nazario tantalo
$\qquad$

FLAG21，arXiv：2111．09849



## FLAG21, arXiv:2111.09849

PDG review, j.rosner, s.stone, r.van de water, 2019 m.moulson, e.passemar CKM(18)
v.cirigliano, m.giannotti, h.neufeld JHEP 11 (2008) v.cirigliano, h.neufeld PLB 700 (2011)
...

$$
\begin{aligned}
& \Gamma_{P_{\ell 2}}=(\text { kin. facts. })\left|V_{C K M}\right|^{2} G_{F}^{2}\left(1+\delta R_{P}\right) f_{P}^{2} \\
& \delta R_{K \pi}^{\chi}=-1.12(21) \% \\
& \Gamma_{K_{\ell 3}}=(\text { kin. facts. })\left|V_{u s}\right|^{2} G_{F}^{2} \underbrace{S_{E W}(1+\overbrace{\delta_{K}^{\ell}+\delta_{S U(2)}}^{\delta_{K}})}\left\{f_{+}(0)\right\}^{2} \\
& \delta R_{K_{e}^{0}}^{\chi}=3.3(2) \%, \quad \delta R_{K_{e}}^{\chi}=2.9(2) \% \\
& \delta R_{K_{K}}^{\chi}=3.7(2) \%, \quad \delta R_{\mu}^{\chi}
\end{aligned}
$$





$$
\mathcal{O}(0) \mapsto L^{\nu}\left\langle H_{F}\right| J_{\nu}^{W}(0)\left|H_{I}\right\rangle
$$

$$
\mathcal{O}(\alpha) \mapsto \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{L^{\mu \nu}(k)}{k^{2}} \int d^{4} x e^{i k x} T\left\langle H_{F}\right| J_{\mu}^{e m}(x) J_{\nu}^{W}(0)\left|H_{I}\right\rangle
$$



extracting QED radiative corrections from a non－perturbative lattice simulation is a challenging problem！
－QED is a long－range unconfined interaction that needs to be consistently defined on a finite volume（see backup）
－finite－volume effects are potentially very large，e．g．of $O\left(L^{-1}\right)$ in the case of the masses of stable hadrons
－in the case of decay rates the problem is much more involved because of the appearance of infrared divergences，$O(\log (L))$ ，at intermediate stages of the calculation：the infrared problem！
－an alternative approach is to calculate the relevant convolution integrals， of the product of the non－local QCD matrix elements with the QED analytical kernels，by estimating the long distance tails of the QCD objects
－from the numerical point of view，it is difficult to disentangle QED radiative corrections from the leading QCD contributions


$$
P_{\ell 2(\gamma)}
$$





- in order to perform this calculation one has to cope, on the lattice, with the well known infrared problem
f.bloch, a.nordsieck, Phys.Rev. 52 (1937) t.d.lee, m.nauenberg, Phys.Rev. 133 (1964) p.p.kulish, I.d.faddeev, Theor.Math.Phys. 4 (1970)
- infrared divergences appear at intermediate stages of the calculation and cancel in physical observables by summing virtual and real photon contributions
- let's consider the infrared-safe observable: at $O(\alpha)$ this is obtained by considering the real contributions with a single photon in the final state

$$
\Gamma(E)=\Gamma_{0}+e^{2} \lim _{L \rightarrow \infty}\left\{\Gamma_{V}(L)+\Gamma_{R}(L, E)\right\}
$$

- the finite-volume calculation of the real contribution is an issue: momenta are quantized!
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- the finite-volume calculation of the real contribution is an issue: momenta are quantized!
- for this reason, by relying on the universality of infrared divergences, it is convenient to rewrite the previous formula as

$$
\Gamma(E)=\Gamma_{0}+e^{2} \lim _{L \rightarrow \infty}\{\Gamma_{V}(L) \overbrace{-\Gamma_{V}^{p t}(L)+\Gamma_{V}^{p t}(L)+\Gamma_{R}^{p t}(L, E)-\Gamma_{R}^{p t}(L, E)}^{=0}+\Gamma_{R}(L, E)\}
$$

where $\Gamma_{V, R}^{p t}$ are evaluated in the point-like effective theory: these have the same infrared behaviour of $\Gamma_{V, R}$

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$$
\Gamma(E)=\Gamma_{0}+e^{2} \lim _{L \rightarrow \infty} \Gamma_{V}^{S D}(L)+e^{2} \lim _{m_{\gamma} \rightarrow 0}\left\{\Gamma_{V}^{p t}\left(m_{\gamma}\right)+\Gamma_{R}^{p t}\left(m_{\gamma}, E\right)\right\}+e^{2} \lim _{m_{\gamma} \rightarrow 0} \Gamma_{R}^{S D}\left(m_{\gamma}, E\right)
$$

where $\Gamma_{V, R}^{p t}$ are evaluated in the point-like effective theory: these have the same infrared behaviour of $\Gamma_{V, R}$


- the RM123+SOTON master formula is a sum of three contributions


$$
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$$

－the RM123＋SOTON master formula is a sum of three contributions
－the calculation started in RM123＋SOTON，PRD 91 （2015）with a generalization of the infinite－volume point－like contribution $\Gamma^{p t}(E)$ ，originally obtained by berman 58 and kinoshita 59 （see backup）


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- the RM123+SOTON master formula is a sum of three contributions
- the calculation started in RM123+SOTON, PRD 91 (2015) with a generalization of the infinite-volume point-like contribution $\Gamma^{p t}(E)$, originally obtained by berman 58 and kinoshita 59 (see backup)
- the analytical calculation of $\Gamma_{V}^{p t}(L)$, up to and including $1 / L$ terms, allows to turn a $\log (L)$ into a $1 / L^{2}$ finite volume effect: this is possible thanks to the universality of soft-photon contributions (Low's theorem)


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established in RM123+SOTON, PRD 95 (2017) checked in RBC-UKQCD PRD 105 (2022)
- the observation that (universality again) $\Gamma_{R}^{S} D(E)$ is negligible in the limit of very small photon energies, together with the challenging numerical calculation of $\Gamma_{V}^{S D}$, allowed to obtain...
$\Gamma_{P}(E)=\Gamma_{P}^{0}\left\{1+\delta R_{P}(E)\right\}$,
- first-principle lattice results:

$$
\begin{aligned}
& \delta R_{K}\left(E_{K}^{m a x}\right)=0.0024(10) \\
& \delta R_{\pi}\left(E_{\pi}^{m a x}\right)=0.0153(19) \\
& \delta R_{K \pi}=-0.0126(14)
\end{aligned}
$$

- to be compared with the result obtained in $\chi$ pt (v.cirigliano and h.neufeld, PLB 700 (2011)) and currently quoted by the PDG:

$$
\begin{aligned}
& \delta R_{K}\left(E_{K}^{m a x}\right)=0.0064(24) \\
& \delta R_{\pi}\left(E_{\pi}^{m a x}\right)=0.0176(21) \\
& \delta R_{K \pi}^{\chi}=-0.0112(21)
\end{aligned}
$$



## some more details

$\qquad$

RM123+SOTON, PRL 120 (2018), PRD 100 (2019)

(a)

(d)

(b)

(c)

(f)


- lattice calculation performed by using the RM123 method, i.e. by expanding the lattice path-integral with respect to $\alpha$ and the up-down quark mass difference
- renormalization constants computed non-perturbatively in the $\mathrm{RI}^{\prime}-\mathrm{MOM}$ scheme and matched perturbatively with the so-called $W$-scheme (a.siritin, NPB 196 (1982); e.braten and c.s.li PRD 42 (1990)) in which $G_{F}$ is defined

(a)

(b)

(c)

(d)

(e)
- contributions corresponding to charged sea-quarks estimated by using $\chi$ pt but not computed: this is the so called electroquenched approximation, there is certainly room for improvement here...


## RM123＋SOTON PRD．100．2019



－excellent numerical signals

## RM123+SOTON PRD.100.2019



RBC-UKQCD, m.dicarlo talk (here and edinburgh)



- excellent numerical signals

RBC-UKQCD, m.dicarlo talk (here and edinburgh)

## RM123+SOTON PRD.100.2019




- excellent numerical signals

- the key point in this game is the universality of FVE RM123+SOTON PRD.100.2019 PRD.95.2017 arXiv:1612.00199

$$
\left(\frac{1}{L^{3}} \sum_{k}-\int \frac{d^{3} k}{(2 \pi)^{3}}\right) \int \frac{d k^{0}}{2 \pi} f(k)
$$

- one has to look at the singularities in $\boldsymbol{k}$ after performing the $k^{0}$ integral: these are the pinched singularities coming from on-shell internal particles

(d)

(e)


(g)

- the key point in this game is the universality of FVE RM123+SOTON PRD.100.2019 PRD.95.2017 arXiv:1612.00199
- in this case the only pinched singularity is the infrared one

$$
\begin{aligned}
& k_{\mu} \Gamma^{\mu}(p, k)=\Delta^{-1}(p+k)-\Delta^{-1}(p) \\
& \Gamma^{\mu}(p, k)=2 p^{\mu}+k^{\mu}+O\left(k^{2}\right) \\
& \left(\frac{1}{L^{3}} \sum_{k}-\int \frac{d^{3} k}{(2 \pi)^{3}}\right) \int \frac{d k^{0}}{2 \pi} \frac{1}{k^{\beta}} \sim O\left(\frac{1}{L^{4-\beta}}\right)
\end{aligned}
$$

(a)

(b)

(d)


(c)

(e)

(g)



- the analysis has been repeated and extended to the non-universal $1 / L^{2}$ FVE

$$
\Gamma_{P}(E)=\Gamma_{P}^{0}\left\{1+\delta R_{P}(E)\right\}
$$

- the RM123+SOTON approach is now really a method:

$$
\begin{aligned}
& \delta R_{K \pi}^{R M 123+S O T O N}=-0.0126(14) \\
& \delta R_{K \pi}^{R B C+U K Q C D}=-0.0088(39) \\
& \delta R_{K \pi}^{\chi}=-0.0112(21)
\end{aligned}
$$



RBC-UKQCD, m.dicarlo talk



- first-principles lattice results for the radiative decays $K^{-} \rightarrow \ell \bar{\nu}_{\ell} \gamma$ and $\pi^{-} \rightarrow \ell \bar{\nu}_{\ell} \gamma$ have then been obtained
see also kane et al., PoS LATTICE2021.162 PoS LATTICE2019.134
- on the one hand, these confirmed the assumptions on $\Gamma_{R}^{S D}(E)$


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$\qquad$
- the problem is more challenging in the case of semileptonic decays because, for generic kinematical configurations, the physical observable cannot be extracted from euclidean correlators by the leading exponential contributions
- nevertheless, the RM123+SOTON method can be extended to the case of semileptonic decays
- the infrared divergence is again proportional to the leading order decay rate (obvious) and the $O\left(L^{-1}\right)$ infrared corrections are again universal although, as expected from Low's theorem, their evaluation requires the knowledge of the derivatives of the form-factors $f_{ \pm}\left(q^{2}\right)$ with respect to $q^{2}=\left(p_{K}-p_{\pi}\right)^{2}$
- there are other finite-volume corrections though, analogous to the ones studied by I.lellouch, m.lüscher CMP 219 (2001) in the case of $K \mapsto \pi \pi$ decays, that appear because of the problem of analytical continuation: the pinched singularities associated with on-shell lepton-hadrons internal states

[^0]

- a more pragmatic approach to the calculation of radiative corrections to $K_{\ell 3}$ decays has been developed and successfully implemented recently
- the idea is that of extracting from lattice simulations in the flavour $S U(3)$ limit the relevant low-energy constants that enter the $\chi \mathbf{p t}$ calculation of the radiative corrections
- the unphysical $p_{\ell}=0, m_{K}=m_{\pi}$ kinematics avoids the problem of analytical continuation
- the use of sirlin decomposition, originally developed for $n \mapsto p \ell \bar{\nu}(\gamma)$,

$$
\begin{aligned}
\delta R_{K_{\ell}} & =\frac{\alpha}{2 \pi}\left\{\tilde{g}+3 \log \frac{m_{Z}}{m_{p}}+\log \frac{m_{Z}}{m_{W}}+\tilde{a}_{g}\right\} \\
& +\delta_{H O}^{Q E D}+2 \square_{\gamma W}^{V A}
\end{aligned}
$$


allows to isolate the poorly known non-perturbative contributions into the so called $\gamma W$-box diagram

- this contribution is both ultraviolet and infrared finite
- the $\gamma W$-box contribution is explicitly given by

$$
H_{\mu \nu}(x)=T\langle\pi| J_{\mu}^{e m}(x) J_{\nu}^{W}(0)|K\rangle
$$

$$
M_{K}\left(Q^{2}\right)=-\frac{\sqrt{Q^{2}}}{6 m_{K}} \int d^{4} x \omega\left(x, Q^{2}\right) \epsilon^{\mu \nu \alpha 0} x_{\alpha} H_{\mu \nu}(x)
$$

$$
\square_{\gamma W}^{V A}=\frac{3 \alpha}{2 \pi} \int \frac{d Q^{2}}{Q^{2}} \frac{m_{W}^{2}}{m_{W}^{2}+Q^{2}} M_{K}\left(Q^{2}\right)
$$

where $\omega\left(x, Q^{2}\right)$ is known analytically

- $M_{K}\left(Q^{2}\right)$ has been matched to perturbation theory in the high $Q^{2}$ region and integrated numerically
- the matching with $\chi$ pt formulae allows the extraction of the relevant low-energy constants
- in this approach, QED is treated in infinite volume. . . it requires a separation of scales and a parametrization/modelling of the QCD kernels at long distances


$$
\left\{\begin{array} { l } 
{ \delta _ { K _ { e } ^ { \chi } } ^ { \chi l a t t } = 1 . 1 6 ( 3 ) \% } \\
{ \delta _ { K _ { e } ^ { \pm } } ^ { \chi l a t t } = 0 . 2 1 ( 5 ) \% } \\
{ \delta _ { K _ { \mu } ^ { \pm } } ^ { \chi l a t t } = 1 . 5 4 ( 4 ) \% } \\
{ K _ { \mu } ^ { 0 } } \\
{ \delta _ { K _ { \mu } ^ { \chi l a t t } } ^ { \chi l a } = 0 . 0 5 ( 5 ) \% }
\end{array} \left\{\begin{array}{l}
\delta_{K_{e}^{0}}^{\chi}=0.99(22) \% \\
\delta_{K_{e}^{\chi}}^{ \pm}=0.10(25) \% \\
\delta_{K_{\mu}^{0}}^{\chi}=1.40(22) \% \\
\delta_{\mu}^{\chi}=0.02(25) \% \\
K_{\mu}^{ \pm}
\end{array}\right.\right.
$$

$$
K_{\ell \bar{\nu}_{\ell} \ell^{\prime} \bar{\ell}^{\prime}}
$$

$$
\Gamma\left[P^{-} \rightarrow \ell \bar{\nu}_{\ell} \ell^{\prime} \bar{\ell}^{\prime}\right]
$$



- weak decays with intermediate virtual photons have also been studied
- also here, with physical meson masses and for generic kinematics, one might have problems of analytical continuation
- to date, kaon decays have been studied with $m_{\pi} \sim 350 \mathrm{MeV}$ where these problems don't arise



## RM123＋SOTON，PRD 105 （2022）

－the decay rates can be computed by extracting the four different form－factors that parametrize the hadronic tensor

$$
\begin{aligned}
H^{\mu \alpha}(k, p) & =\int d^{4} y e^{i k \cdot y} \mathrm{~T}\langle 0| j_{W}^{\alpha}(0) j_{e m}^{\mu}(y)|P(p)\rangle \\
& =H_{1}\left[k^{2} g^{\mu \alpha}-k^{\mu} k^{\alpha}\right] \\
& +H_{2}\left[\left(p \cdot k-k^{2}\right) k^{\mu}-k^{2}(p-k)^{\mu}\right](p-k)^{\alpha} \\
& -i \frac{F_{V}}{m_{P}} \varepsilon^{\mu \alpha \gamma \beta} k_{\gamma} p_{\beta} \\
& +\frac{F_{A}}{m_{P}}\left[\left(p \cdot k-k^{2}\right) g^{\mu \alpha}-(p-k)^{\mu} k^{\alpha}\right] \\
& +f_{P}\left[g^{\mu \alpha}+\frac{(2 p-k)^{\mu}(p-k)^{\alpha}}{2 p \cdot k-k^{2}}\right]
\end{aligned}
$$





$x_{q}$

$$
t_{s}|t| \uparrow \underset{\text { Lattice }}{\substack{\delta_{I V R}^{(1)}}} \delta_{\substack{(2) \\ H^{(L), \mu v}(x) \rightarrow H^{\mu v}(x)}}|\vec{x}|
$$

or the problem can be addressed by using infinite-volume QED

$$
\begin{aligned}
& H^{\mu \alpha}(k, p)=\int d^{4} y e^{i k \cdot y} H^{\mu \alpha}(y) \\
& =\int_{L} d^{4} y e^{i k \cdot y} H_{L}^{\mu \alpha}(y) \\
& +\int_{L} d^{4} y e^{i k \cdot y}\left\{H^{\mu \alpha}(y)-H_{L}^{\mu \alpha}(y)\right\} \\
& +\int_{>L} d^{4} y e^{i k \cdot y} H^{\mu \alpha}(y)
\end{aligned}
$$





| Channel | This work | Tuo et al. | ChPT | Experiment |
| :---: | :---: | :---: | :---: | :---: |
| $\operatorname{Br}\left[K \rightarrow \mu \nu_{\mu} e^{+} e^{-}\right]$ | $8.26(13) 10^{-8}$ | $10.59(33) 10^{-8}$ | $8.2510^{-8}$ | $7.93(33) 10^{-8}$ |
| $\operatorname{Br}\left[K \rightarrow e \nu_{e} \mu^{+} \mu^{-}\right]$ | $0.762(49) 10^{-8}$ | $0.72(5) 10^{-8}$ | $0.6210^{-8}$ | $1.72(45) 10^{-8}$ |
| $\operatorname{Br}\left[K \rightarrow e \nu_{e} e^{+} e^{-}\right]$ | $1.95(11) 10^{-8}$ | $1.77(16) 10^{-8}$ | $1.7510^{-8}$ | $2.91(23) 10^{-8}$ |
| $\operatorname{Br}\left[K \rightarrow \mu \nu_{\mu} \mu^{+} \mu^{-}\right]$ | $1.178(35) 10^{-8}$ | $1.45(6) 10^{-8}$ | $1.1010^{-8}$ | - |

$\qquad$

- lattice QCD calculations of $\pi_{\ell 2}, K_{\ell 2}$ and $K_{\ell 3}$ decay rates reached the impressive precision of $0.2 \%$
- at this level QED radiative corrections are relevant and must be computed with the required non-perturbative accuracy
- first-principle lattice results for $\pi_{\ell 2(\gamma)}, K_{\ell 2(\gamma)}$ are now available from different groups
- a new method to extract from the lattice the low energy constant entering $K_{\ell 3(\gamma)}$ decays has been recently developed and successfully implemented
- new techniques to calculate radiative decay rates with both real and virtual photons have been developed and interesting first-principles results are already available
- very likely i had to skip many of the things i wanted to say... for sure i have not touched heavy-light and heavy-heavy meson decays and many other things...
- from the lattice point of view, nowadays precision means $0.1 \%$ and QCD+QED: much more to come in the near future...


backup material
$\qquad$
- in order to compare results for QED radiative corrections we must first agree on what we call QCD...
- indeed, when electromagnetic interactions are taken into account the physical theory is QCD+QED
- the QCD action is no longer expected to reproduce physics and, consequently, its renormalization becomes prescription dependent
- a natural matching prescription is to use again physical experimental inputs to set the QCD parameters
 two approaches and found that the difference, nowadays, is smaller than the statistical uncertainties
- this will rapidly became an important issue
－power－law finite volume effects arise when internal states can go on－shell，e．g．

$$
\boldsymbol{k}=\frac{2 \pi \boldsymbol{n}+\boldsymbol{\theta}}{L}
$$

$$
\Delta \mathcal{O}(p, L)=\mathcal{O}(p, L)-\mathcal{O}(p, \infty)
$$

$$
=\left(\frac{1}{L^{3}} \sum_{\boldsymbol{k}}-\int \frac{d^{3} k}{(2 \pi)^{3}}\right) \int \frac{d k^{0}}{2 \pi} f_{\mathcal{O}}(p, k)
$$



- power-law finite volume effects arise when internal states can go on-shell, e.g.

$$
\boldsymbol{k}=\frac{2 \pi \boldsymbol{n}+\boldsymbol{\theta}}{L}, \quad \alpha>0
$$

$$
\Delta \mathcal{O}(p, L)=\mathcal{O}(p, L)-\mathcal{O}(p, \infty)
$$



$$
=\left(\frac{1}{L^{3}} \sum_{k}-\int \frac{d^{3} k}{(2 \pi)^{3}}\right) \int \frac{d k^{0}}{2 \pi} f_{\mathcal{O}}(p, k)
$$

$$
=\left(\frac{1}{L^{3}} \sum_{\boldsymbol{k}}-\int \frac{d^{3} k}{(2 \pi)^{3}}\right)\left\{\frac{g_{\mathcal{O}}(p)+O(\boldsymbol{k})}{(\boldsymbol{k} \cdot \boldsymbol{p})^{\alpha}}\right\}
$$



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$$
\boldsymbol{k}=\frac{2 \pi \boldsymbol{n}+\boldsymbol{\theta}}{L}, \quad \alpha>0
$$

$$
\begin{aligned}
& \Delta \mathcal{O}(p, L)=\mathcal{O}(p, L)-\mathcal{O}(p, \infty) \\
& =\left(\frac{1}{L^{3}} \sum_{\boldsymbol{k}}-\int \frac{d^{3} k}{(2 \pi)^{3}}\right) \int \frac{d k^{0}}{2 \pi} f_{\mathcal{O}}(p, k) \\
& =\left(\frac{1}{L^{3}} \sum_{\boldsymbol{k}}-\int \frac{d^{3} k}{(2 \pi)^{3}}\right)\left\{\frac{g_{\mathcal{O}}(p)+O(\boldsymbol{k})}{(\boldsymbol{k} \cdot \boldsymbol{p})^{\alpha}}\right\} \\
& =\frac{g_{\mathcal{O}}(p) \xi(p, \boldsymbol{\theta})}{L^{3-\alpha}}+O\left(\frac{1}{L^{4-\alpha}}\right)
\end{aligned}
$$



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$$



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$$



$$
=\frac{g_{\mathcal{O}}(p) \xi(p, \boldsymbol{\theta})}{L^{3-\alpha}}+O\left(\frac{1}{L^{4-\alpha}}\right)
$$

$$
\begin{aligned}
& \left(\frac{1}{L^{3}} \sum_{k}-\int \frac{d^{3} k}{(2 \pi)^{3}}\right) \int \frac{d k^{0}}{2 \pi} \frac{1}{k^{\beta}} \\
& \sim O\left(\frac{1}{L^{4-\beta}}\right)
\end{aligned}
$$

$$
\xi(p, \boldsymbol{\theta})=\left\{\sum_{\boldsymbol{n}}-\int \frac{d^{3} n}{(2 \pi)^{3}}\right\} \frac{1}{(2 \pi \boldsymbol{n} \cdot \boldsymbol{p}+\boldsymbol{\theta} \cdot \boldsymbol{p})^{\alpha}}
$$



$$
\sim \frac{1}{2 p \cdot k+k^{2}}
$$

- the key point of our method is the universality of infrared divergences
- to see how this works, let's consider the contribution to the decay rate coming from the diagrams shown in the figure

$$
\Gamma_{V}^{P \ell}=\int \frac{d^{4} k}{(2 \pi)^{4}} H^{\alpha \mu}(k, p) \frac{1}{k^{2}} \frac{\mathcal{L}_{\alpha \mu}(k)}{2 p_{\ell} \cdot k+k^{2}}
$$

- infrared divergences (and power-law finite volume effects) come from the singularity at $k^{2}=0$ of the integrand
- the tensor $\mathcal{L}_{\alpha \mu}$ is a regular function, it contains the numerator of the lepton propagator and the appropriate normalization factors

$$
\mathcal{L}_{\alpha \mu}(k) \equiv \mathcal{L}_{\alpha \mu}\left(k, p_{\nu}, p_{\ell}\right)=O(1)
$$

- the hadronic tensor is a QCD quantity

$$
H^{\alpha \mu}(k, p)=i \int d^{4} x e^{i k \cdot x} T\langle 0| J_{W}^{\alpha}(0) j^{\mu}(x)|P\rangle
$$

- it satisfies the WIs coming from QED gauge invariance, e.g.

$$
k_{\mu} H^{\alpha \mu}(k, p)=-f_{P} p^{\alpha}
$$



- and, given the kinematics of the process, it is singular only at the single-meson pole
- the hadronic tensor is a QCD quantity

$$
H^{\alpha \mu}(k, p)=i \int d^{4} x e^{i k \cdot x} T\langle 0| J_{W}^{\alpha}(0) j^{\mu}(x)|P\rangle
$$



- it satisfies the WIs coming from QED gauge invariance, e.g.

$$
k_{\mu} H^{\alpha \mu}(k, p)=-f_{P} p^{\alpha}
$$



- and, given the kinematics of the process, it is singular only at the single-meson pole
- the singularity can be isolated by considering the point-like tensor, built in such a way to satisfy the same WIs of the full theory

$$
\begin{aligned}
& H_{p t}^{\alpha \mu}(k, p)=f_{P}\left\{\delta^{\alpha \mu}-\frac{(p+k)^{\alpha}(2 p+k)^{\mu}}{2 p \cdot k+k^{2}}\right\} \\
& H_{S D}^{\alpha \mu}(k, p)=H^{\alpha \mu}(k, p)-H_{p t}^{\alpha \mu}(k, p), \quad k_{\mu} H_{p t}^{\alpha \mu}(k, p)=-f_{P} p^{\alpha}, \quad k_{\mu} H_{S D}^{\alpha \mu}(k, p)=0
\end{aligned}
$$

－the hadronic tensor is a QCD quantity

$$
H^{\alpha \mu}(k, p)=i \int d^{4} x e^{i k \cdot x} T\langle 0| J_{W}^{\alpha}(0) j^{\mu}(x)|P\rangle
$$


－it satisfies the WIs coming from QED gauge invariance，e．g．

$$
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\end{aligned}
$$

－the structure dependent contributions are regular and，since there is no constant two－index tensor orthogonal to $k$ ，

$$
H_{S D}^{\alpha \mu}(k, p)=\left(p \cdot k \delta^{\alpha \mu}-k^{\alpha} p^{\mu}\right) F_{A}+\epsilon^{\alpha \mu \rho \sigma} p_{\rho} k_{\sigma} F_{V}+\cdots=O(k)
$$

- at $O\left(e^{2}\right)$ with massive charged particles, singularities arise only at

$$
k^{2}=( \pm i|\boldsymbol{k}|)^{2}+\boldsymbol{k}^{2}=0
$$


(a)

- the blobs on the right are QCD vertexes, e.g.

$$
\begin{aligned}
& \Delta(p+k) \Gamma^{\mu}(p, k) \Delta(p)= \\
& i N(p) \int d^{4} x d^{4} y e^{-i p y-i k x} T\langle 0| P(y) j^{\mu}(x) P^{\dagger}(0)|0\rangle
\end{aligned}
$$

$$
\Delta(p)=N(p) \int d^{4} y e^{-i p y} T\langle 0| P(y) P^{\dagger}(0)|0\rangle
$$

$$
\left.N^{-1}(p)=\left|\langle P(\boldsymbol{p})| P^{\dagger}(0)\right| 0\right\rangle\left.\right|^{2}
$$


(b)


(c)

(e)
gauge WIs constrain the first two terms in the expansion, e.g.

$$
\begin{aligned}
& k_{\mu} \Gamma^{\mu}(p, k)=\Delta^{-1}(p+k)-\Delta^{-1}(p) \\
& \Gamma^{\mu}(p, k)=2 p^{\mu}+k^{\mu}+O\left(k^{2}\right)
\end{aligned}
$$


(f)

(g)

- it is always a good idea to address the issue of analytical continuation by starting from correlators, it is usually more cumbersome to locate singularities in the amplitudes
- the reason is that correlators (Schwinger's functions) can always be Wick rotated without any problem
- euclidean reduction formulae work straightforwardly only for the lightest states, i.e. the leading exponentials appearing in the correlators, because the corresponding integrals are convergent
- problems arise when one is interested in processes corresponding to non-leading exponentials (notice that at finite $L$ the spectrum of $H$ is discrete)
- the first step in a lattice calculation of a new observable is to understand if the leading exponentials correspond to the external states for the process of interest
- the lightest state appearing in a correlator is readily found by using the quantum numbers of the theory (in p.t. by using the quantum numbers of the full theory)
in minkowsky time:

$$
\begin{aligned}
& C(t)=\mathrm{T}\langle 0| \cdots \bar{O}(t) O(0)|0\rangle \\
& =\langle 0| \cdots e^{-i t(H-i \epsilon)} O|0\rangle+\text { o.t.o. }
\end{aligned}
$$

$$
\mathcal{A}(E)=2 E\left(p^{0}-E\right) \int_{0}^{\infty} d t e^{i p^{0} t} C(t)+\text { o.t.o. }
$$

in euclidean time:

$$
C_{E}(\tau)=\langle 0| \cdots e^{-\tau H} O|0\rangle+\text { o.t.o. }
$$

$$
\mathcal{A}(E)=-2 i E\left(p^{0}-E\right) \int_{0}^{\infty} d \tau e^{p^{0} \tau} C_{E}(\tau)+\text { o.t.o. }
$$

from the spectral decomposition of correlators at $O(\alpha)$ one gets expressions that are rather involved but their structure is easy to understand and somehow illuminating

$$
\begin{aligned}
C(t) & =e^{-t E(\boldsymbol{p})} \int \frac{d^{4} q}{(2 \pi)^{4}} A^{v i r t}(q) \\
& +\int \frac{d^{3} q}{(2 \pi)^{3}} A^{\text {real }}(\boldsymbol{q}) e^{-t\left[E(\boldsymbol{p}-\boldsymbol{q})+E_{\gamma}(\boldsymbol{q})\right]} \\
& +\cdots
\end{aligned}
$$

when the spatial momentum $\boldsymbol{q}$ of the photon goes to zero wave

$$
\begin{aligned}
& |\boldsymbol{q}| \mapsto 0 \\
& E(\boldsymbol{p}-\boldsymbol{q})+E_{\gamma}(\boldsymbol{q}) \mapsto E(\boldsymbol{p}) \\
& A^{\text {virt }}(q) \mapsto c^{v i r t}-c_{I R} \log \frac{|\boldsymbol{q}|}{m} \\
& A^{\text {real }}(q) \mapsto c^{\text {real }}+c_{I R} \log \frac{|\boldsymbol{q}|}{m}
\end{aligned}
$$

for each charged particle emitting a photon one has the exponential corresponding to the charged particle itself as an external state (the virtual photon contribution)
but also the exponential corresponding to the external states with the photon on-shell (the real photon contribution)
since

$$
|\boldsymbol{q}|+\sqrt{M^{2}+|\boldsymbol{p}-\boldsymbol{q}|^{2}} \geq \sqrt{M^{2}+|\boldsymbol{p}|^{2}}
$$

with an infrared regulator the blue exponentials are sub-leading and, if one is interested in the virtual contribution, there is no problem of analytical continuation
in the case of the $O\left(e^{2}\right)$ QED radiative corrections to the leptonic decays of pseudoscalar mesons

since as we have seen

$$
|\boldsymbol{q}|+\sqrt{M^{2}+|\boldsymbol{p}-\boldsymbol{q}|^{2}} \geq \sqrt{M^{2}+|\boldsymbol{p}|^{2}}
$$

here there is a problem of analytical continuation! but this diagram can be factorized and the leptonic part can be computed analytically

at fixed total momentum and with an infrared regulator the pseudoscalar meson is the lightest state in QED+QCD with the given quantum numbers
therefore, no problems of analytical continuation arise in the self-energy diagrams and in the diagram in which the real photon is emitted from the meson!
notice that this is true for a pion but also in the case of flavoured pseudoscalar mesons such as $K, B, D$ !

- problems of analytical continuation do arise in the case of semileptonic decays because of electromagnetic final state interactions
- the internal meson-lepton pair, and eventually multi-hadrons-lepton internal states, can be lighter than the external meson-lepton state
- this is a big issue, particularly in the case of $B$ decays because of the presence of many kinematically-allowed multi-hadron states

- the starting point is the hadronic tensor $\left(p^{2}=m_{P}^{2}\right)$

$$
H^{\mu \alpha}(k, p)=\int d^{4} y e^{i k \cdot y} \mathrm{~T}\langle 0| j_{W}^{\alpha}(0) j_{e m}^{\mu}(y)|P(p)\rangle
$$

- this can be conveniently decomposed in terms of form-factors as follows

$$
\begin{aligned}
H^{\mu \alpha}(k, p)= & H_{S D}^{\mu \alpha}(k, p)+H_{p t}^{\mu \alpha}(k, p) \\
H_{S D}^{\mu \alpha}(k, p)= & H_{1}\left[k^{2} g^{\mu \alpha}-k^{\mu} k^{\alpha}\right]+H_{2}\left[\left(p \cdot k-k^{2}\right) k^{\mu}-k^{2}(p-k)^{\mu}\right](p-k)^{\alpha} \\
& -i \frac{F_{V}}{m_{P}} \varepsilon^{\mu \alpha \gamma \beta} k_{\gamma} p_{\beta}+\frac{F_{A}}{m_{P}}\left[\left(p \cdot k-k^{2}\right) g^{\mu \alpha}-(p-k)^{\mu} k^{\alpha}\right] \\
H_{p t}^{\mu \alpha}(k, p)= & f_{P}\left[g^{\mu \alpha}+\frac{(2 p-k)^{\mu}(p-k)^{\alpha}}{2 p \cdot k-k^{2}}\right]
\end{aligned}
$$

- the choice of the basis is of course not unique and, moreover, the separation of the point-like contribution can also depend upon the conventions: our definition of $H_{p t}^{\mu \alpha}(k, p)$ is consistent with the point-like effective lagrangian and it is what we used to compute $\Gamma_{R}^{p t}(E)$; notice that

$$
k_{\mu} H^{\mu \alpha}(k, p)=f_{P} p^{\alpha}, \quad k_{\mu} H_{p t}^{\mu \alpha}(k, p)=f_{P} p^{\alpha}, \quad k_{\mu} H_{S D}^{\mu \alpha}(k, p)=0
$$

i.e. $H_{p t}^{\mu \alpha}(k, p)$ satisfies the same ward identity of the full-theory tensor

- in the case of real photons, $k^{2}=0$, the previous expressions simplify as follows

$$
\begin{aligned}
H^{\mu \alpha}(k, p)= & H_{S D}^{\mu \alpha}(k, p)+H_{p t}^{\mu \alpha}(k, p) \\
H_{S D}^{\mu \alpha}(k, p)= & k^{\mu}\left\{-H_{1} k^{\alpha}+H_{2} p \cdot k(p-k)^{\alpha}\right\} \\
& -i \frac{F_{V}}{m_{P}} \varepsilon^{\mu \alpha \gamma \beta} k_{\gamma} p_{\beta}+\frac{F_{A}}{m_{P}}\left[p \cdot k g^{\mu \alpha}-(p-k)^{\mu} k^{\alpha}\right] \\
H_{p t}^{\mu \alpha}(k, p)= & f_{P}\left[g^{\mu \alpha}+\frac{(2 p-k)^{\mu}(p-k)^{\alpha}}{2 p \cdot k}\right]
\end{aligned}
$$

- the form factors $H_{1,2}$ do not enter into the physical decay rate for $P \mapsto \ell \bar{\nu} \gamma$ and can be conveniently separated by considering the projector onto the transverse (and therefore physical) degrees of freedom of the photon that is attached to the vector current

$$
n=(1, \mathbf{0}), \quad P^{\mu \nu}(k, n)=-g^{\mu \nu}+n^{\mu} n^{\nu}+\frac{\left[k^{\mu}-n \cdot k n^{\mu}\right]\left[k^{\nu}-n \cdot k n^{\nu}\right]}{k^{2}-(n \cdot k)^{2}}
$$

- the projector $P^{\mu \nu}(k, n)$ is such that

$$
\begin{array}{ll}
P^{\mu \nu}(k, n) k_{\nu}=P^{\mu \nu}(k, n) n_{\nu}=0, & P^{\mu \beta}(k, n) P_{\beta}^{\nu}(k, n)=P^{\mu \nu}(k, n) \\
P^{\mu \nu}(k, n)=P^{\nu \mu}(k, n), & P^{00}(k, n)=P^{0 i}(k, n)=0 \\
P^{i j}(k, n)=\delta^{i j}-\frac{k^{i} k^{j}}{k^{2}}
\end{array}
$$

- in fact $P^{\mu \nu}(k, n)$ is nothing but the numerator of the photon propagator in the Coulomb's gauge that forbids the propagation of unphysical degrees of freedom; we have

$$
P_{\nu \mu}(k, n) H_{S D}^{\mu \alpha}(k, p)=P_{\nu \mu}(k, n)\left\{-i \frac{F_{V}}{m_{P}} \varepsilon^{\mu \alpha \gamma \beta} k_{\gamma} p_{\beta}+\frac{F_{A}}{m_{P}}\left[p \cdot k g^{\mu \alpha}-(p-k)^{\mu} k^{\alpha}\right]\right\}
$$

- by introducing the polarization vectors as follows (that depend upon $n$ and $k$ )

$$
\begin{aligned}
& \epsilon_{0}=n=(1, \mathbf{0}), \quad \epsilon_{1,2}=\left(0, \boldsymbol{\epsilon}_{1,2}\right), \quad \epsilon_{3}=(0, \boldsymbol{k} /|\boldsymbol{k}|) \\
& \epsilon_{r} \cdot \epsilon_{s}=g_{r s}, \quad g^{r s} \epsilon_{r}^{\mu} \epsilon_{s}^{\nu}=g^{\mu \nu}
\end{aligned}
$$

- the projector $P^{\mu \nu}(k, n)$ can be rewritten in terms of the transverse polarization vectors $\epsilon_{1,2}$ as follows

$$
\sum_{r=1,2} \epsilon_{r}^{\mu} \epsilon_{r}^{\nu}=P^{\mu \nu}(k, n), \quad \epsilon_{r, \mu} P^{\mu \nu}(k, n)=-\epsilon_{r}^{\nu}, \quad r=1,2
$$

- explicit expressions for the transverse polarization vectors are given below

$$
\begin{aligned}
& \epsilon_{1}^{\mu}(\boldsymbol{k})=\left(0, \frac{-k_{1} k_{3}}{|\boldsymbol{k}| \sqrt{k_{1}^{2}+k_{2}^{2}}}, \frac{-k_{2} k_{3}}{|\boldsymbol{k}| \sqrt{k_{1}^{2}+k_{2}^{2}}}, \frac{\sqrt{k_{1}^{2}+k_{2}^{2}}}{|\boldsymbol{k}|}\right), \\
& \epsilon_{2}^{\mu}(\boldsymbol{k})=\left(0, \frac{k_{2}}{\sqrt{k_{1}^{2}+k_{2}^{2}}},-\frac{k_{1}}{\sqrt{k_{1}^{2}+k_{2}^{2}}}, 0\right)
\end{aligned}
$$

- in light of the previous discussion, one can either use the (formally) covariant expressions given above for $P^{\mu \nu}$ ( $k, n$ ) or the explicit expressions for the transverse polarizations $\epsilon_{1,2}$ in order to isolate the physical contributions appearing into $H^{\mu \alpha}(k, p)$
- in particular, since the axial and vector part of the weak current can be computed separately, we have

$$
\begin{aligned}
& \epsilon_{r, \mu} H_{A}^{\mu \alpha}(k, p)=\frac{p \cdot k \epsilon_{r}^{\alpha}-\epsilon_{r} \cdot p k^{\alpha}}{m_{P}}\left\{F_{A}+\frac{m_{P} f_{P}}{p \cdot k}\right\}+p^{\alpha} \epsilon_{r} \cdot p \frac{f_{P}}{p \cdot k} \\
& \epsilon_{r, \mu} H_{V}^{\mu \alpha}(k, p)=i \frac{F_{V}}{m_{P}} \varepsilon^{\alpha \mu \gamma \beta} \epsilon_{r, \mu} k_{\gamma} p_{\beta} \\
& r=1,2
\end{aligned}
$$

- the infrared problem has been analyzed by many authors over the years
- electrically-charged asymptotic states are not eigenstates of the photon-number operator
- the perturbative expansion of decay-rates and cross-sections with respect to $\alpha$ is cumbersome because of the infinitely many degenerate states
- the block \& nordsieck approach consists in lifting the degeneracies by introducing an infrared regulator, say $m_{\gamma}$, and in computing infrared-safe observables
- at any fixed order in $\alpha$, infrared-safe observables are obtained by adding the appropriate number of photons in the final states and by integrating over their energy in a finite range, say $[0, E]$
- in this framework, infrared divergences appear at intermediate stages of the calculations and cancel in the sum of the so-called virtual and real contributions

$(p+k)^{2}+m_{P}^{2}=2 p \cdot k+k^{2} \sim 2 p \cdot k$,
$\int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{\left(k^{2}+m_{\gamma}^{2}\right)(2 p \cdot k)\left(2 p_{\ell} \cdot k\right)} \sim c_{I R} \log \left(\frac{m_{P}}{m_{\gamma}}\right)$,
$c_{I R}\left\{\log \left(\frac{m_{P}}{m_{\gamma}}\right)+\log \left(\frac{m_{\gamma}}{E}\right)\right\}=c_{I R} \log \left(\frac{m_{P}}{E}\right)$
- infrared divergences can be computed in the so called point-like effective theory

$$
\mathcal{L}_{p t}=\phi_{P}^{\dagger}\left\{-D_{\mu}^{2}+m_{P}^{2}\right\} \phi_{P}+f_{P}\left\{2 i G_{F} V_{C K M} D_{\mu} \phi_{P}^{\dagger} \bar{\ell}_{\gamma}^{\mu} \nu+\text { h.c. }\right\}, \quad D_{\mu}=\partial_{\mu}-i e A_{\mu}
$$

- properly matched effective field theories have, by definition, the same infrared behaviour of the fundamental theory: at leading order the matching is obtained by using $\Gamma_{0}$

$$
\Gamma_{0}^{p t}=\Gamma_{0}=\frac{G_{F}^{2}\left|V_{C K M}\right|^{2} f_{P}^{2}}{8 \pi} m_{P}^{3} r_{\ell}^{2}\left(1-r_{\ell}^{2}\right)^{2}, \quad r_{\ell}=\frac{m_{\ell}}{m_{P}}, \quad D_{\mu} \mapsto \partial_{\mu}
$$

- infrared divergences can be computed in the so called point-like effective theory

$$
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$$

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$$
\Gamma_{0}^{p t}=\Gamma_{0}=\frac{G_{F}^{2}\left|V_{C K M}\right|^{2} f_{P}^{2}}{8 \pi} m_{P}^{3} r_{\ell}^{2}\left(1-r_{\ell}^{2}\right)^{2}, \quad r_{\ell}=\frac{m_{\ell}}{m_{P}}, \quad D_{\mu} \mapsto \partial_{\mu}
$$

- structure-dependent terms can also be understood in the effective field theory language, e.g.

$$
\mathcal{O}_{V}(x)=F_{V} \epsilon^{\mu \nu \rho \sigma} D_{\mu} \phi_{P} F_{\nu \rho} \bar{\ell} \gamma_{\sigma} \nu, \quad F_{\nu \rho}=\partial_{\nu} A_{\rho}-\partial_{\rho} A_{\nu}, \quad \text { subleading in } \quad \frac{E_{\gamma}}{m_{\pi}}
$$

- by exploiting the full set of constraints coming from the WIs and from the e.o.m one can rigorously show that in the expansion around vanishing photon energies both the leading (infrared divergent) and the next-to-leading terms are universal: this implies that $O\left(L^{-1}\right)$ finite volume effects are universal
- concerning the perturbative point-like calculation in infinite volume, we have generalized the results obtained in the early days of quantum field theory by berman 58, kinoshita 59

$$
\begin{aligned}
\Gamma^{p t}(E)=e^{2} & \lim _{\gamma \rightarrow \infty}\left\{\Gamma_{V}^{p t}\left(m_{\gamma}\right)+\Gamma_{R}^{p t}\left(m_{\gamma}, E\right)\right\} \\
= & \Gamma_{0} \frac{\alpha_{e m}}{4 \pi}
\end{aligned} \begin{aligned}
& \left\{3 \log \left(\frac{m_{P}^{2}}{m_{W}^{2}}\right)+\log \left(r_{\ell}^{2}\right)-4 \log \left(r_{E}^{2}\right)+\frac{2-10 r_{\ell}^{2}}{1-r_{\ell}^{2}} \log \left(r_{\ell}^{2}\right)\right. \\
& -2 \frac{1+r_{\ell}^{2}}{1-r_{\ell}^{2}} \log \left(r_{E}^{2}\right) \log \left(r_{\ell}^{2}\right)-4 \frac{1+r_{\ell}^{2}}{1-r_{\ell}^{2}} \operatorname{Li}_{2}\left(1-r_{\ell}^{2}\right)-3 \\
& +\frac{3+r_{E}^{2}-6 r_{\ell}^{2}+4 r_{E}\left(-1+r_{\ell}^{2}\right)}{\left(1-r_{\ell}^{2}\right)^{2}} \log \left(1-r_{E}\right)+\frac{r_{E}\left(4-r_{E}-4 r_{\ell}^{2}\right)}{\left(1-r_{\ell}^{2}\right)^{2}} \log \left(r_{\ell}^{2}\right) \\
& \left.-\frac{r_{E}\left(-22+3 r_{E}+28 r_{\ell}^{2}\right)}{2\left(1-r_{\ell}^{2}\right)^{2}}-4 \frac{1+r_{\ell}^{2}}{1-r_{\ell}^{2}} \operatorname{Li}_{2}\left(r_{E}\right)\right\}
\end{aligned}
$$

where

$$
r_{E}=\frac{2 E}{m_{P}}, \quad r_{\ell}=\frac{m_{\ell}}{m_{P}}
$$

- notice that $\Gamma_{V}(L)$ and $\Gamma_{V}^{p t}(L)$ are ultraviolet divergent in the Fermi theory
- the divergence can be reabsorbed into a renormalization of $G_{F}$, both in the full theory and in the point-like effective theory
- we have analyzed the renormalization of the four-fermion weak operator on the lattice in details and calculated non-perturbatively the renormalization constants in the RI-MOM scheme
- we have then matched the non-perturbative results to the so-called W-regularization at $O$ ( $\alpha$ ) (a.sirlin, NPB 196 (1982); e.braaten and c.s.li PRD 42 (1990))

$$
\begin{aligned}
& \frac{1}{k^{2}} \mapsto \frac{1}{k^{2}}-\frac{1}{k^{2}+m_{W}^{2}}, \quad H_{W}=\frac{G_{F} V_{C K M}}{\sqrt{2}}\left\{1+\frac{\alpha}{\pi} \log \frac{m_{Z}}{m_{W}}\right\} O_{1}^{\mathrm{W}-\mathrm{reg}} \\
& O_{1}^{\mathrm{W}-\mathrm{reg}}=\sum_{i=1}^{5} Z_{1 i} O_{i}^{l a t t}(a)
\end{aligned}
$$

- indeed, this is the scheme conventionally used to extract $G_{F}$ from the muon decay

$$
\frac{1}{\tau_{\mu}}=\frac{G_{F}^{2} m_{\mu}^{5}}{192 \pi^{3}}\left[1-\frac{8 m_{e}^{2}}{m_{\mu}^{2}}\right]\left[1+\frac{\alpha}{2 \pi}\left(\frac{25}{4}-\pi^{2}\right)\right]
$$

- we performed an analytical calculation of $\Gamma_{V}^{p t}(L)$

RM123+SOTON, PRD 95 (2017), arXiv:1612.00199

$$
\frac{\Gamma_{V}^{p t}(L)-\Gamma_{V}^{\ell \ell}(L)}{\Gamma_{0}}=c_{I R} \log \left(L^{2} m_{P}^{2}\right)+c_{0}+\frac{c_{1}}{\left(m_{P} L\right)}+O\left(\frac{1}{L^{2}}\right)
$$

where

$$
\begin{aligned}
& c_{I R}=\frac{1}{8 \pi^{2}}\left\{\frac{\left(1+r_{\ell}^{2}\right) \log \left(r_{\ell}^{2}\right)}{\left(1-r_{\ell}^{2}\right)}+1\right\} \\
& c_{0}=\frac{1}{16 \pi^{2}}\left\{2 \log \left(\frac{m_{P}^{2}}{m_{W}^{2}}\right)+\frac{\left(2-6 r_{\ell}^{2}\right) \log \left(r_{\ell}^{2}\right)+\left(1+r_{\ell}^{2}\right) \log ^{2}\left(r_{\ell}^{2}\right)}{1-r_{\ell}^{2}}-\frac{5}{2}\right\}+\frac{\zeta_{C}(\mathbf{0})-2 \zeta_{C}\left(\boldsymbol{\beta}_{\ell}\right)}{2} \\
& c_{1}=-\frac{2\left(1+r_{\ell}^{2}\right)}{1-r_{\ell}^{2}} \zeta_{B}(\mathbf{0})+\frac{8 r_{\ell}^{2}}{1-r_{\ell}^{4}} \zeta_{B}\left(\boldsymbol{\beta}_{\ell}\right)
\end{aligned}
$$

and we have shown that $c_{I R}, c_{0}$ and $c_{1}$ are universal, i.e. they are the same in the point-like and in the full theories! this means that in $\Gamma_{V}^{S D}(L)=\Gamma_{V}(L)-\Gamma_{V}^{p t}(L)$ we subtract exactly, together with the infrared divergence, the leading $O(1 / L)$ terms and we have $O\left(1 / L^{2}\right)$ finite size effects

- notice: the lepton wave-function contribution, $\Gamma_{V}^{\ell \ell}(L)$, does not contribute to $\Gamma_{V}^{S D}(L)$

- it is impossible to have a net electric charge in a periodic box
- this is a consequence of gauss's law

$$
S=\int_{L^{3}} d^{4} x\left\{\frac{1}{4} F_{\mu \nu} F_{\mu \nu}+\bar{\psi}_{f}\left(\gamma_{\mu} D_{\mu}^{f}+m_{f}\right) \psi_{f}\right\}
$$

$$
\partial_{k} \underbrace{F_{0 k}(x)}_{E_{k}(x)}-\underbrace{i e q_{f} \bar{\psi}_{f} \gamma_{0} \psi_{f}(x)}_{e \rho(x)}=0
$$

$$
Q=\int_{L^{3}} d^{3} x \rho(x)=\frac{1}{e} \int_{L^{3}} d^{3} x \partial_{k} E_{k}(x)=0
$$



- one may think to overcome this problem by gauge fixing but large gauge transformations survive a local gauge fixing procedure $\left(n \in \mathbb{Z}^{4}\right)$

$$
\begin{array}{ll}
\psi(x) \mapsto e^{2 \pi i \sum_{\mu} \frac{x_{\mu} n_{\mu}}{L_{\mu}}} \psi(x), & A_{\mu}(x) \mapsto A_{\mu}(x)+\frac{2 \pi n_{\mu}}{L_{\mu}} \\
\psi(x) \bar{\psi}(0) \mapsto e^{2 \pi i \sum_{\mu} \frac{x_{\mu} n_{\mu}}{L_{\mu}}} \psi(x) \bar{\psi}(0), & \langle\psi(x) \bar{\psi}(0)\rangle=0, \quad x \neq 0
\end{array}
$$

- in order to study charged particles in a periodic box it has been suggested long ago (duncan et al. 96) to quench (a set of) the zero momentum modes of the gauge field, for example

$$
\langle\mathcal{O}\rangle=\int_{\text {pbc in space }} \mathcal{D} \psi \mathcal{D} \bar{\psi} \mathcal{D} A_{\mu} \prod_{\mu} \delta\left\{\int_{T L^{3}} d^{4} x A_{\mu}(x)\right\} e^{-S} \mathcal{O}
$$

- by using this procedure one is also quenching large gauge transformations that are no longer a symmetry and charged particles can propagate
- the assumption is that the induced modifications on the infrared dynamics of the theory should disappear once the infinite volume limit is taken

- the point to note is that the resulting finite volume theory, although it may admit an hamiltonian description, is non-local
m.hayakawa, s.uno Prog.Theor.Phys. 120 (2008) BMW, Science 347 (2015), Phys.Lett. B755 (2016) z.davoudi, m.j.savage PRD90 (2014)

$$
\mathrm{QED}_{\mathrm{L}}: \quad \prod_{\mu, t} \delta\left\{\int_{L^{3}} d^{3} x A_{\mu}(t, \boldsymbol{x})\right\} \quad \mapsto \quad \int_{\mathrm{pbc} \text { in space }} \mathcal{D} \alpha_{\mu}(t) e^{-\int_{L} 3^{4} x \alpha_{\mu}(t) A_{\mu}(t, \boldsymbol{x})}
$$

b．lucini，a．patella，a．ramos，n．t，JHEP 1602（2016）
－consider $C^{\star}$ boundary conditions（first suggested by wise and polley 91 ）

$$
\begin{aligned}
& \psi_{f}(x+L \boldsymbol{k})=C^{-1} \bar{\psi}_{f}^{T}(x) \\
& \bar{\psi}_{f}(x+L \boldsymbol{k})=-\psi_{f}^{T}(x) C \\
& A_{\mu}(x+L \boldsymbol{k})=-A_{\mu}(x), \quad U_{\mu}(x+L \boldsymbol{k})=U_{\mu}^{*}(x)
\end{aligned}
$$

－the gauge field is anti－periodic $(|\boldsymbol{p}| \geq \pi / L)$ ：no zero modes by construction！
－this means no large gauge transformations and

$$
Q=\int_{L^{3}} d^{3} x \rho(x)=\frac{1}{e} \int_{L^{3}} d^{3} x \partial_{k} E_{k}(x) \neq 0
$$


－a fully gauge invariant formulation is possible：technically this is a consequence of the fact that the electrostatic potential is unique with anti－periodic boundary conditions（see backup）

- electrically charged states can be probed by considering (Dirac's factor)

$$
\Psi_{f}(t, \boldsymbol{x})=\underbrace{e^{-i q_{f} \int d^{3} y \Phi(\boldsymbol{y}-\boldsymbol{x}) \partial_{k} A_{k}(t, \boldsymbol{y})}}_{\Theta(t, \boldsymbol{x})} \psi_{f}(t, \boldsymbol{x}), \quad \partial_{k} \partial_{k} \Phi(\boldsymbol{x})=\delta^{3}(\boldsymbol{x})
$$

- these interpolating operators are invariant under $U(1)$ local gauge transformations

$$
\begin{aligned}
& \psi_{f}(x) \mapsto e^{i q_{f} \alpha(x)} \psi_{f}(x), \quad A_{\mu}(x) \mapsto A_{\mu}(x)+\partial_{\mu} \alpha(x) \\
& \Theta(t, \boldsymbol{x}) \mapsto e^{-i q_{f} \int d^{3} y \Phi(\boldsymbol{y}-\boldsymbol{x}) \partial_{k} \partial_{k} \alpha(t, \boldsymbol{y})} \Theta(t, \boldsymbol{x})=e^{-i q_{f} \alpha(t, \boldsymbol{x})} \Theta(t, \boldsymbol{x})
\end{aligned}
$$

- the gauge factor is not unique, for example one can consider

$$
\Psi_{f}(t, \boldsymbol{x})=e^{-i q_{f} \int_{-\infty}^{x_{1}} d y A_{1}\left(t, y, x_{2}, x_{3}\right)} \psi_{f}(t, \boldsymbol{x})
$$

- for any consistent gauge-fixing condition one can build the Dirac factor that provides the unique gauge-invariant extension of matter fields in that gauge
- notice though: interpolating operators can be non-local in space but must be localized in time!
- $\mathrm{QED}_{\infty}$ : at any fixed order in $\alpha$ radiative corrections can be represented as the convolution of hadronic correlators with QED kernels, e.g.
x.feng et al PRD 100 (2019), LATTICE19

$$
\begin{aligned}
& O(L)=\int_{L^{3}} d^{4} x H_{Q C D}^{L}(x) D_{\gamma}^{L}(x) \\
& \mapsto \int d^{4} x H_{Q C D}(x) D_{\gamma}(x)
\end{aligned}
$$

the subtle issue here is the parametrization of the long-distance tails of the hadronic part;
in fact the proposal is an extension of the spectacular applications of the convolution approach to the $g_{\mu}-2$,


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- which is the best approach?
- QED $\infty$ : at any fixed order in $\alpha$ radiative corrections can be represented as the convolution of hadronic correlators with QED kernels, e.g.
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- $\mathrm{QED}_{\mathrm{L}}$ : very attractive for its formal simplicity; generally, at $O(\alpha)$ the systematics associated with non-localities can be understood
- $\mathrm{QED}_{\mathrm{m}}$ : formally, the simplest way to solve the problem in a local framework is to give a mass to the photon; the $L \mapsto \infty$ limit must be taken before restoring gauge invariance $\left(m_{\gamma} \mapsto 0\right)$
m.endres et al. PRL 117 (2016)

QED $_{\mathrm{C}}$ a local and fully gauge invariant solution, formally a bit cumbersome, flavour symmetries reduced to discrete subgroups (no spurious operator mixings though) and fully recovered in the infinite volume limit

$$
\begin{aligned}
& O(L)=\int_{L^{3}} d^{4} x H_{Q C D}^{L}(x) D_{\gamma}^{L}(x) \\
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- which is the best approach?
- in my opinion this is not the relevant point: what really matters is that one must be able to estimate reliably the systematic uncertainties associated with the chosen approach!


[^0]:    - a detailed analysis of these contributions is currently underway

