif: 2015-2021  
Yoon, Bhattacharya, Cirigliano, Gupta: 2015-2021

# ChPT-inspired fit

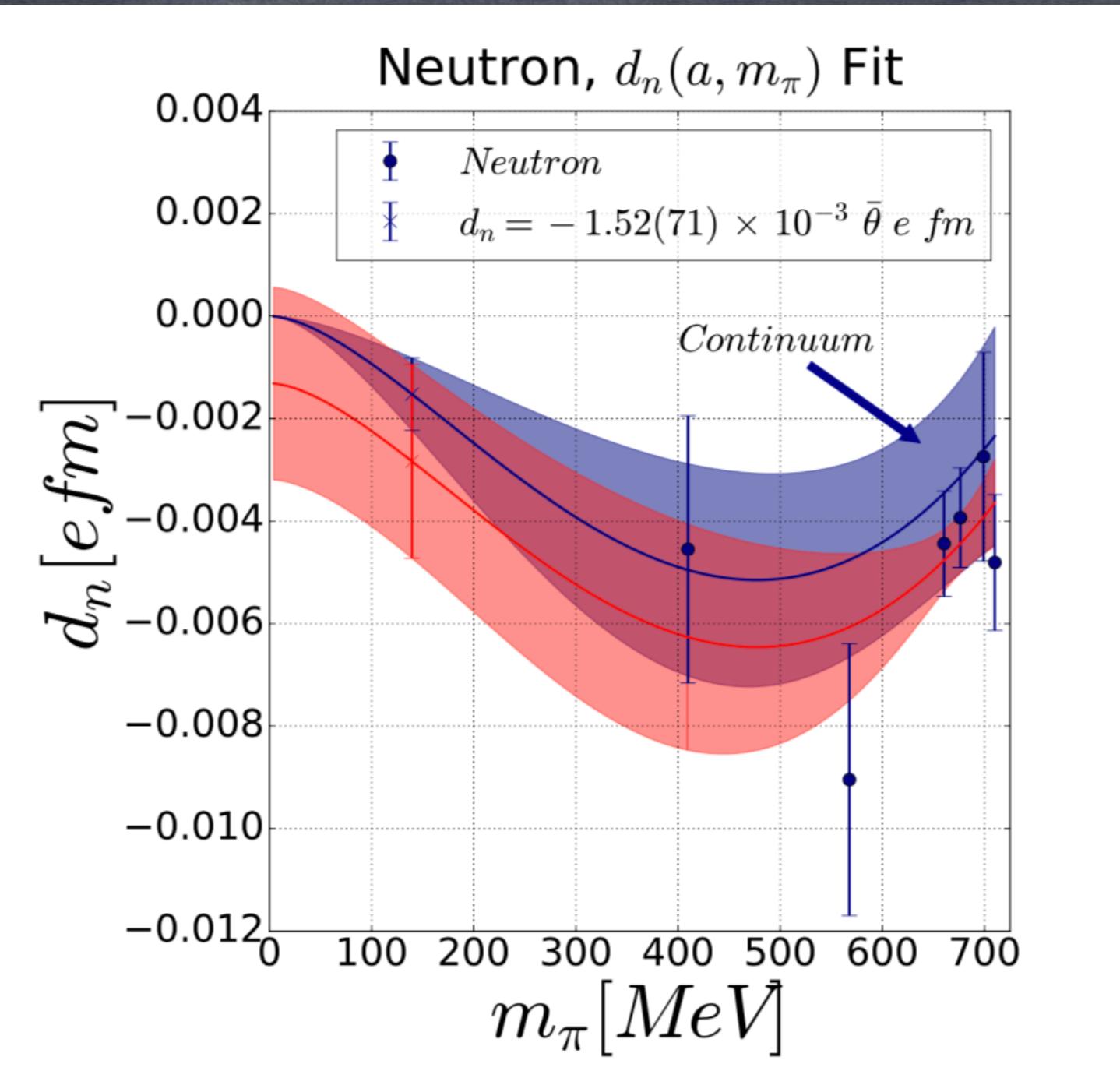


$$d_{n/p}(m_\pi) = C_1^{n/p} m_\pi^2 + C_2^{n/p} m_\pi^2 \ln \frac{m_\pi^2}{M_N^2}$$

$$d_{n/p}(m_\pi) = C_1^{n/p} m_\pi^2 + C_2^{n/p} m_\pi^4$$

Data naturally favor the ChPT-inspired pion mass dependence ==> log dominance

# Chiral interpolation



$$d_n(\bar{\theta}) = \bar{d}_n - \frac{eg_A \bar{g}_0^{\bar{\theta}}}{8\pi^2 F_\pi} \ln \frac{m_\pi^2}{M_N^2}$$

$$d_p(\bar{\theta}) = \bar{d}_p + \frac{eg_A \bar{g}_0^{\bar{\theta}}}{8\pi^2 F_\pi} \ln \frac{m_\pi^2}{M_N^2}$$

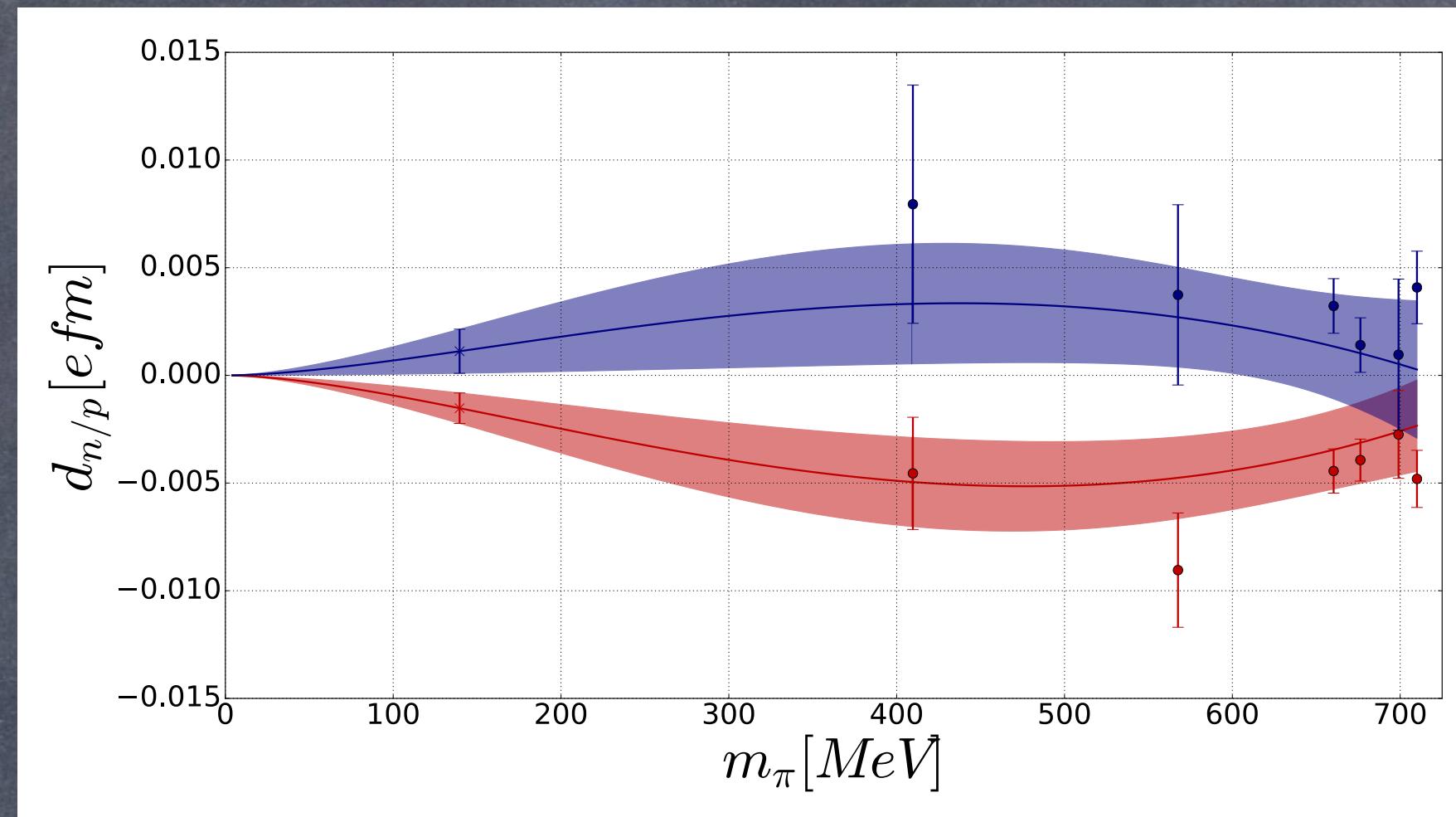
Ottnad, Kubis, Meißner, Gut: 2010  
 Mereghetti, de Vries, Hockings,  
 Maekawa, van Kolck: 2011

$$d_{n/p}(a, m_\pi) = C_1^{n/p} m_\pi^2 + C_2^{n/p} m_\pi^2 \ln \frac{m_\pi^2}{M_N^2}$$

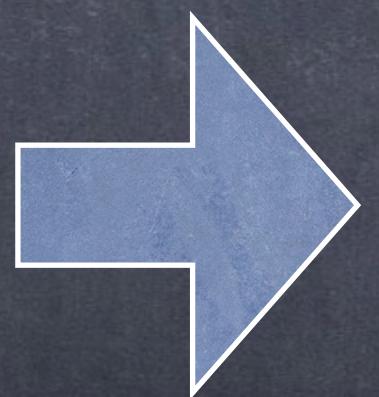
Dragos, Luu, A.S.,  
 de Vries, Yousif: 2019

# EDM from $\theta$ -term

Dragos, Luu,  
A.S.,  
de Vries, Yousif:  
2019



$$d_n^{\text{phys}} = -0.00152(71) \bar{\theta} e \text{ fm}$$



$$|\bar{\theta}| < 1.98 \times 10^{-10} (90\% \text{ CL})$$

$$\bar{g}_0^{\bar{\theta}} = -1.28(64) \cdot 10^{-2} \bar{\theta}$$

Ab-initio determination of  $\bar{g}_0^{\bar{\theta}}$

$$\bar{g}_0^{\bar{\theta}} = -1.47(23) \cdot 10^{-2} \bar{\theta}$$

Crewther, di Vecchia,  
Veneziano, Witten: 1979

de Vries, Mereghetti, Walker-Loud: 2015

# 4+1 chiral symmetry

A.S. 2013

$$S_{F,fl} = \int_0^\infty dt \int d^4x \left[ \bar{\lambda}(t, x) (\partial_t - \Delta) \chi(t, x) + \bar{\chi}(t, x) \left( \overleftarrow{\partial}_t - \overleftarrow{\Delta} \right) \lambda(t, x) \right]$$

$$\begin{cases} \chi(t, x) \rightarrow \exp \left\{ i \left( \alpha_V^a \frac{T^a}{2} + \alpha_A^a \frac{T^a}{2} \gamma_5 \right) \right\} \chi(t, x) \\ \bar{\chi}(t, x) \rightarrow \bar{\chi}(t, x) \exp \left\{ i \left( -\alpha_V^a \frac{T^a}{2} + \alpha_A^a \frac{T^a}{2} \gamma_5 \right) \right\} . \end{cases} \quad \begin{cases} \lambda(t, x) \rightarrow \exp \left\{ i \left( \alpha_V^a \frac{T^a}{2} - \alpha_A^a \frac{T^a}{2} \gamma_5 \right) \right\} \lambda(t, x) \\ \bar{\lambda}(t, x) \rightarrow \bar{\lambda}(t, x) \exp \left\{ i \left( -\alpha_V^a \frac{T^a}{2} - \alpha_A^a \frac{T^a}{2} \gamma_5 \right) \right\} \end{cases}$$

$$\langle \mathcal{O}_t \delta S \rangle = \langle \delta \mathcal{O}_t \rangle$$

Chiral variation before integrating  
the Lagrange multipliers

$$\begin{cases} \left\langle \left[ \partial_\mu A_{R,\mu}^a(x) - 2m P^a(x) + \tilde{P}_R^a(0, x) \right] \mathcal{O}_R(\{t_0\}, x) \right\rangle = 0 & t = 0 \\ \\ \left\langle \left[ \partial_s \tilde{P}^a(s, x) + \partial_\mu \mathcal{A}_\mu^a(s, x) \right] \mathcal{O}_R(\{t_0\}, x) \right\rangle = 0 & s > 0, \quad s < \{t_0\} \end{cases} \quad \text{Lüscher: 2013}$$

$$\tilde{P}^a(t, x) = \bar{\lambda}(t, x) \frac{T^a}{2} \gamma_5 \chi(t, x) + \bar{\chi}(t, x) \frac{T^a}{2} \gamma_5 \lambda(t, x)$$

# Ward identities

A.S. 2013

$$\begin{cases} \delta\psi(x) = [i\alpha_V^a(x)\frac{T^a}{2} + i\alpha_A^a(x)\frac{T^a}{2}\gamma_5] \psi(x) \\ \delta\bar{\psi}(x) = \bar{\psi}(x) [-i\alpha_V^a(x)\frac{T^a}{2} + i\alpha_A^a(x)\frac{T^a}{2}\gamma_5] \end{cases} \quad \text{Chiral variation after integrating the Lagrange multipliers}$$

$$\langle \mathcal{O}_t \delta S \rangle = \langle \delta \mathcal{O}_t \rangle \Rightarrow \langle \langle \left[ \partial_\mu A_{IR,\mu}^a(x) - 2m_R P_R^a P_R^a(x) \tilde{P}_R^a \mathcal{O}_R(t) \right] \mathcal{G}_R(t, y) \rangle \rangle = 0$$

$$\left\langle i \left[ \frac{\delta \mathcal{O}_t}{\delta \alpha_A^a(x)} \right]_R \right\rangle = - \left\langle \mathcal{O}_R \tilde{P}_R^a(0, x) \right\rangle$$

**VWI**

$$\begin{cases} \left\langle \left[ \partial_\mu V_{R,\mu}^a(x) + (\bar{\psi}(x) [\frac{T^a}{2}, M] \psi)_R(x) + \bar{S}^a(0, x) \right] \mathcal{O}_R \right\rangle = 0 & t = 0 \\ \left\langle \left[ \partial_t \bar{S}^a(t, x) + \partial_\mu \mathcal{V}_\mu^a(t, x) \right] \mathcal{O}(\{t_0\}) \right\rangle = 0 & t > 0 \quad t < \{t_0\} \end{cases}$$

# Integrated WI

A.S. 2013

$$\left\{ \begin{array}{l} \left\langle \left[ \partial_\mu A_{R,\mu}^a(x) - 2m_R P_R^a(x) + \tilde{P}_R^a(0,x) \right] P_R^a(t,y) \right\rangle = 0 \quad t=0 \\ \\ \left\langle \left[ \partial_s \tilde{P}^a(s,x) + \partial_\mu \mathcal{A}_\mu^a(s,x) \right] P_R^a(t,y) \right\rangle = -\frac{1}{2} \left\langle S_R^0(s,y) \right\rangle \delta(t-s) \delta(x-y) \quad s,t > 0 \end{array} \right.$$

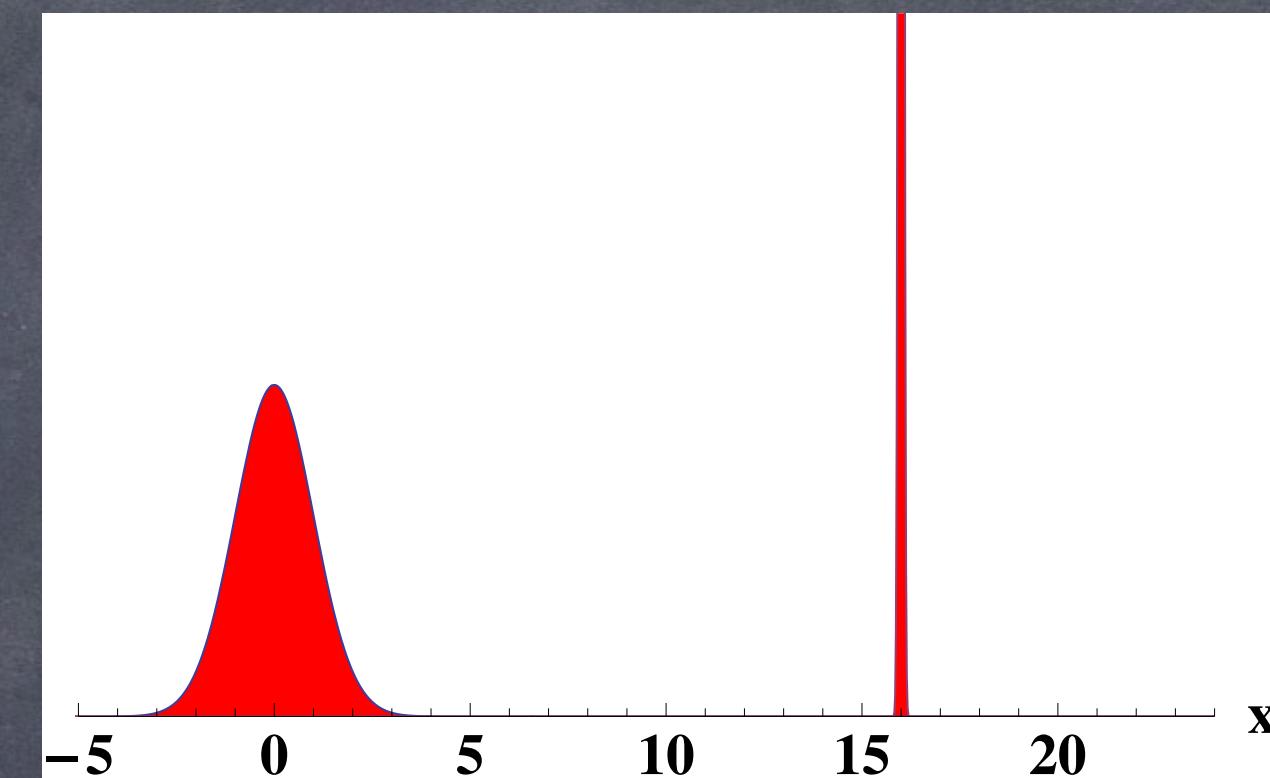
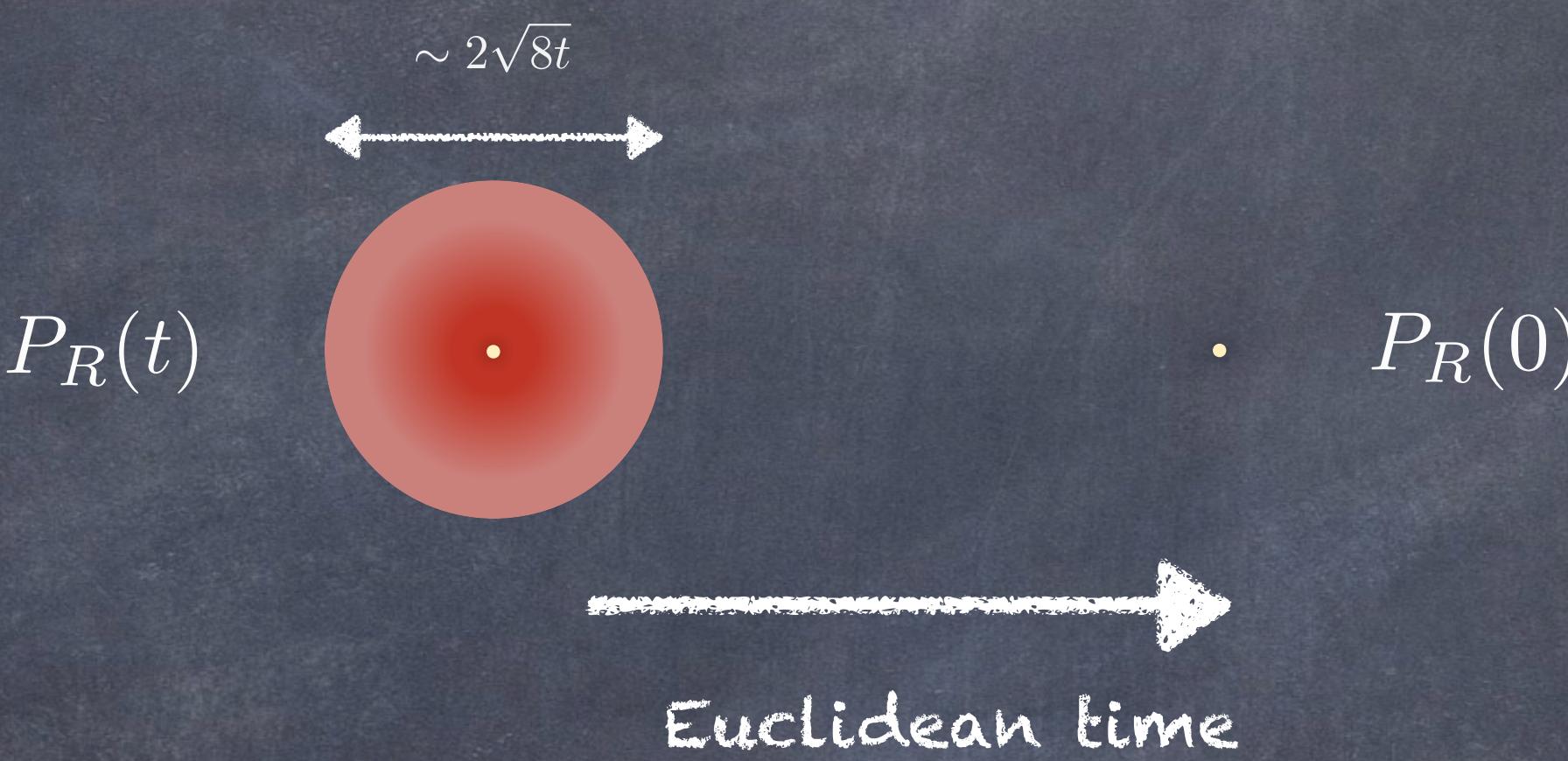
$$\int d^4x \left\langle \tilde{P}^a(0,x) P_R^a(t,y) \right\rangle = \frac{1}{2} \left\langle S_R^0(t,y) \right\rangle$$

Lüscher:2013

$$2m_R \int d^4x \left\langle P_R^a(0,x) P_R^a(t,y) \right\rangle = \frac{1}{2} \left\langle S_R^0(t,y) \right\rangle \equiv \Sigma_{R,t}$$

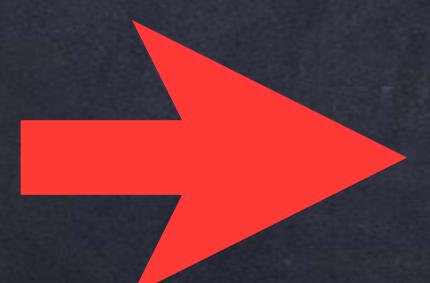
# Chiral condensate

$$\int d^3x \langle P_R^a(x) P_R^a(0, t) \rangle = -\frac{G_{R,\pi} G_{R,\pi,t}}{m_\pi} e^{-m_\pi x_0} + \dots$$



Integrated W.I.

$$2m_R \int d^4x \langle P_R^a(0, x) P_R^a(t, y) \rangle = \frac{1}{2} \langle S_R^0(t, y) \rangle \equiv \Sigma_{R,t}$$



$$\Sigma_{R,t} = 2m_R \frac{G_{R,\pi} G_{\pi,t}}{m_\pi^2}$$

Lüscher: 2013  
A.S.: 2013

# Chiral condensate

$$\Sigma_R = \lim_{m_R \rightarrow 0} [F_\pi G_\pi]_R$$

$$\Sigma_{R,t} = 2m_R \frac{G_{R,\pi} G_{\pi,t}}{m_\pi^2}$$

PCAC

$$2m_R G_{R,\pi} = m_\pi^2 F_\pi \quad \rightarrow$$

$$\Sigma_R = \lim_{m_R \rightarrow 0} \frac{2m_R G_{R,\pi}^2}{m_\pi^2}$$

$$\boxed{\Sigma_R = Z_P \lim_{m_R \rightarrow 0} \frac{G_\pi}{G_{\pi,t}} \Sigma_t}$$

Lüscher: 2013  
A.S.: 2013

No power divergences  
Regularization independent

# Strategy

A.S., de Vries, Luu: 2014

$$S^{rs}(t, x) = \bar{\chi}_r(t, x)\chi_s(t, x)$$

$$S^{rs}(t, x) = c_0(t)M^{rs} + c_1(t)M^{rs}\text{Tr} [M^2] + c_2(t) (M^3)^{rs} + c_3(t)S^{rs}(0, x) + O(t)$$

$$P^{rs}(t, x) = c_3(t)P^{rs}(0, x) + O(t)$$

$$\int d^3x \left\langle P^{ud}(0, x)P^{du}(t, x) \right\rangle = -\frac{G_\pi G_{\pi,t}}{M_\pi} e^{-M_\pi x_0}$$



$$c_3(t) = \frac{G_{\pi,t}}{G_\pi} + O(t)$$

# Strategy

$$\mathcal{C}^{\text{sub}}(t, x) = \langle \mathcal{N} S^{rs}(t) \mathcal{N}^\dagger \rangle - \langle S^{rs}(t) \rangle \langle \mathcal{N} \mathcal{N}^\dagger \rangle$$

$$\mathcal{C}^{\text{sub}}(t, x) = c_3(t) \mathcal{C}^{\text{sub}}(0, x) + O(t)$$

$$\mathcal{C}_{\text{sub}}(0, x) = \frac{G_\pi}{G_{\pi,t}} \cdot [\langle \mathcal{N} S^{rs}(t) \mathcal{N}^\dagger \rangle - \langle S^{rs}(t) \rangle \langle \mathcal{N} \mathcal{N}^\dagger \rangle] + O(t)$$

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No renormalization

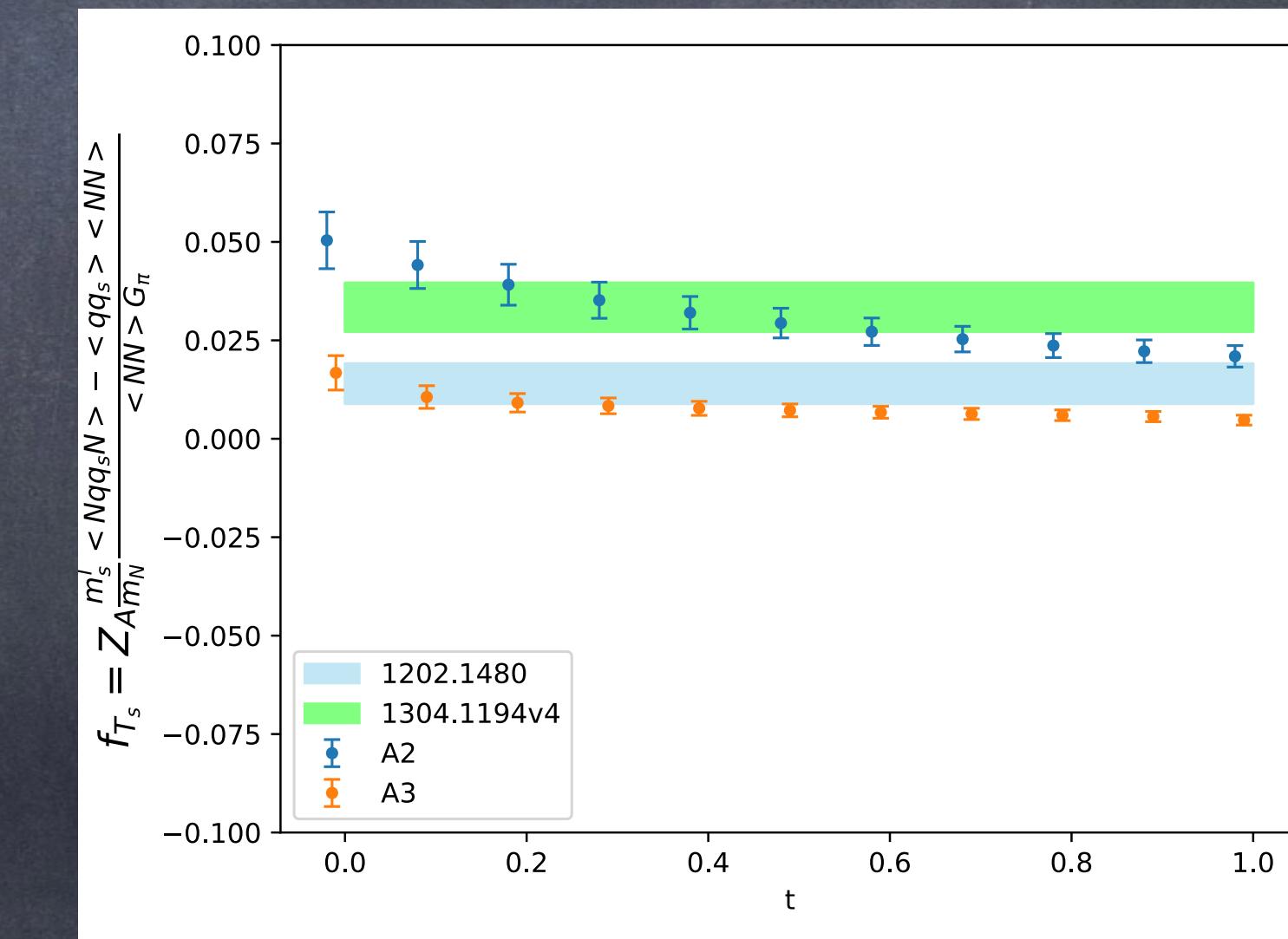
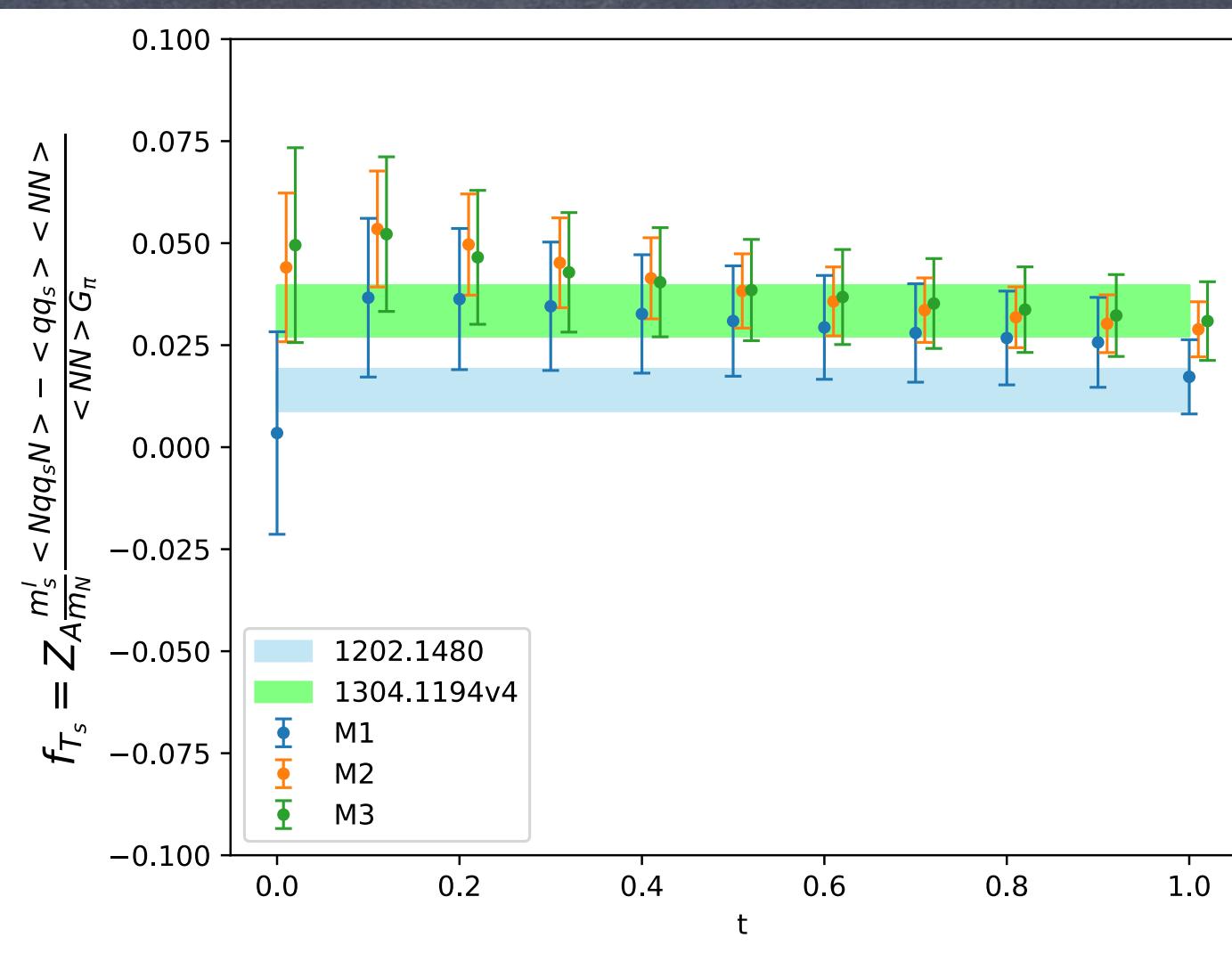
No power divergent subtraction

# Strange content of the nucleon

Kim, Luu, Pederiva, Rizik, A.S.:2022

$$\mathcal{C}_{\text{sub}}(0, x) = \frac{G_\pi}{G_{\pi,t}} \cdot [\langle \mathcal{N} S^{rs}(t) \mathcal{N}^\dagger \rangle - \langle S^{rs}(t) \rangle \langle \mathcal{N} \mathcal{N}^\dagger \rangle] + O(t)$$

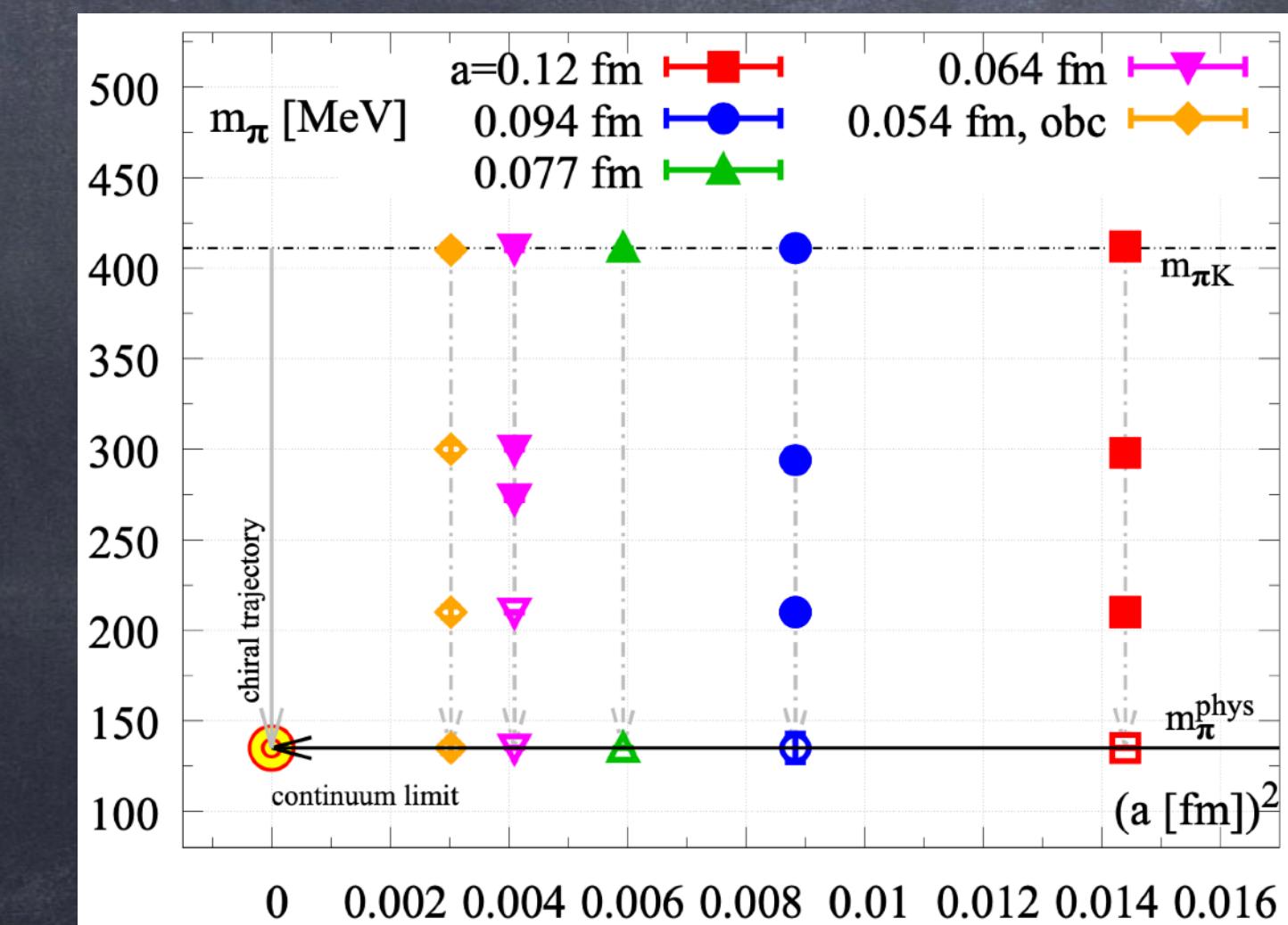
$$f_{T_s} = \frac{m_s}{m_N} \langle N | \bar{s}s | N \rangle$$



# Ongoing work

- ⦿ Perturbation theory with the Gradient Flow      Mereghetti, Monahan, Rizik, A.S., Stoffer
- ⦿ Non-perturbative determination of expansion coefficients      Kim, Luu, Pederiva, Rizik, A.S.
- ⦿ Non-perturbative renormalization scheme with GF      Hasenfratz A., Monahan, Rizik, A.S., Witzel  
(in progress)
- ⦿ OpenLat: open science initiative. Gauges with SWF open to the whole community

Cuteri, Francis, Fritzsch, Pederiva, Rago, A.S.,  
Walker-Loud, Zafeiropoulos



# Quark-Chromo EDM: non-perturbative renormalization (power divergences)

Kim, Luu, Rizik, A.S.:2020

- ⦿ Non-perturbative determination of power divergences
- ⦿ Continuum limit impossible with other methods

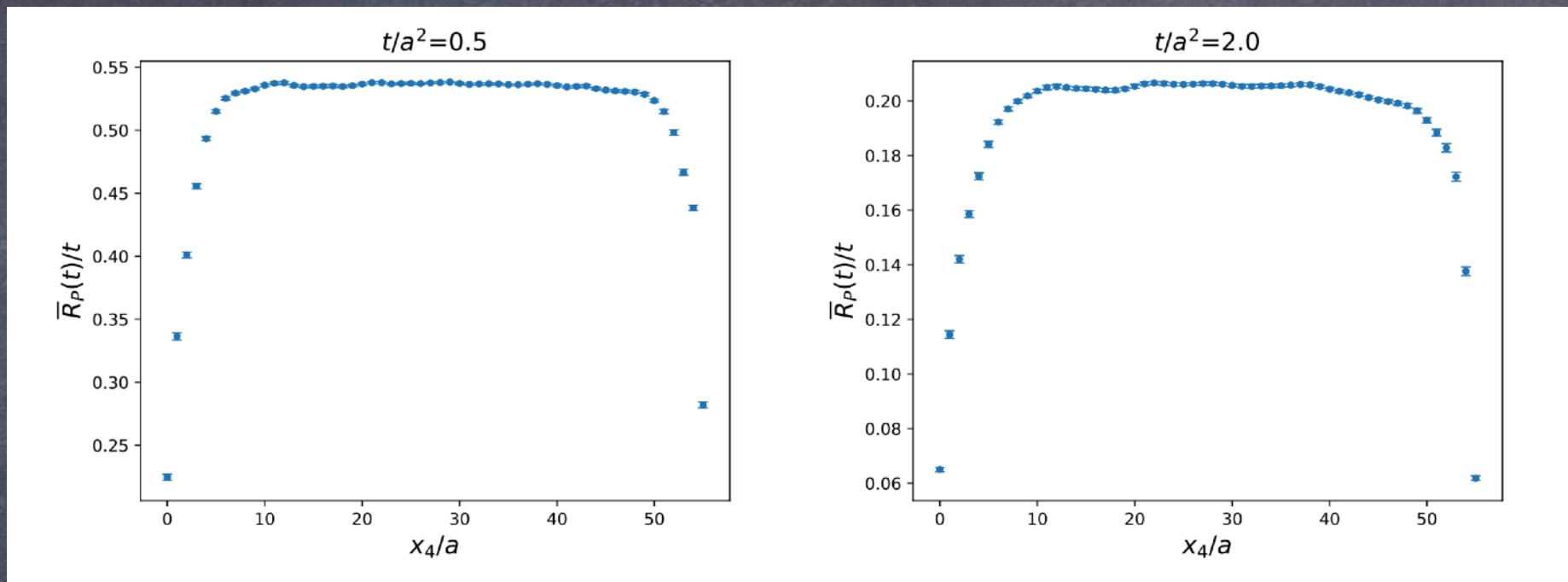
$$\Gamma_{CP}(x_4; t) = a^3 \sum_{\mathbf{x}} \left\langle \mathcal{O}_{CE}^{ij}(x_4, \mathbf{x}; t) P^{ji}(0, \mathbf{0}; 0) \right\rangle$$

$$\Gamma_{PP}(x_4; t) = a^3 \sum_{\mathbf{x}} \left\langle P^{ij}(x_4, \mathbf{x}; t) P^{ji}(0, \mathbf{0}) \right\rangle$$

$$[\overline{R}_P(x_4; t)]_R = t \frac{[\Gamma_{CP}(x_4; t)]_R}{[\Gamma_{PP}(x_4; t)]_R} \longrightarrow \text{Coefficient linear divergence}$$

# Quark-Chromo EDM: non-perturbative renormalization (power divergences)

Kim, Luu, Rizik, A.S.:2020



$$\frac{[\bar{R}_P(x_4; t)]_R}{t}$$

$$c_P(t, \mu) = \frac{\alpha_s C_F}{4\pi} \frac{6i}{t}$$

Coefficient linear divergence

