The plaquette with exponential accuracy from lattice

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Nonperturbative OPE (Novikov et al.)

$$\text{Observable}(\frac{Q}{\Lambda_{\text{QCD}}}) = S_{\text{pert}}(\alpha_X(Q)) + \sum_d C_{O,d}(\alpha_X(Q))\frac{\langle O_d\rangle}{Q^d}\,.$$

What is $S_{\text{pert}}(\alpha_X(Q))$?
What is $C_{O,d}(\alpha_X(Q))\frac{\langle O_d\rangle}{Q^d}$?

$$\text{Observable}(\frac{Q}{\Lambda_{\text{QCD}}}) =$$

$$= \sum_{n=0}^{\infty} p_n^{(X)}\alpha_X^{n+1}(Q) + \left(K + \sum_{n=0}^{\infty} p_n^{(X,d)}\alpha_X^{n+1}(Q)\right)\alpha_X^{\gamma}(Q)\frac{\Lambda_X^d}{Q^d} + \cdots$$

$$= \sum_{n=0}^{\infty} p_n^{(X)}\alpha_X^{n+1}(Q) + \left(K' + \sum_{n=0}^{\infty} p_n^{'(X,d)}\alpha_X^{n+1}(Q)\right)\alpha_X^{\gamma-db}(Q)e^{-d\frac{2\pi}{\beta_0\alpha_X(Q)}} + \cdots$$

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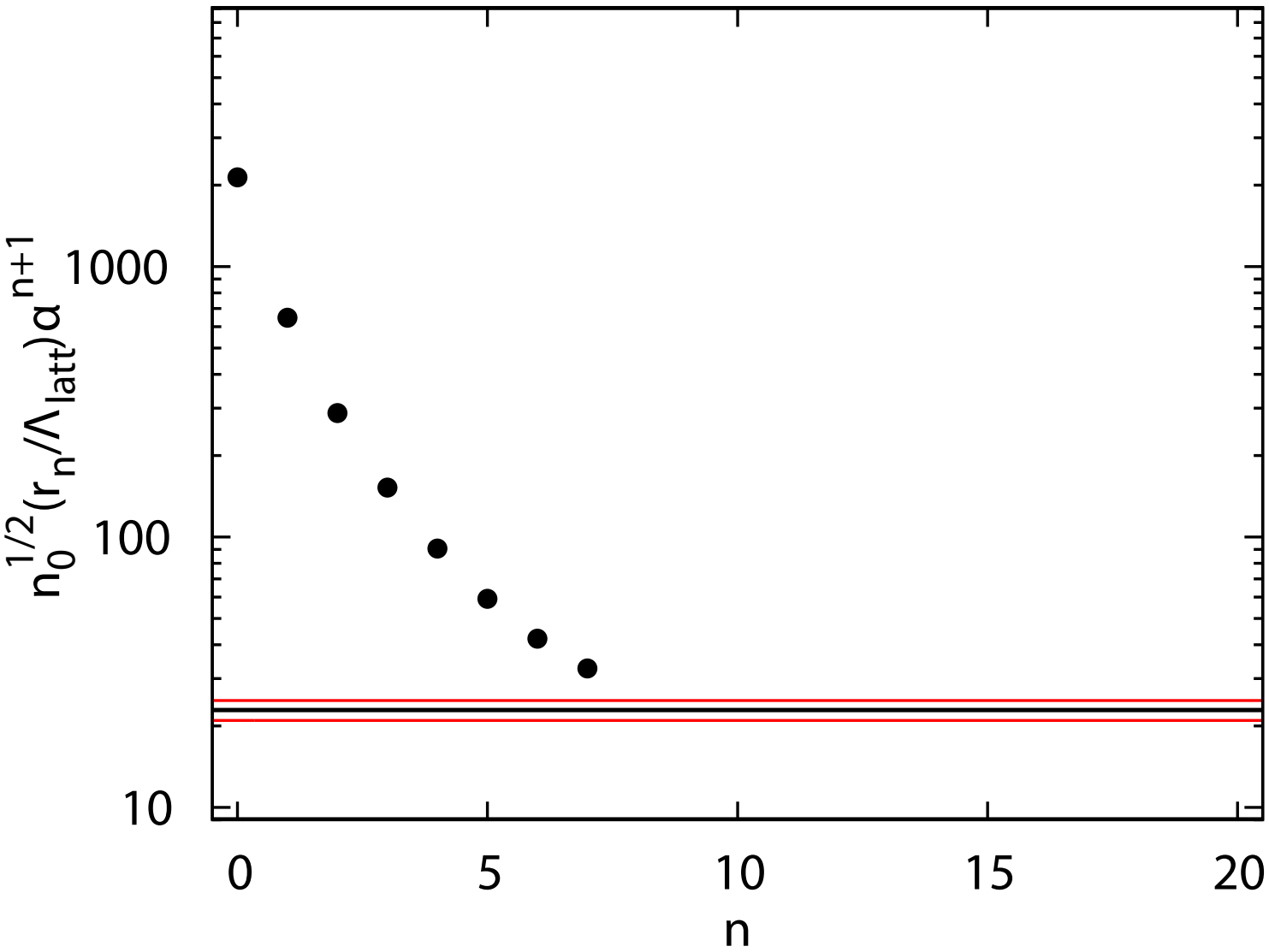

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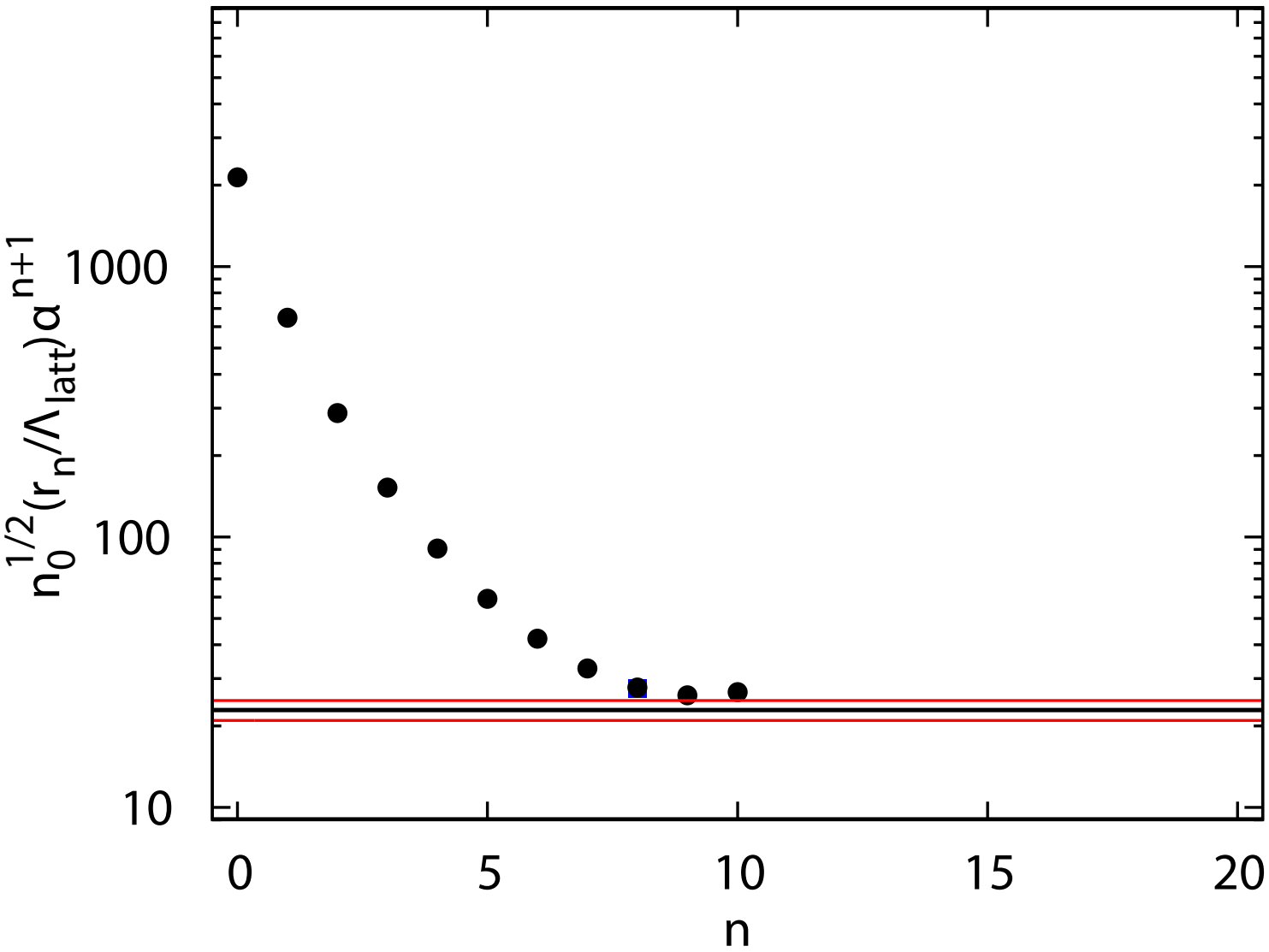

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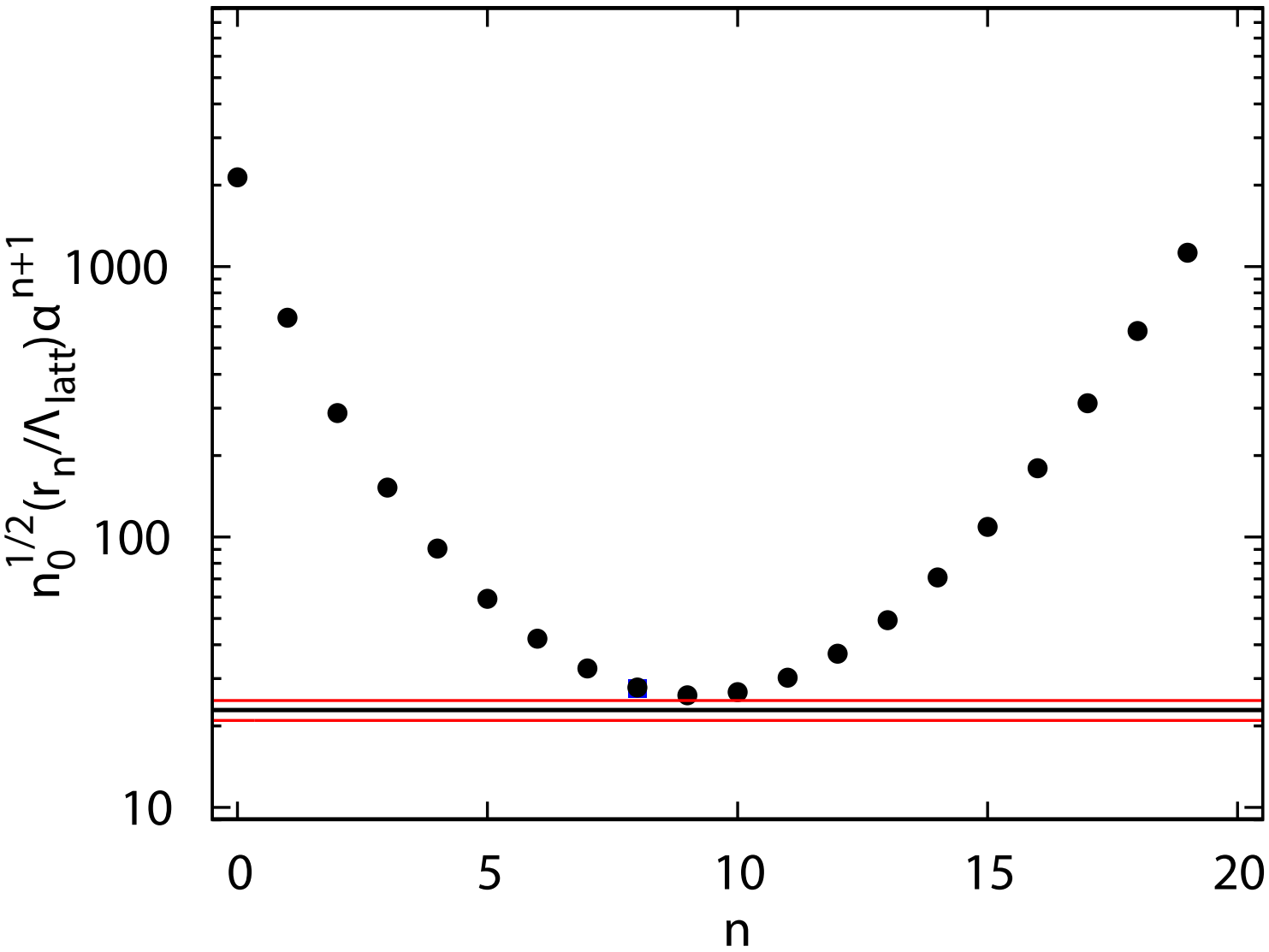

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One possible way out: Organize the computation using superasymptotic and hyperasymptotic approximations (ODE: Berry and Howls)

It allows for a parametric control of the error.

$$\text{Observable}(\frac{Q}{\Lambda_{\text{QCD}}}) - \sum_{n=0}^{N} p_n^{(X)}(\frac{\mu}{Q})\alpha_X^{n+1}(\mu) \sim \mathcal{O}(\alpha_X^{N+2})$$

but with large coefficient!!

Truncate the sum at the minimal term $\rightarrow$ superasymptotic approximation:

$$N = N_P \equiv |d| \frac{2\pi}{\beta_0 \alpha_X(\mu)} (1 - c\,\alpha_X(\mu)) \,,$$

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Connection with EFT's=Factorization

Two scales: $Q \gg \Lambda_{\rm QCD}$

$$\langle O \rangle = \sum_{n=0}^{N_P} p_n^X \alpha_X^{n+1}(\mu) + \# \frac{\mu^d}{Q^d} e^{-d\frac{2\pi}{\beta_0 \alpha_X(\mu)}} + o\left(e^{-\frac{\#}{\alpha_X(\mu)}}\right)$$

p_n: n-loop $\rightarrow Qe^{-n} \Rightarrow Qe^{-N_P} \sim \Lambda_{\rm QCD}$

Lattice:

$$\frac{1}{a} \quad \& \quad \frac{1}{N_L a} \quad \& \quad \frac{1}{a} e^{-\frac{2\pi}{\beta_0 \alpha(1/a)}}$$

All these scales are potentially α modulated.

Numerical Stochastic Perturbation Theory (NSPT) allows to go to high orders in perturbation theory (Di Renzo et al.).

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Plaquette (Bali, Bauer, AP: 1401.7999, 1403.6477)

$$\langle P\rangle_{\rm MC} = \frac{1}{N_L^4}\sum_{x\in\Lambda_E}\langle P_x\rangle = \frac{1}{Z}\int[dU_{x,\mu}]\,e^{-S[U]}P[U]\Big|_{\rm MC}\,,$$

where Λ_E is a Euclidean spacetime lattice and

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$$\frac{1}{a} \gg \frac{1}{N_L a} \to \langle P\rangle_{\rm pert}(N_L) = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36}C_{\rm G}(\alpha)\,a^4\langle G^2\rangle_{\rm soft} + \mathcal{O}\left(\frac{1}{N_L^6}\right)\,,$$

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$C_G(\alpha) \to \beta$-function (Di Giacomo et al.)

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$f_n^{(0)} = f_n$. The coefficients $f_n^{(i)}$ for $i > 0$ are determined by f_m and β_j, ...

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"Physical interpretation"

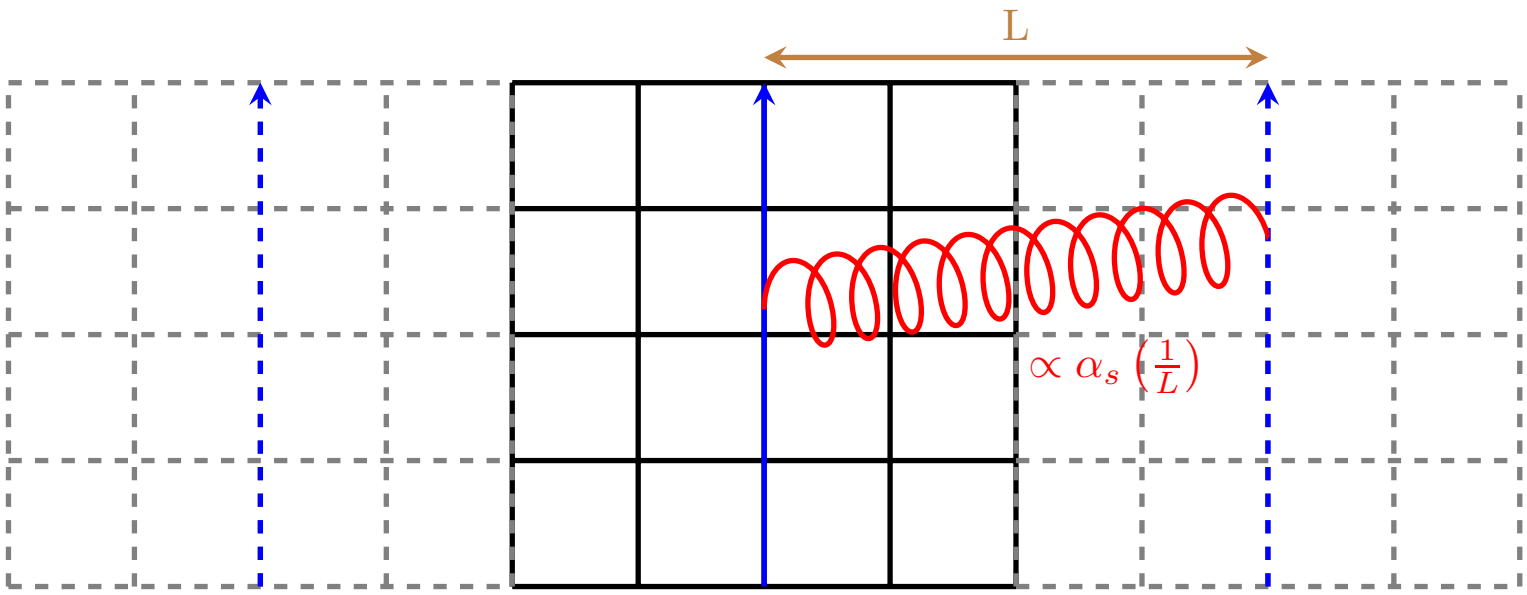

Figure: *Self-interactions with replicas producing* $1/L = 1/(aN_S)$ *Coulomb terms.*

$$\delta m^{(R)}(N_S) \propto \int_{1/(aN_S)}^{1/a} dk\, \alpha(k) \sim \frac{1}{a}\sum_n c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_n c_n \alpha^{n+1}\left((aN_S)^{-1}\right) ,$$

$$c_n \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n n! , \qquad f_n^{(i)}(N_S) \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{n!}{i!} .$$

$$\langle P \rangle = \sum_{n=0}^{N} p_n \alpha^{n+1}(a^{-1}) + a^4 \frac{\pi^2}{36} \langle G^2 \rangle + \cdots$$

$$d = 1 (n_0 \sim 7) \longrightarrow d = 4 (n_0 \sim 28)$$

$$N + 1 = 35$$

(before Di Renzo et al. N+1=8; Horsley et al. N+1=20)
Renormalon expectations:

$$p_n^{\rm latt} \stackrel{n\to\infty}{=} N_P^{\rm latt} \left(\frac{\beta_0}{2\pi d}\right)^n \frac{\Gamma(n+1+db)}{\Gamma(1+db)} \left\{1 + \frac{20.09}{n+db} + \frac{505 \pm 33}{(n+db)^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right\}.$$

$$\frac{p_n}{n p_{n-1}} = \frac{\beta_0}{2\pi d} \left\{1 + \frac{db}{n} + \frac{db(1-ds_1)}{n^2} + \frac{\#}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right\}.$$

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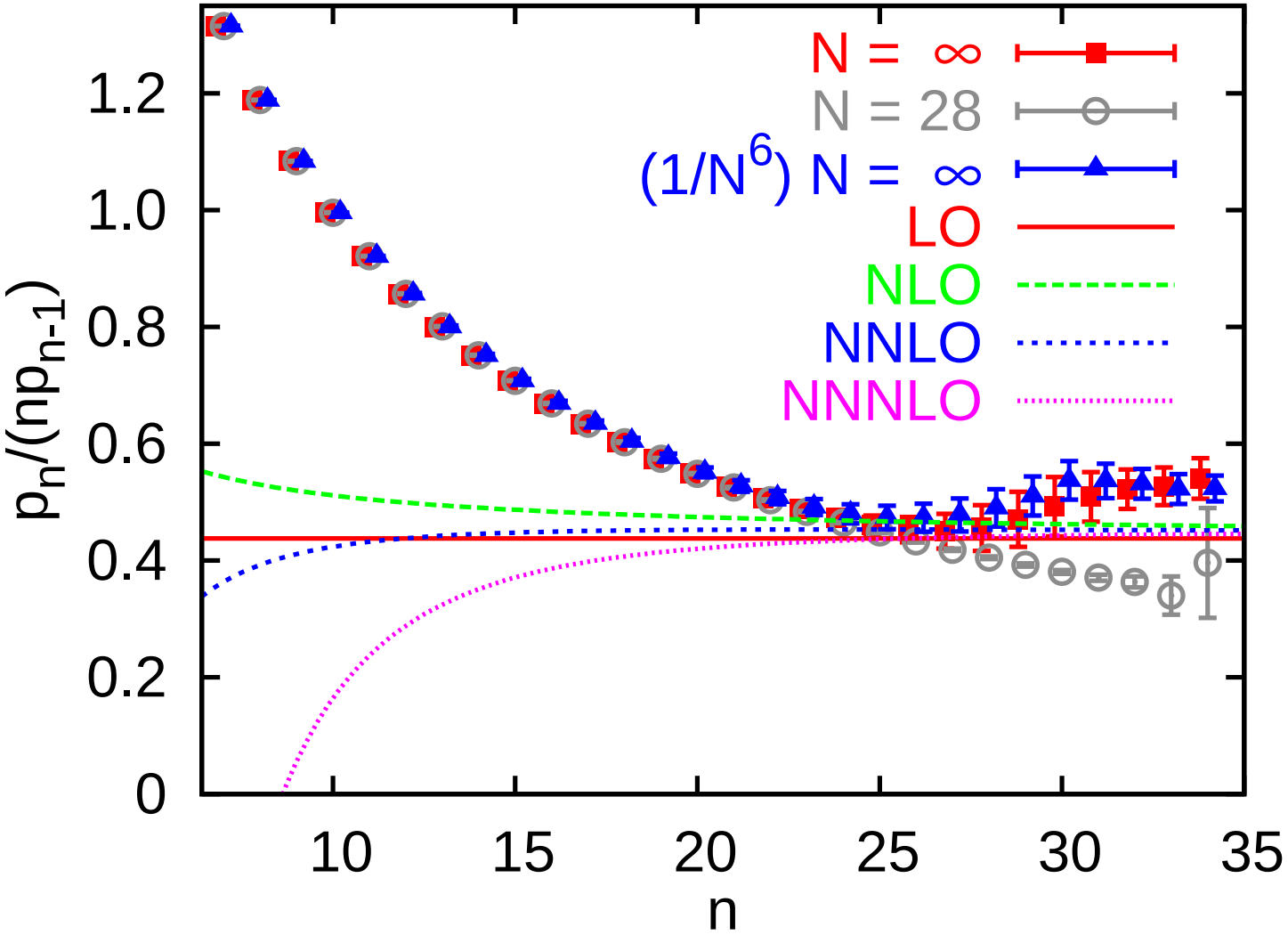


Figure: Ratios $p_n/(np_{n-1})$ of the plaquette coefficients p_n ($N=\infty$, $N=28$) in comparison to the theoretical prediction at different orders in the $1/n$ expansion. Bali, Bauer, AP

Pole mass renormalon seen in full glory: $\sim$ 20 standard deviations

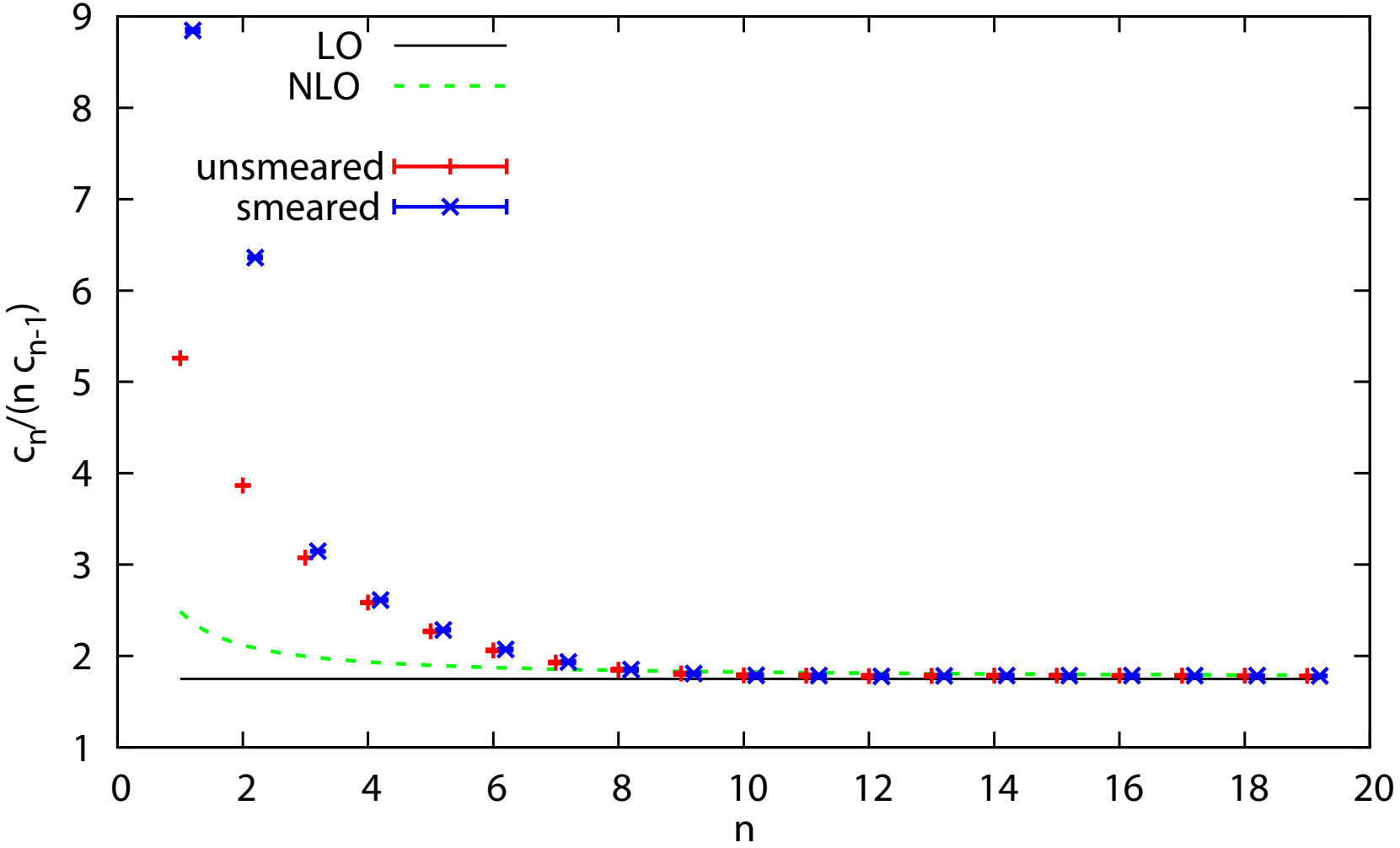


Figure: Ratios $c_n/(nc_{n-1})$ of the smeared (blue) and unsmeared (red) triplet static self-energy coefficients c_n in comparison to the theoretical prediction at different orders in the $1/n$ expansion. Bali, Bauer, AP.

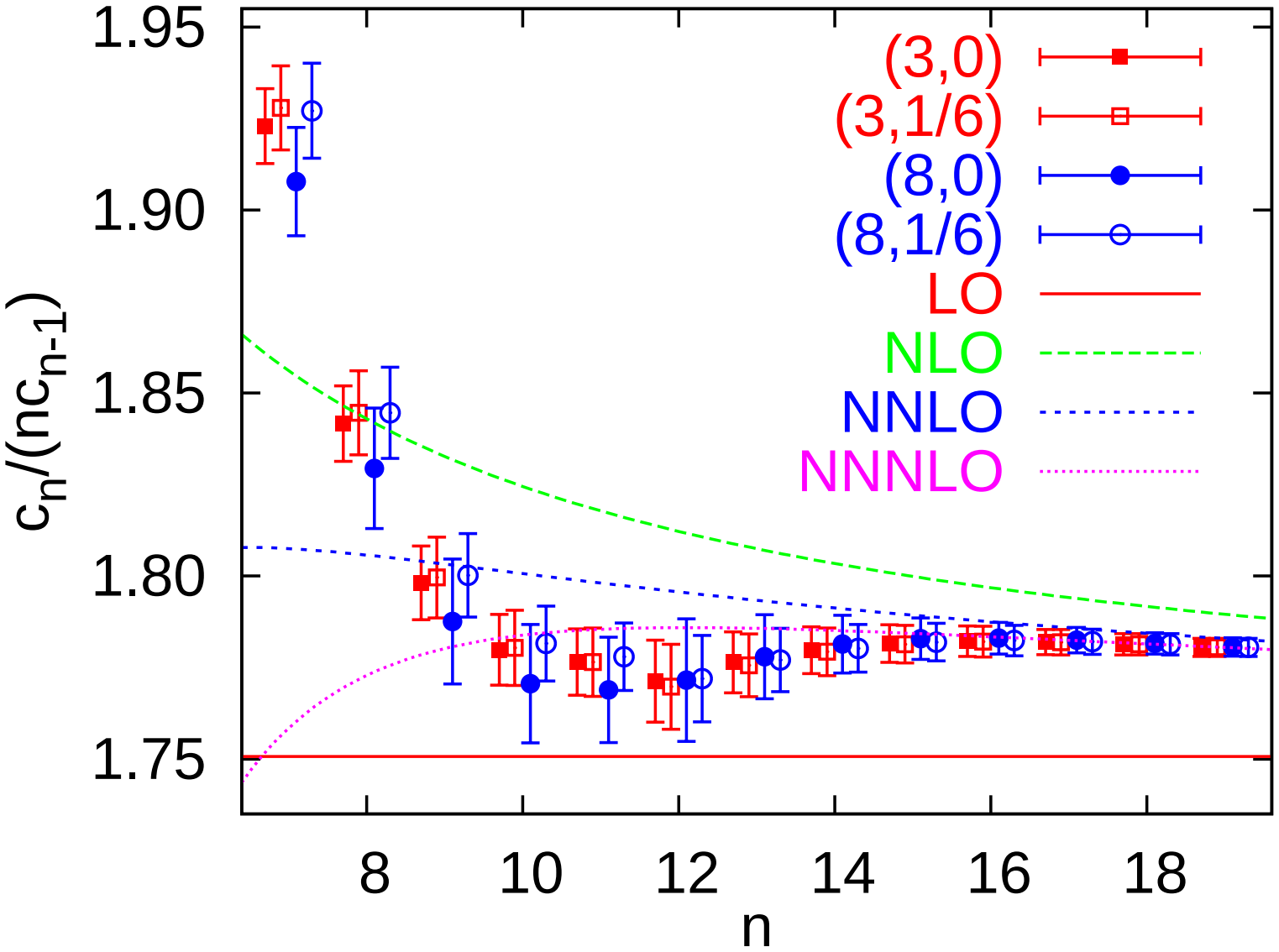


Figure: The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet static self-energies, compared to the prediction for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion. Bali, Bauer, AP, Torrero.

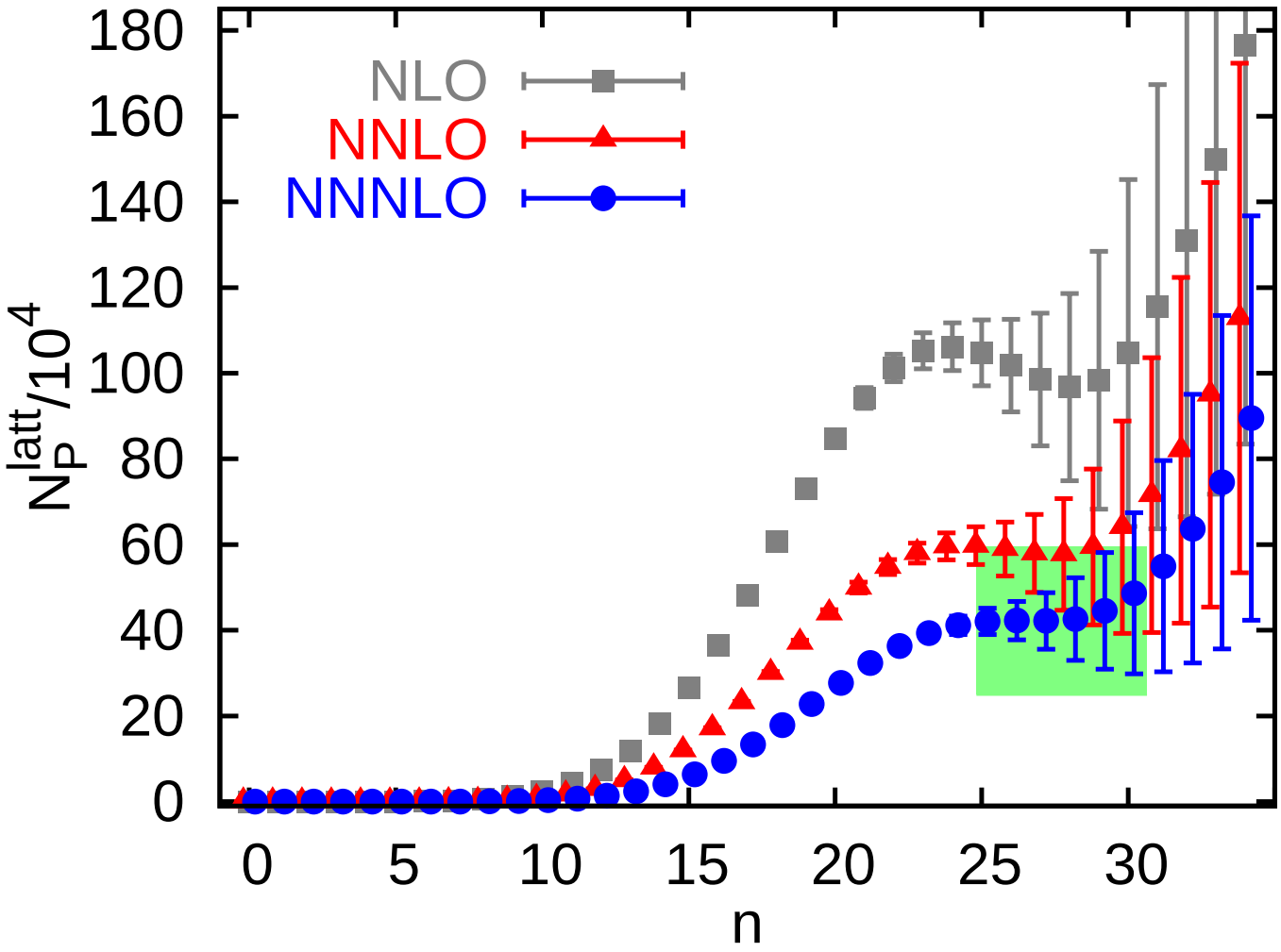
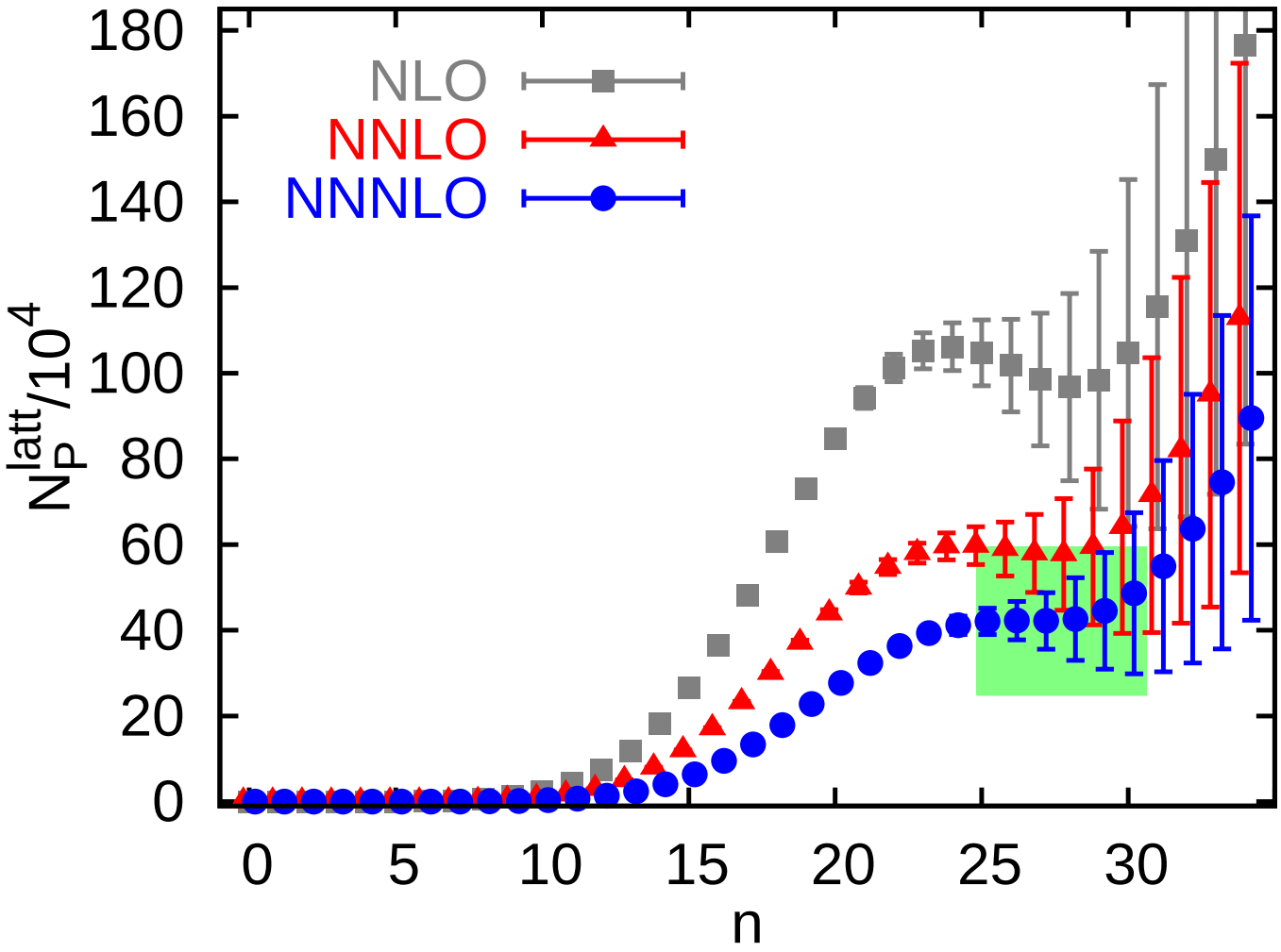


Figure: N_P, determined from the coefficients p_n truncated at NLO, NNLO and NNNLO. The green box marks our final result. Bali, Bauer, AP.

$$N_P^{\overline{\rm MS}} = 0.61(25) \qquad N_{\rm G}^{\overline{\rm MS}} = \frac{36}{\pi^2} N_P^{\overline{\rm MS}} = 2.24(92)\,.$$



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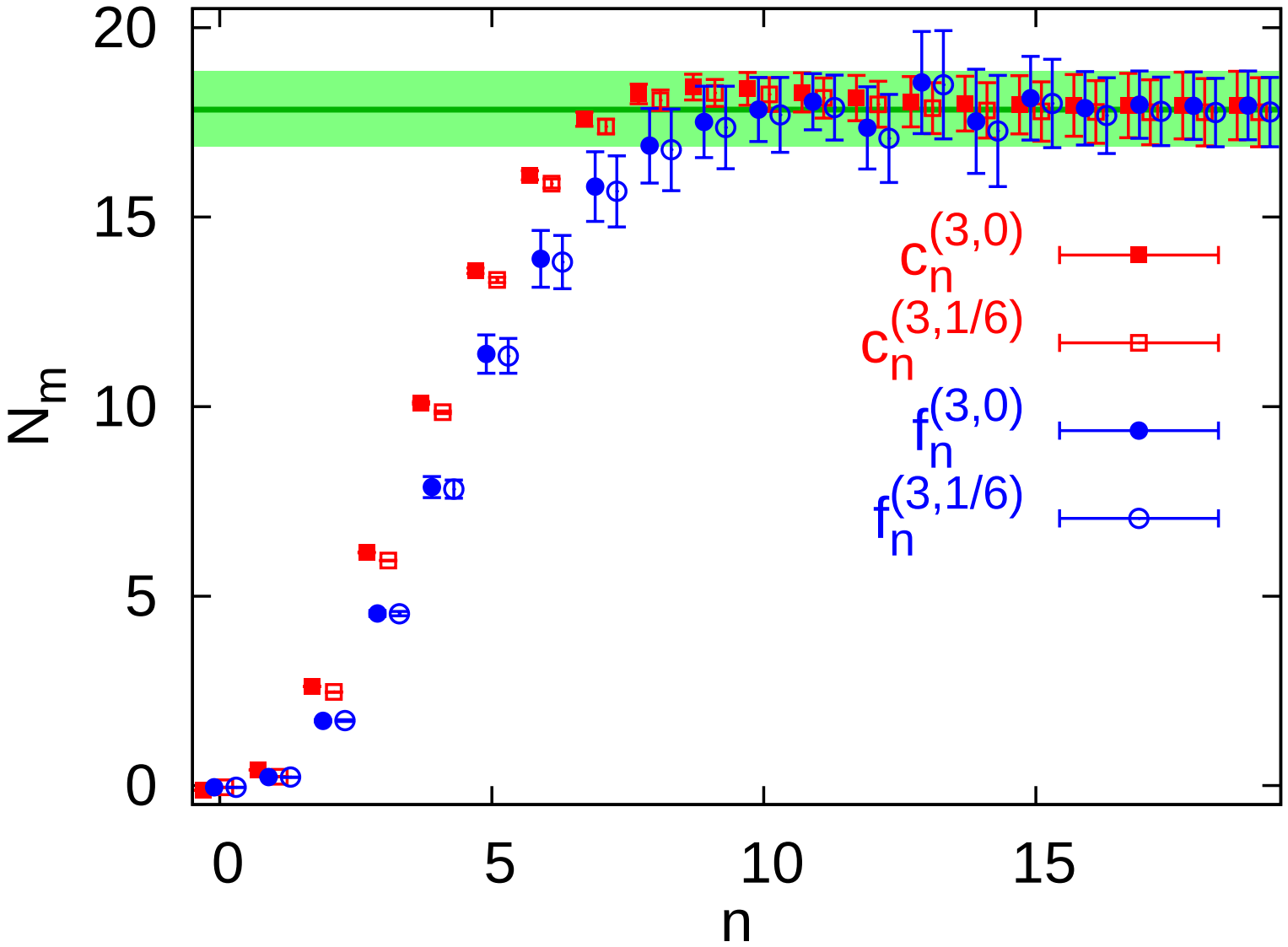

Figure: N_m^{latt} determined via r_n truncated at NNNLO, from the coefficients $c_n^{(3,0)}$, $c_n^{(3,1/6)}$, $f_n^{(3,0)}$ and $f_n^{(3,1/6)}$. The horizontal band is our final result: $N_m = 17.9(1.0)$. Bali, Bauer, AP.

Beyond perturbation theory (at last...)

$$\langle P\rangle_{\rm pert} = \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm NSPT} = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle O_{\rm G}\rangle_{\rm soft} + \mathcal{O}\left(a^6\right) .$$

$$\frac{1}{a} \gg \frac{1}{Na}$$

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$$\frac{1}{a} \gg \frac{1}{Na} \gg \Lambda_{\rm QCD} \quad \rightarrow \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm soft}\left[1 + \mathcal{O}(\Lambda^2_{\rm QCD}(Na)^2)\right]$$

$$\frac{1}{a} \gg \Lambda_{\rm QCD} \gg \frac{1}{Na} \quad \rightarrow \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm NP}\left[1 + \mathcal{O}\left(\frac{1}{\Lambda^2_{\rm QCD}(Na)^2}\right)\right] ,$$

where $\langle G^2\rangle_{\rm NP} \sim \Lambda^4_{\rm QCD}$ is the NP gluon condensate (Vainshtein, Zakharov, Shifman).

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Determination of the gluon condensate: $\frac{1}{a} \gg \Lambda_{\rm QCD} \gg \frac{1}{Na}$

$$\langle G^2 \rangle_{\rm NP} = \frac{36 C_{\rm G}^{-1}(\alpha)}{\pi^2 a^4(\alpha)} \left[\langle P \rangle_{\rm MC}(\alpha) - S_P(\alpha) \right] + \mathcal{O}(a^2 \Lambda_{\rm QCD}^2) \,.$$

$$S_P(\alpha) \equiv S_{n_0}(\alpha) \,, \quad \text{where} \quad S_n(\alpha) = \sum_{j=0}^{n} p_j \alpha^{j+1} \,.$$

$n_0 \equiv n_0(\alpha)$ is the order for which $p_{n_0} \alpha^{n_0+1}$ is minimal.
MC simulation from Boyd et al.

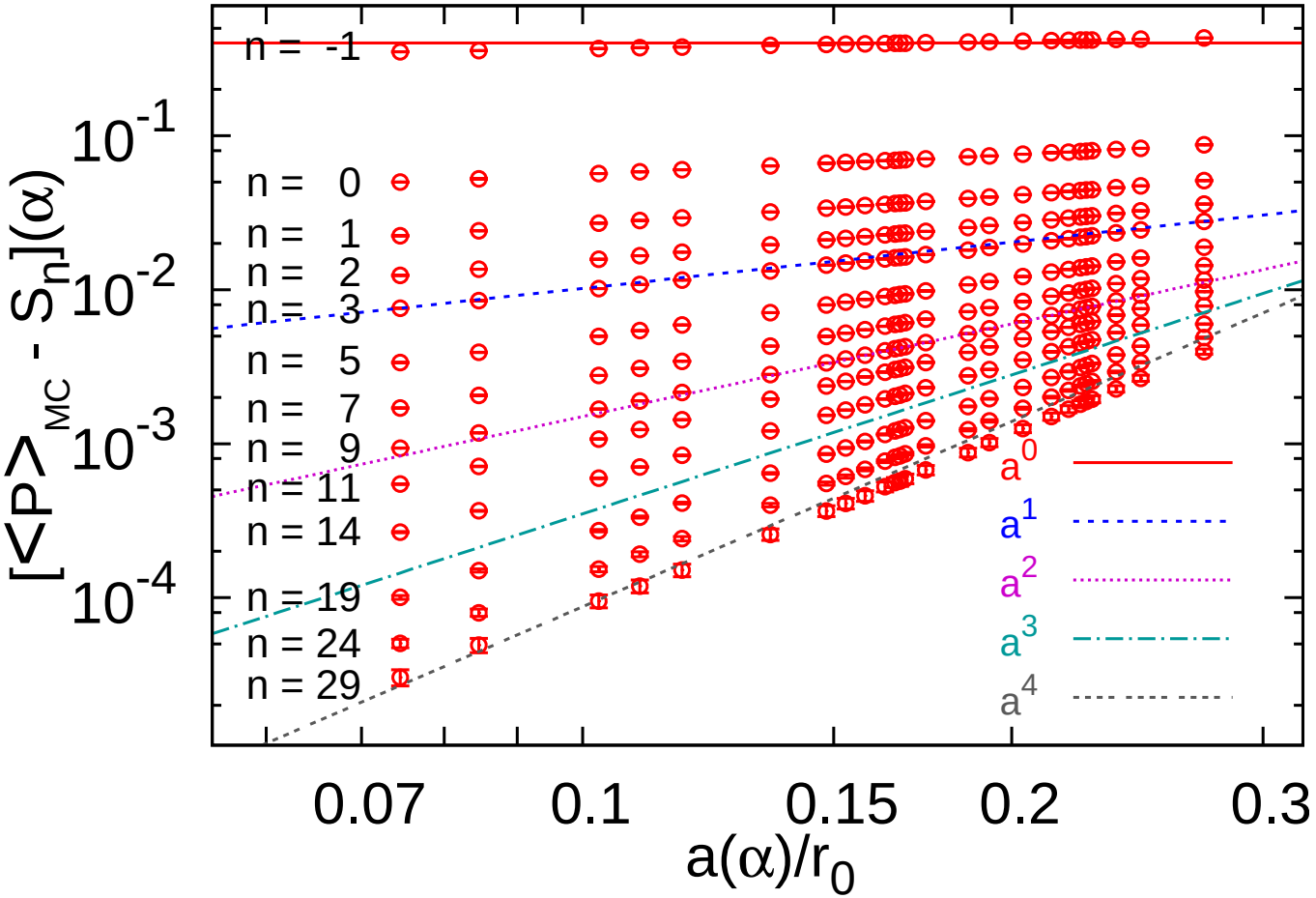

Figure: $\langle P\rangle_{\rm MC}(\alpha) - S_n(\alpha)$ between MC data and sums truncated at orders α^{n+1} ($S_{-1} = 0$) vs. $a(\alpha)/r_0$. The lines $\propto a^j$ are drawn to guide the eye. Bali, Bauer, AP.

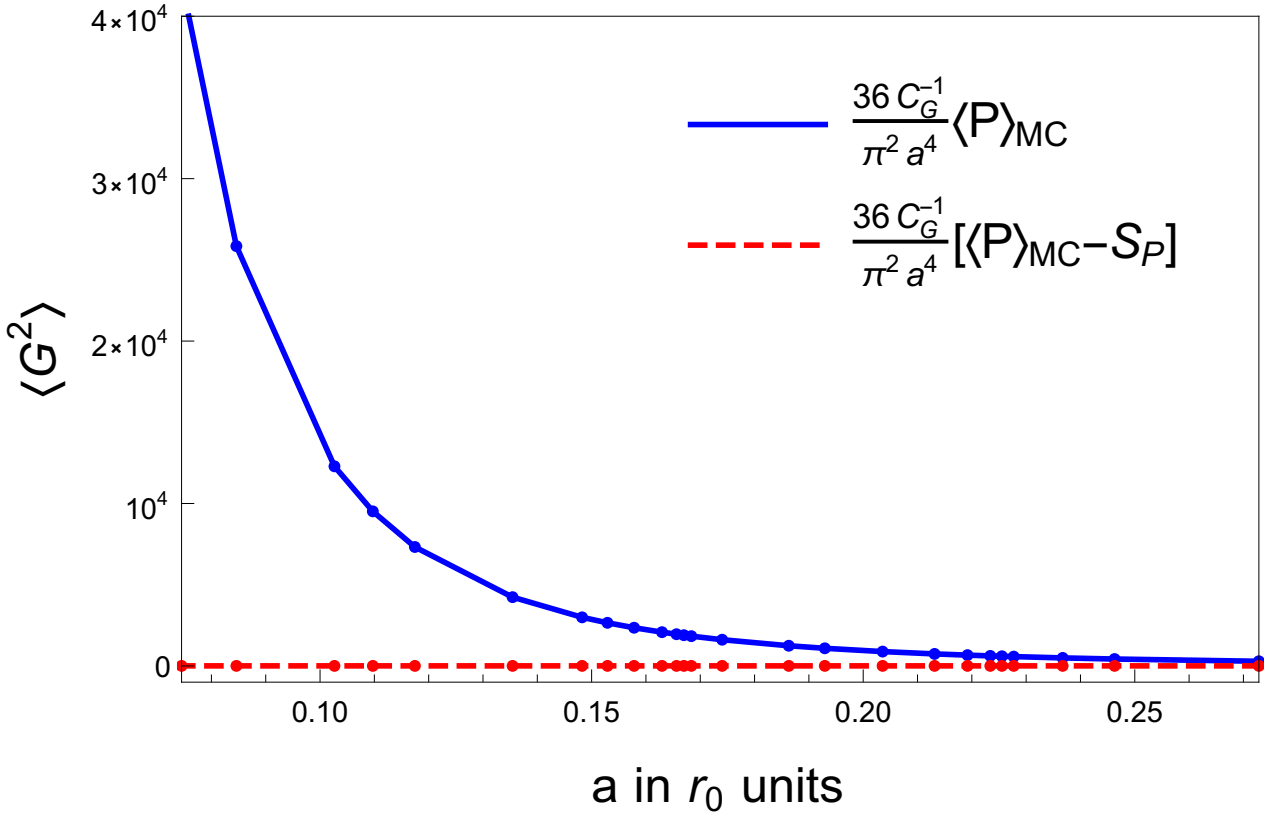

Figure: $\frac{36 C_G^{-1}}{\pi^2 a^4} \langle P \rangle_{\rm MC}$ (continuous blue line) and $\frac{36 C_G^{-1}}{\pi^2 a^4} [\langle P \rangle_{\rm MC} - S_P]$ (dashed red line). The second line is basically indistinguishable with respect to zero with the scale resolution of this plot. The statistical errors are smaller than the size of the points. Ayala, Lobregat, AP.

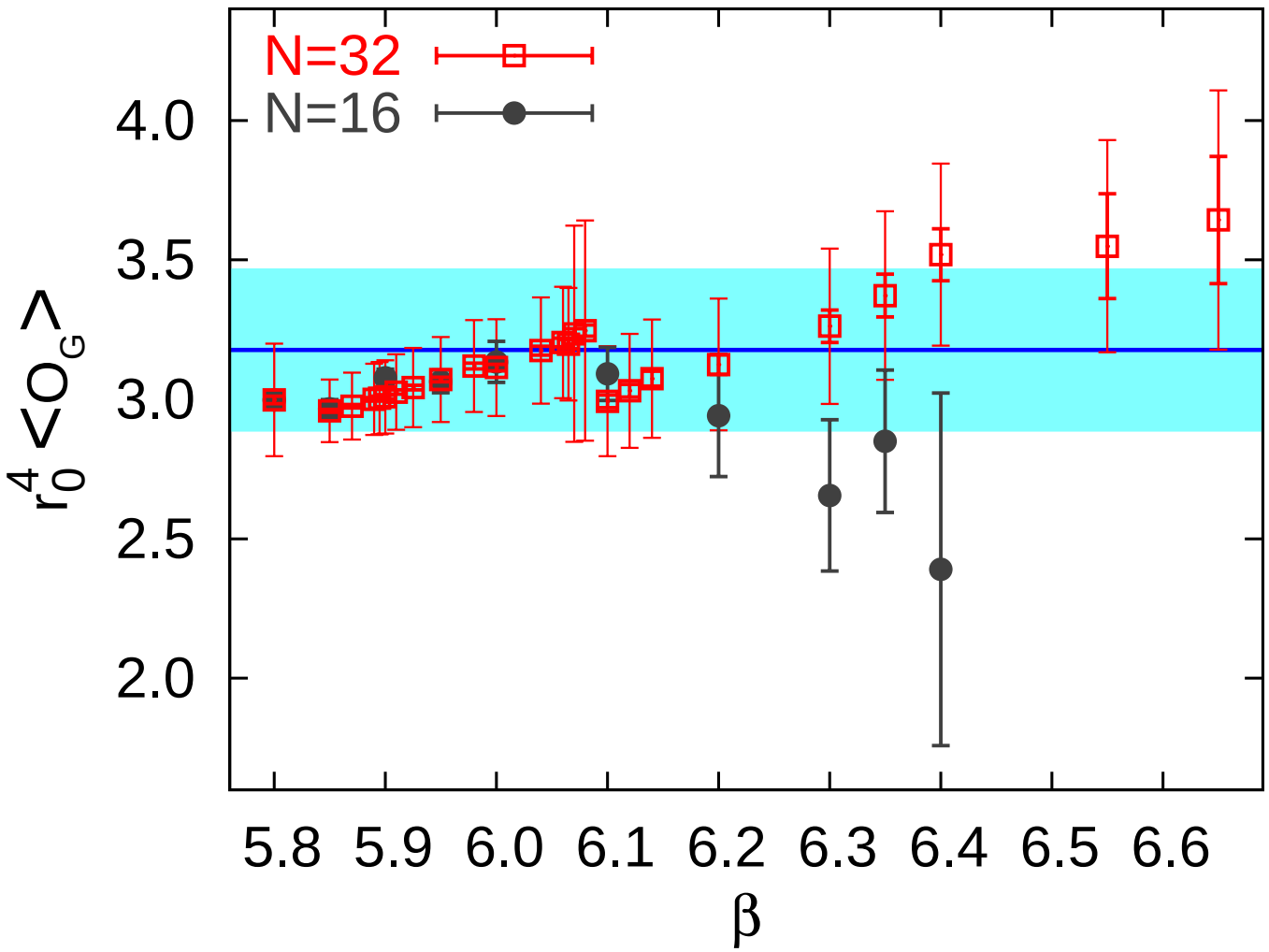

Figure: $\langle G^2 \rangle$ evaluated using the $N = 16$ and $N = 32$ MC data of Boyd et al. The error band is our prediction for $\langle G^2 \rangle$. Bali, Bauer, AP.

$$\langle G^2 \rangle = 3.18(29) r_0^{-4} = 24.2(8.0) \Lambda_{\overline{\rm MS}}^4 \simeq 0.077\,{\rm GeV}^4 \,.$$

How to go beyond superasymptotic approximation? (Ayala, Lobregat, AP)

1. Predict observables with $e^{-A\frac{2\pi}{\beta_0\alpha(Q)}}$ precision ($A > |d|$).

We need to define the perturbative sum with higher precision than superasymptotic approximation.

Borel transform

$$\sum_{n=0}^{\infty} p_n^{(X)}(\frac{\mu}{Q})\alpha_X^{n+1}(\mu) \to B[O](t) = \sum_{n=0}^{\infty} \frac{p_n^{(X)}(\frac{\mu}{Q})}{n!} t^n$$

Inverse Borel transform

$$\int_0^{\infty} dt e^{-t/\alpha_X(Q)} B[O](t)$$

PV regularization

$$S_{\rm PV}(\alpha(Q)) \equiv \int_{0,\rm PV}^{\infty} dt e^{-t/\alpha_X(Q)} B[O](t)$$

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$$\text{Observable}(\frac{Q}{\Lambda_{\text{QCD}}}) = S_{\text{PV}}(\alpha(Q))$$
$$+K_X^{(\text{PV})}\alpha_X^{\gamma}(Q)\frac{\Lambda_X^d}{Q^d}\,(1+\mathcal{O}(\alpha_X(Q)))+\mathcal{O}(\frac{\Lambda_X^{d'}}{Q^{d'}})\,,$$

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Ayala, Lobregat, AP (2009.01285)

$$\langle P\rangle_{\rm MC} = S_{\rm PV}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle G^2\rangle_{\rm MC} + \mathcal{O}\left(a^6\right) .$$

$$S_{\rm PV} = \sum_{n=0}^{N_P} p_n \alpha^{n+1} + \Omega_{G^2} + \sum_{n=N_P+1}^{N'} [p_n - p_n^{(\rm as)}]\alpha^{n+1} + \cdots ,$$

$$N_P = 4\frac{2\pi}{\beta_0 \alpha(1/a)}\left(1 - c\alpha(1/a)\right) ,$$

$$\Omega_{G^2} = \sqrt{\alpha(1/a)} K^{(P)} \Lambda^4 \Big(1 + \bar{K}_1^{(P)} \alpha(1/a) + \bar{K}_2^{(P)} \alpha^2(1/a) + \mathcal{O}(\alpha^3(1/a)) \Big) ,$$

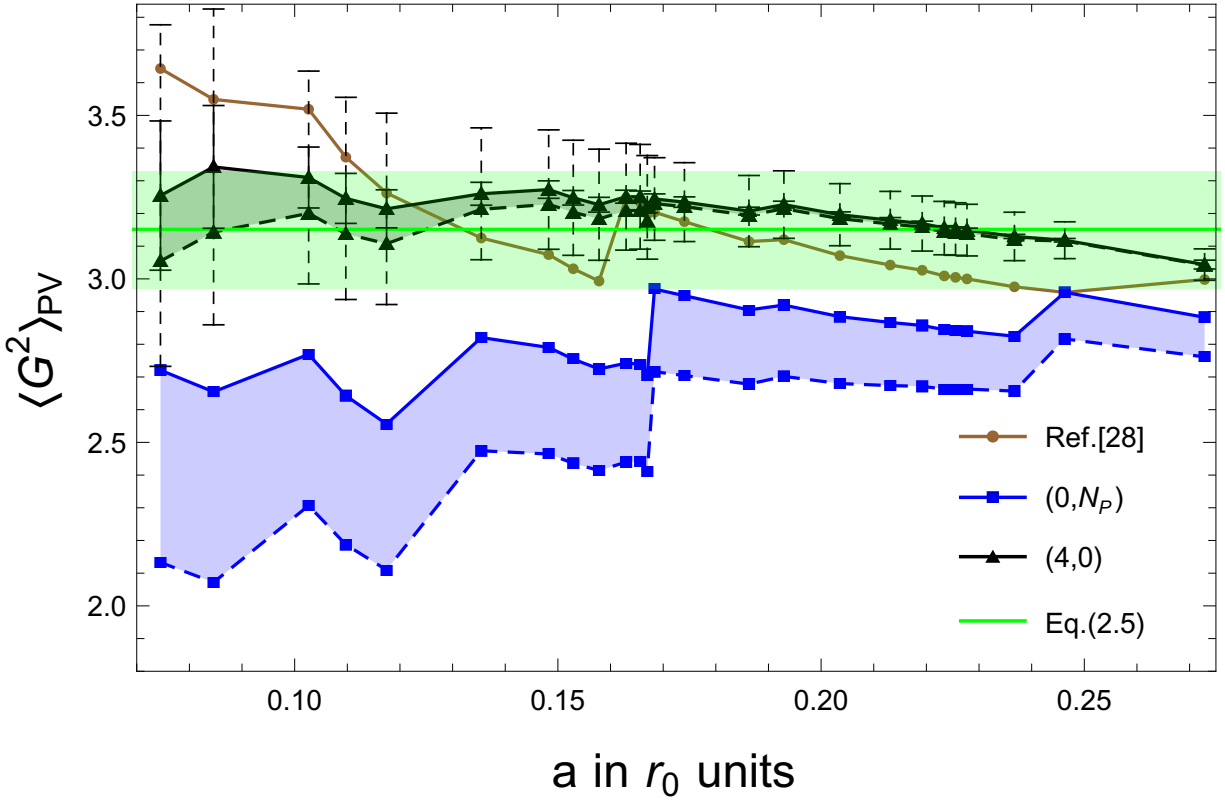

Figure: Gluon condensate with superasymptotic approximation $(0, N_P)$ and with hyperasymptotic accuracy $(4, 0)$. We show the value obtained for the gluon condensate with the values of N_P using the smallest positive (upper line) and negative (lower line) value of c that yields an integer value of N_P. For the hyperasymptotic approximation with c positive we also show the statistical errors of the MC determination of the plaquette (inner error) and its combination in quadrature with the statistical error of the partial sum (outer error). We also show the superasymptotic approximation obtained by Bali, Bauer, AP truncating at the minimal term determined numerically. The horizontal green band is our final prediction. Ayala, Lobregat, AP.

CONCLUSIONS: Non-perturbative OPE OK

$$d = 4 \text{ renormalon OK} \rightarrow N_P^{\overline{\rm MS}} = 0.61(25) \qquad N_{\rm G}^{\overline{\rm MS}} = \frac{36}{\pi^2} N_P^{\overline{\rm MS}} = 2.24(92)\,.$$

Nonperturbative quantities can only be defined after subtracting the divergent perturbative series. $\longrightarrow$ We have organized the OPE along an hyperasymptotic expansion. We use the PV prescription of the Borel integral to regularize the perturbative sum (scheme/scale independence)

$$\langle G^2 \rangle_{\rm PV}(n_f = 0) = 3.15(18)\, r_0^{-4}.$$

Dimension two condensates: artifacts of incomplete subtractions

- unquantifiable error due to the simplified parameterization of higher order perturbation theory
- short distance effect $\rightarrow$ process dependent

FUTURE:
n_f dependence $\longrightarrow$ Del Debbio et al. (preliminary)
Application to sum rules
Model independent/systematic procedure to get ALL condensates

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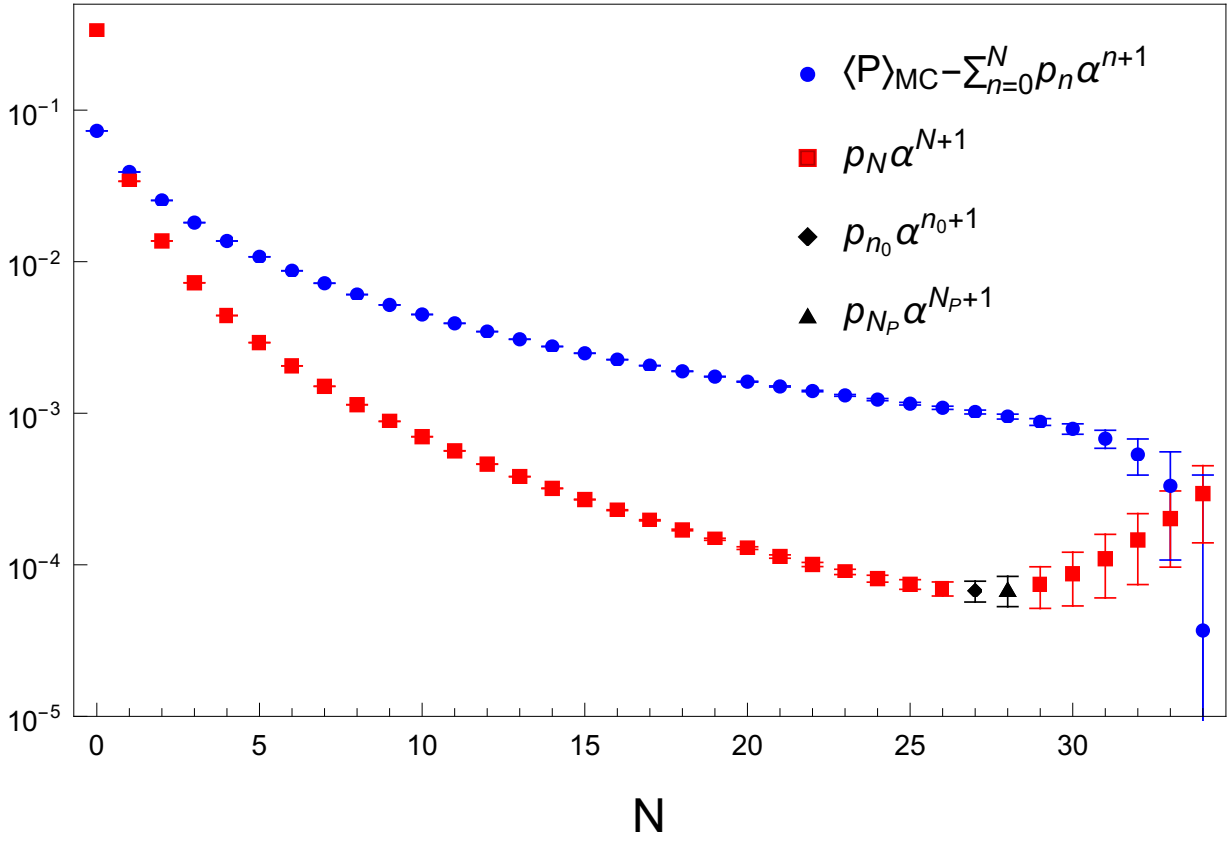

Figure: $\langle P\rangle_{\rm MC} - \sum_{n=0}^{N} p_n\alpha^{n+1}$ (full blue circle points), and $p_N\alpha^{N+1}$ (full red squares points) for different values of N for $\beta = 6$. The error of the blue points is the statistical error of of the MC simulation and of the sum $\sum_{n=0}^{N} p_n\alpha^{n+1}$ combined in quadrature (for large N the error of the perturbative sum is dominant). The error displayed for the red points is the complete error (statistical plus systematic combined in quadrature) of the p_N coefficient times α^{N+1}. The black diamond stands for the numerical minimal value of $p_N\alpha^{N+1}$. The black triangle is $p_{N_P}\alpha^{N_P+1}$ using the smallest positive c that makes N_P to be integer. Ayala, Lobregat, AP.

Hyperasymptotic expansion (note that D is positive):

$$S_{\rm PV}(Q) = S_P(Q;\mu) + \Omega(\mu) + \sum_{n=N_P+1}^{N'_P} (p_n - p_n^{(as)})\alpha_X^{n+1}(\mu) + \Omega'(\mu) + \cdots ,$$

$$S_{\rm PV}^{(D,N)}(Q) = \sum_{\{|d|\}} S_{|d|<D} + \sum_{\{|d|\leq D\}} \Omega_d + \sum_{n=N_P(D)+1}^{N_P(D)+N} (p_n - p_n^{(as)})\alpha^{n+1}(\mu) .$$

where

$$S_P \equiv \sum_{n=0}^{N_P(|d_{min}|)} p_n \alpha^{n+1}(\mu) \equiv S_{|d|=0} ,$$

$$\Omega_{d>0} \sim \sqrt{\alpha(\mu)}\Lambda_{\rm QCD}^d \qquad \Omega_{d<0} \sim \sqrt{\alpha(\mu)}\frac{\Lambda_{\rm QCD}^{|d|} Q^{|d|}}{\mu^{2|d|}}$$

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$$S_{|d|} = \sum_{n=N_P(|d|)+1}^{N_P(|d'|)} (p_n - p_n^{(as)})\alpha^{n+1}(\mu) ,$$

$$S_{\rm PV}(Q) = S_P + \sum_{\{|d|\}} S_{|d|} + \sum_{\{d>0\}} \Omega_d + \sum_{\{d<0\}} \Omega_d ,$$

For each value of the couple (D, N) we can state the parametric accuracy of $S_{\rm PV}^{(D,N)}(Q)$. For $S^{(0,N_P)}$ the error would be (up to a numerical and a $\sqrt{\alpha_X}$ factor)

$$\delta S^{(0,N_P)} \sim \mathcal{O}\left(e^{-|d_{min}|\frac{2\pi}{\beta_0\alpha_X(Q)}}\right),$$

and for $S^{(|d_{min}|,0)}$ (up to a numerical and a possible $\alpha_X^{3/2}$ factor):

$$\delta S^{(|d_{min}|,0)} \sim \mathcal{O}\left(\left(|\frac{d}{d_{min}}|e\right)^{-|d_{min}|\frac{2\pi}{\beta_0\alpha_X(Q)}} = e^{-|d_{min}|\frac{2\pi}{\beta_0\alpha_X(Q)}(1+\ln(|d/d_{min}|)}\right),$$

where d is the location of the next renormalon closest to the origin. This corresponds to the first term in the hyperasymptotic approximation. The expression for the error in the general case $S_{\rm PV}^{(D,N)}(Q)$ reads ($N \neq N_P$ but large)

$$\delta S^{(D,N)} \sim \mathcal{O}\left(e^{-D\frac{2\pi}{\beta_0\alpha_X(Q)}(1+\ln(|d/D|)}\alpha_X^N\right),$$

where d is the location of the next renormalon closest to the origin after D.

POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the lattice: δm of HQET.
We use Numerical Stochastic Perturbation Theory (Di Renzo et al.).

$$L^{(R)}(N_S, N_T) = \frac{1}{N_S^3} \sum_{\mathbf{n}} \frac{1}{d_R} \mathrm{tr} \left[\prod_{n_4=0}^{N_T-1} U_4^R(n) \right] \qquad U_\mu^R(n) \approx e^{iA_\mu^R[(n+1/2)a]}$$

We implement triplet and octet representations R ($d_R = 3, 8$).

$$\langle L^{(R)}(N_S, N_T) \rangle \overset{N_T \to \infty}{\sim} e^{-N_T a \delta m^{(R)}(N_S)}$$

$$\delta m^{(R)}(N_S) = \lim_{N_T \to \infty} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S, N_T) \alpha^{n+1} = \frac{1}{a} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S) \alpha^{n+1}(1/a)$$

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Perturbative OPE (Zimmermann) at finite volume ($N_S \to \infty$)

$$\delta m = \lim_{N_S\to\infty} \delta m(N_S) \quad c_n = \lim_{N_S\to\infty} c_n(N_S) \qquad \left(\lim_{n\to\infty} c_n^{(3,\rho)} = r_n(\nu)/\nu\right).$$

For large N_S, we write (OPE: $\frac{1}{a} \gg \frac{1}{N_S a}$)

$$\delta m(N_S) = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n \alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right).$$

Taylor expansion of $\alpha\left((aN_S)^{-1}\right)$ in powers of $\alpha\left(a^{-1}\right)$:

$$c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + \mathcal{O}\left(\frac{1}{N_S^2}\right); \qquad f_n(N_S) = \sum_{i=0}^{n} f_n^{(i)} \ln^i(N_S),$$

$f_n^{(0)} = f_n$. The coefficients $f_n^{(i)}$ for $i > 0$ are determined by f_m and β_j.

$$f_1(N_S) = f_1 + f_0 \frac{\beta_0}{2\pi} \ln(N_S),$$

$$f_2(N_S) = f_2 + \left[2 f_1 \frac{\beta_0}{2\pi} + f_0 \frac{\beta_1}{8\pi^2}\right] \ln(N_S) + f_0 \left(\frac{\beta_0}{2\pi}\right)^2 \ln^2(N_S),$$

and so on.

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and so on.

	$c_n^{(3,0)}$	$c_n^{(3,1/6)}$	$c_n^{(8,0)} C_F/C_A$	$c_n^{(8,1/6)} C_F/C_A$
c_0	2.117274357	0.72181(99)	2.117274357	0.72181(99)
c_1	11.136(11)	6.385(10)	11.140(12)	6.387(10)
$c_2/10$	8.610(13)	8.124(12)	8.587(14)	8.129(12)
$c_3/10^2$	7.945(16)	7.670(13)	7.917(20)	7.682(15)
$c_4/10^3$	8.215(34)	8.017(33)	8.197(42)	8.017(36)
$c_5/10^4$	9.322(59)	9.160(59)	9.295(76)	9.139(64)
$c_6/10^6$	1.153(11)	1.138(11)	1.144(13)	1.134(12)
$c_7/10^7$	1.558(21)	1.541(22)	1.533(25)	1.535(22)
$c_8/10^8$	2.304(43)	2.284(45)	2.254(51)	2.275(45)
$c_9/10^9$	3.747(95)	3.717(97)	3.64(11)	3.703(98)
$c_{10}/10^{10}$	6.70(22)	6.65(22)	6.49(25)	6.63(22)
$c_{11}/10^{12}$	1.316(52)	1.306(53)	1.269(59)	1.303(53)
$c_{12}/10^{13}$	2.81(13)	2.79(13)	2.71(14)	2.78(13)
$c_{13}/10^{14}$	6.51(35)	6.46(35)	6.29(37)	6.45(35)
$c_{14}/10^{16}$	1.628(96)	1.613(97)	1.57(10)	1.614(97)
$c_{15}/10^{17}$	4.36(28)	4.32(28)	4.22(29)	4.33(28)
$c_{16}/10^{19}$	1.247(86)	1.235(86)	1.206(89)	1.236(86)
$c_{17}/10^{20}$	3.78(28)	3.75(28)	3.66(28)	3.75(28)
$c_{18}/10^{22}$	1.215(93)	1.204(94)	1.176(95)	1.205(94)
$c_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

	$f_n^{(3,0)}$	$f_n^{(3,1/6)}$	$f_n^{(8,0)} C_F/C_A$	$f_n^{(8,1/6)} C_F/C_A$
f_0	0.7696256328	0.7810(59)	0.7696256328	0.7810(69)
f_1	6.075(78)	6.046(58)	6.124(87)	6.063(68)
$f_2/10$	5.628(91)	5.644(62)	5.60(11)	5.691(78)
$f_3/10^2$	5.87(11)	5.858(76)	6.00(18)	5.946(91)
$f_4/10^3$	6.33(22)	6.29(17)	6.57(40)	6.26(23)
$f_5/10^4$	7.73(35)	7.71(26)	7.67(66)	7.78(42)
$f_6/10^5$	9.86(53)	9.80(42)	9.68(99)	9.79(69)
$f_7/10^7$	1.388(81)	1.378(71)	1.35(15)	1.38(11)
$f_8/10^8$	2.12(12)	2.11(12)	2.06(22)	2.10(17)
$f_9/10^9$	3.54(20)	3.52(20)	3.40(37)	3.51(27)
$f_{10}/10^{10}$	6.49(33)	6.44(34)	6.23(67)	6.44(43)
$f_{11}/10^{12}$	1.296(64)	1.286(66)	1.24(13)	1.286(74)
$f_{12}/10^{13}$	2.68(19)	2.64(18)	2.65(33)	2.65(21)
$f_{13}/10^{14}$	6.70(54)	6.68(52)	6.36(90)	6.66(57)
$f_{14}/10^{16}$	1.58(14)	1.56(14)	1.55(22)	1.57(15)
$f_{15}/10^{17}$	4.41(34)	4.37(33)	4.24(47)	4.37(35)
$f_{16}/10^{19}$	1.241(92)	1.230(91)	1.20(11)	1.231(94)
$f_{17}/10^{20}$	3.79(28)	3.75(28)	3.67(30)	3.76(28)
$f_{18}/10^{22}$	1.215(94)	1.204(94)	1.176(97)	1.205(94)
$f_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

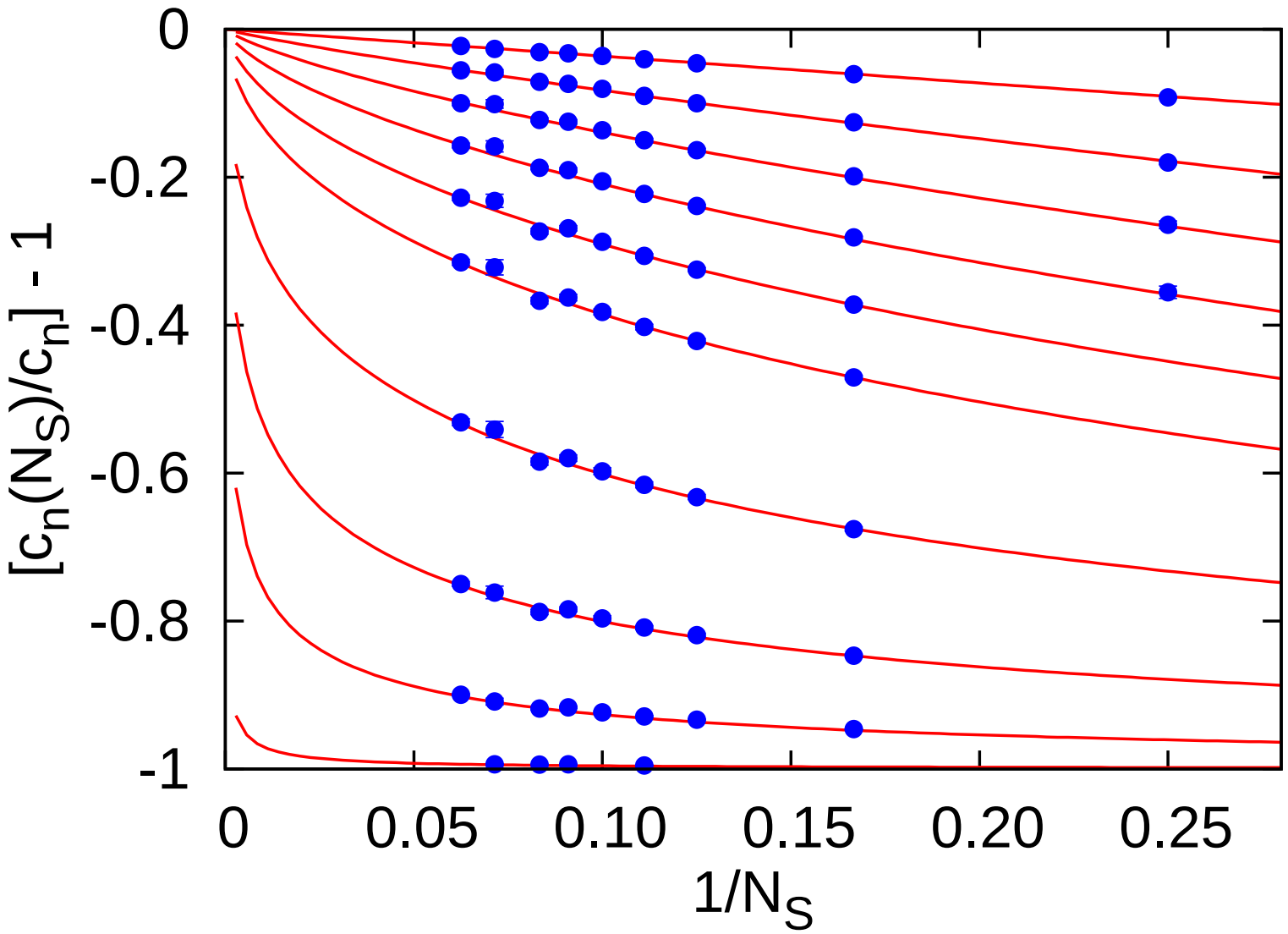

Figure: $c_n^{(3,0)}(N_S)/c_n^{(3,0)} - 1$ *for* $n \in \{0, 1, 2, 3, 4, 5, 7, 9, 11, 15\}$ *(top to bottom). For each value of* N_S *we have plotted the data point with the maximum value of* N_T*. The curves represent the global fit.* $-(1/N_S) f^{(3,0)}_{0\ \mathrm{DLPT}}/c^{(3,0)}_{0\ \mathrm{DLPT}}$ *is shown for* $n = 0$.

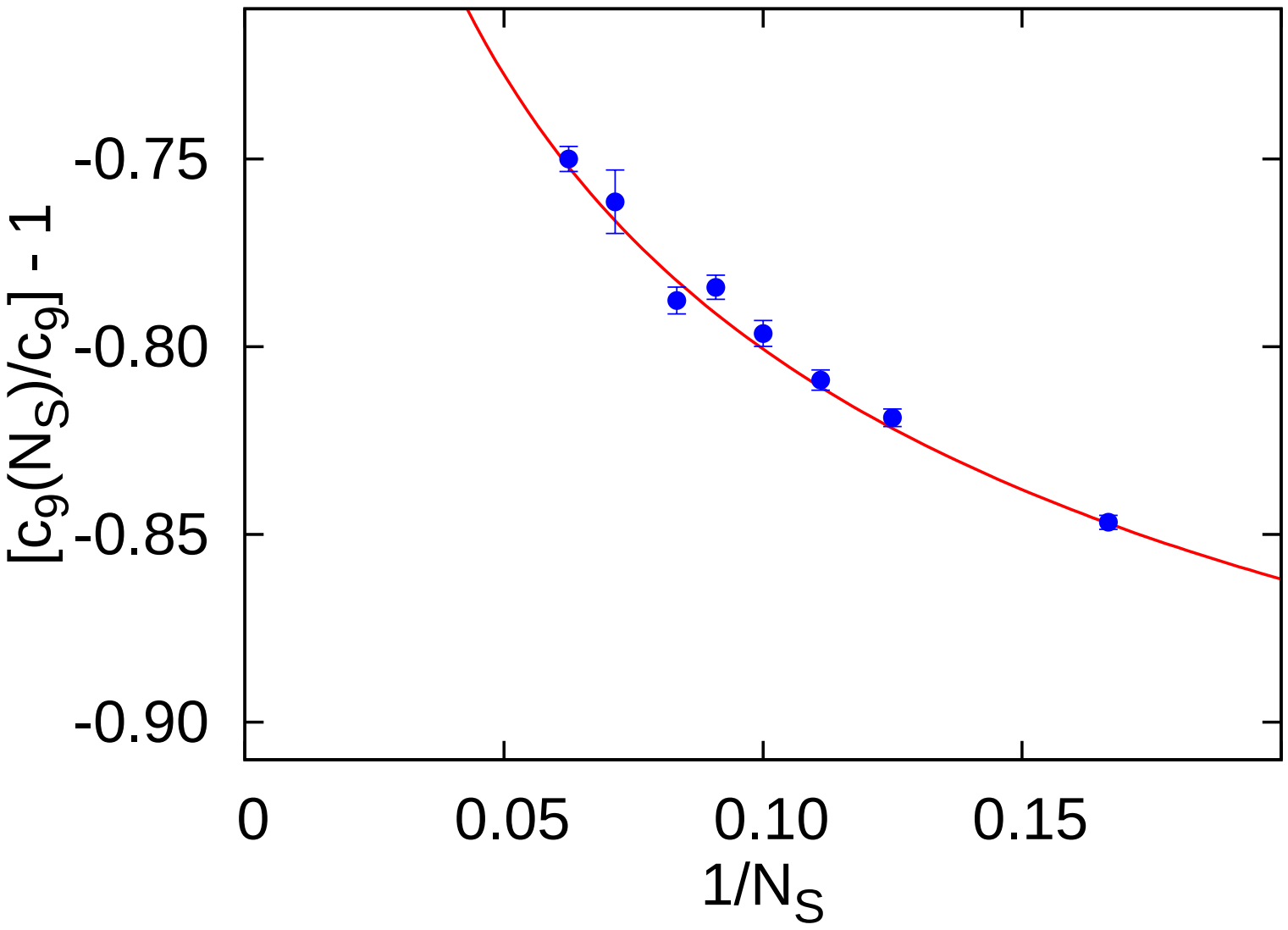

Figure: *Zoom of previous Figure for $n = 9$.*

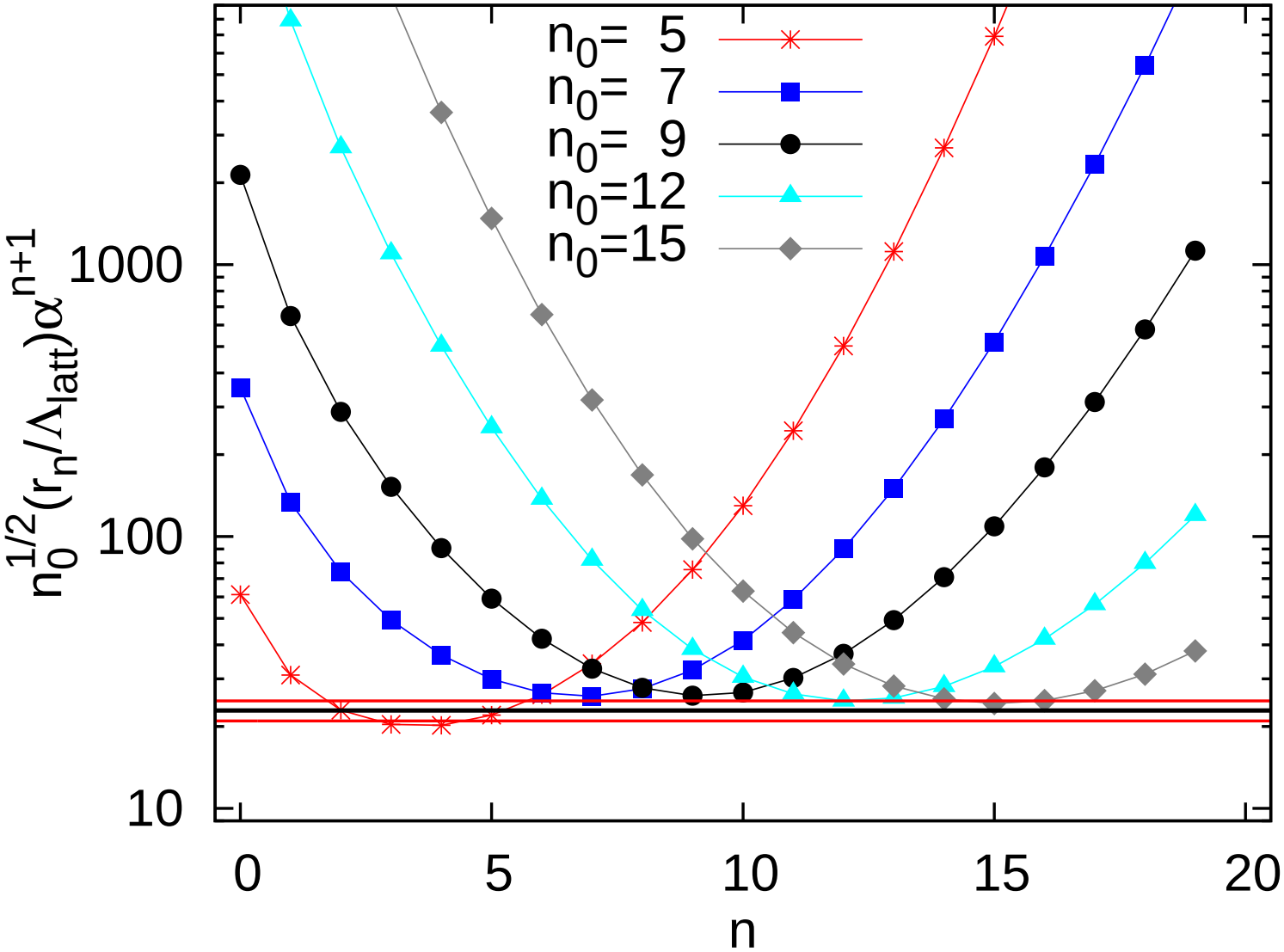

Figure: *c_n times $\sqrt{n_0}$, for five different values of the lattice scheme coupling constant α, ranging from $\alpha(\nu) \approx 0.096$ ($n_0 = 5$) to $\alpha(\nu) \approx 0.036$ ($n_0 = 15$). Bali, Bauer, AP, Torrero, 1303.3279.*

Ratios

$$\frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}}\frac{1}{n} = \frac{c_n^{(8,\rho)}}{c_{n-1}^{(8,\rho)}}\frac{1}{n}$$
$$= \frac{\beta_0}{2\pi}\left\{1+\frac{b}{n}-\frac{bs_1}{n^2}+\frac{1}{n^3}\left[b^2 s_1^2 + b(b-1)(s_1-2s_2)\right]+\mathcal{O}\left(\frac{1}{n^4}\right)\right\} \ .$$

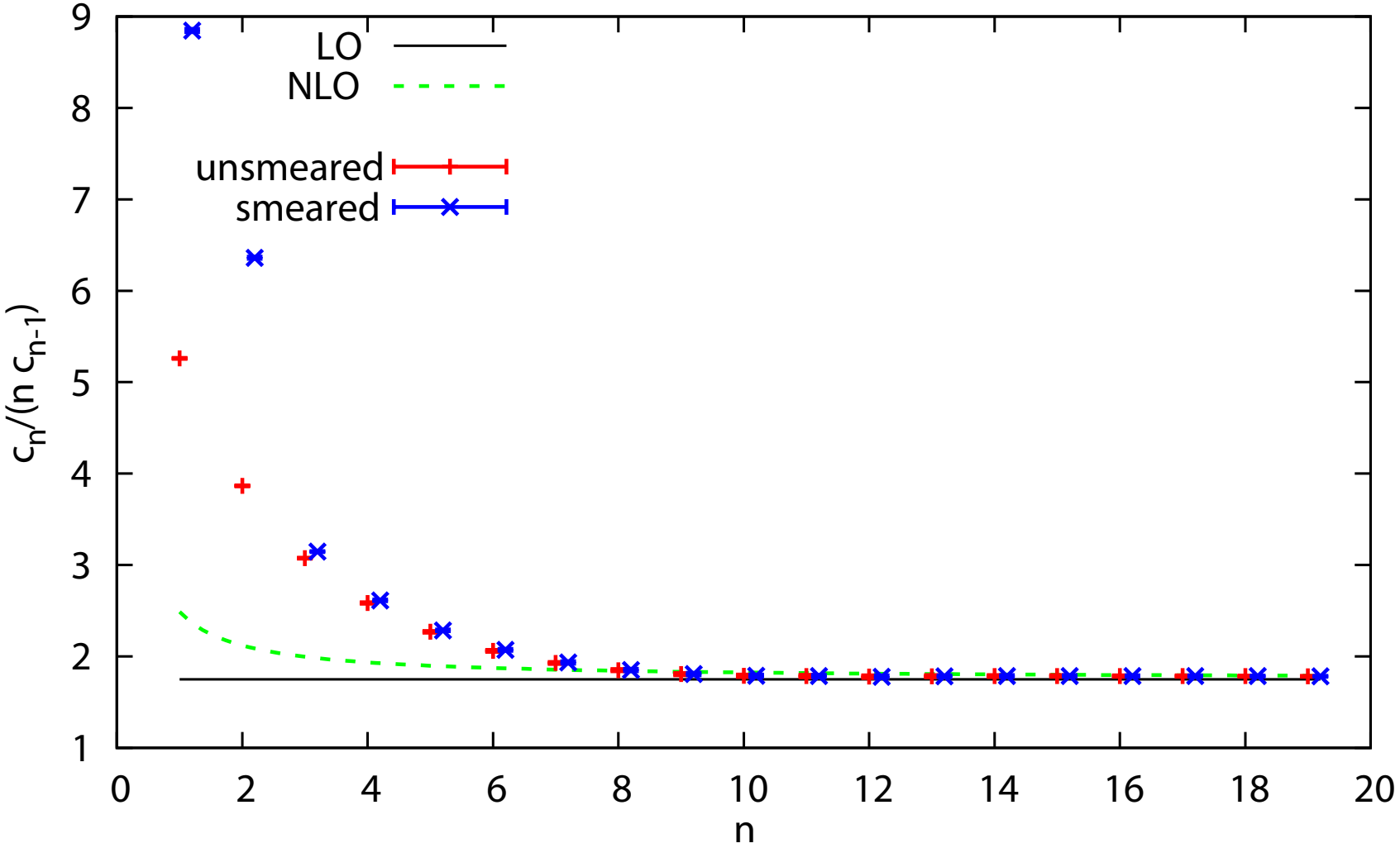

Figure: Ratios $c_n/(nc_{n-1})$ of the smeared (blue) and unsmeared (red) triplet static self-energy coefficients c_n in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

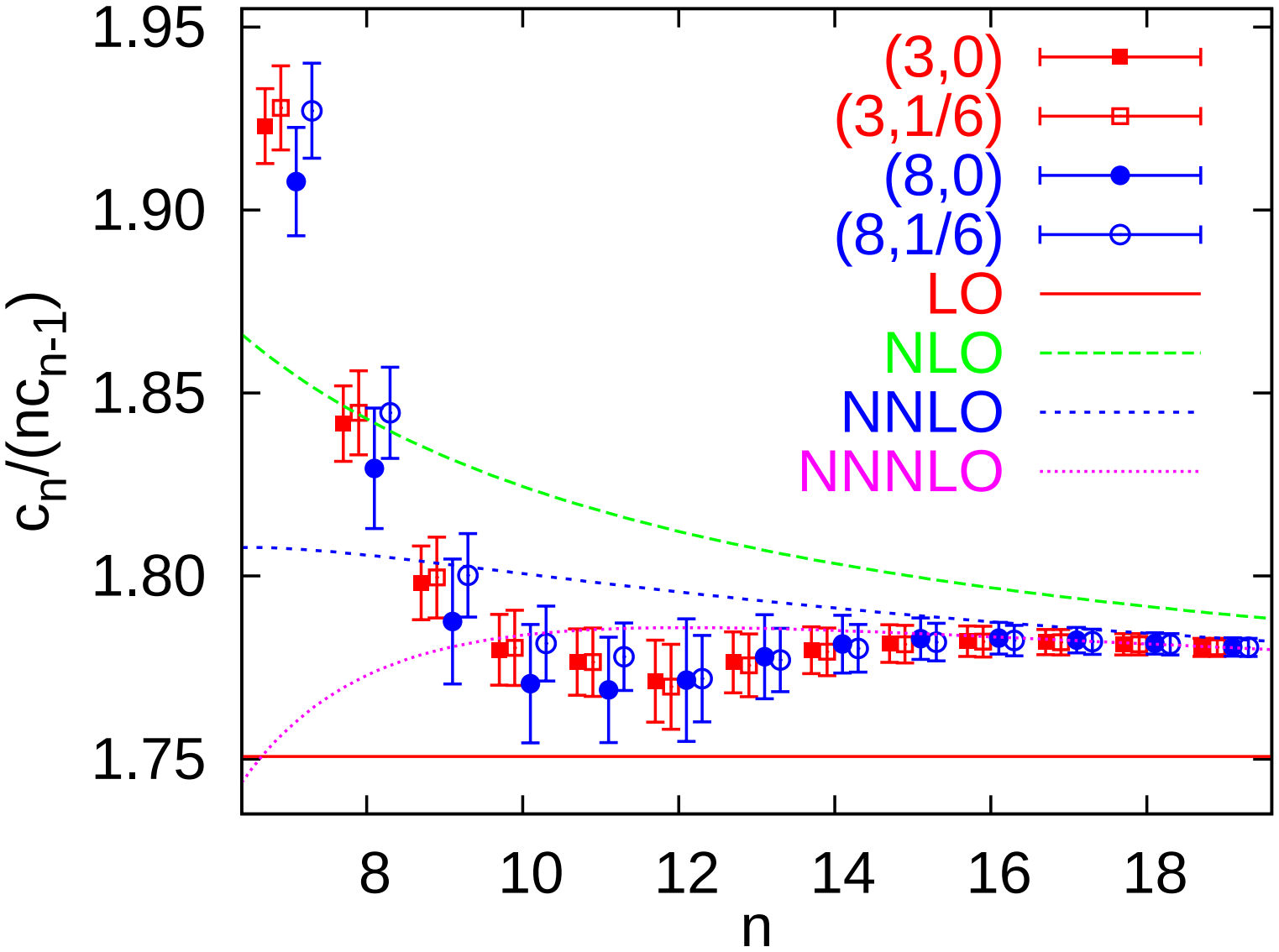

Figure: The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet static self-energies, compared to the prediction for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion.

N_m

$$c_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right).$$

$$f_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right).$$

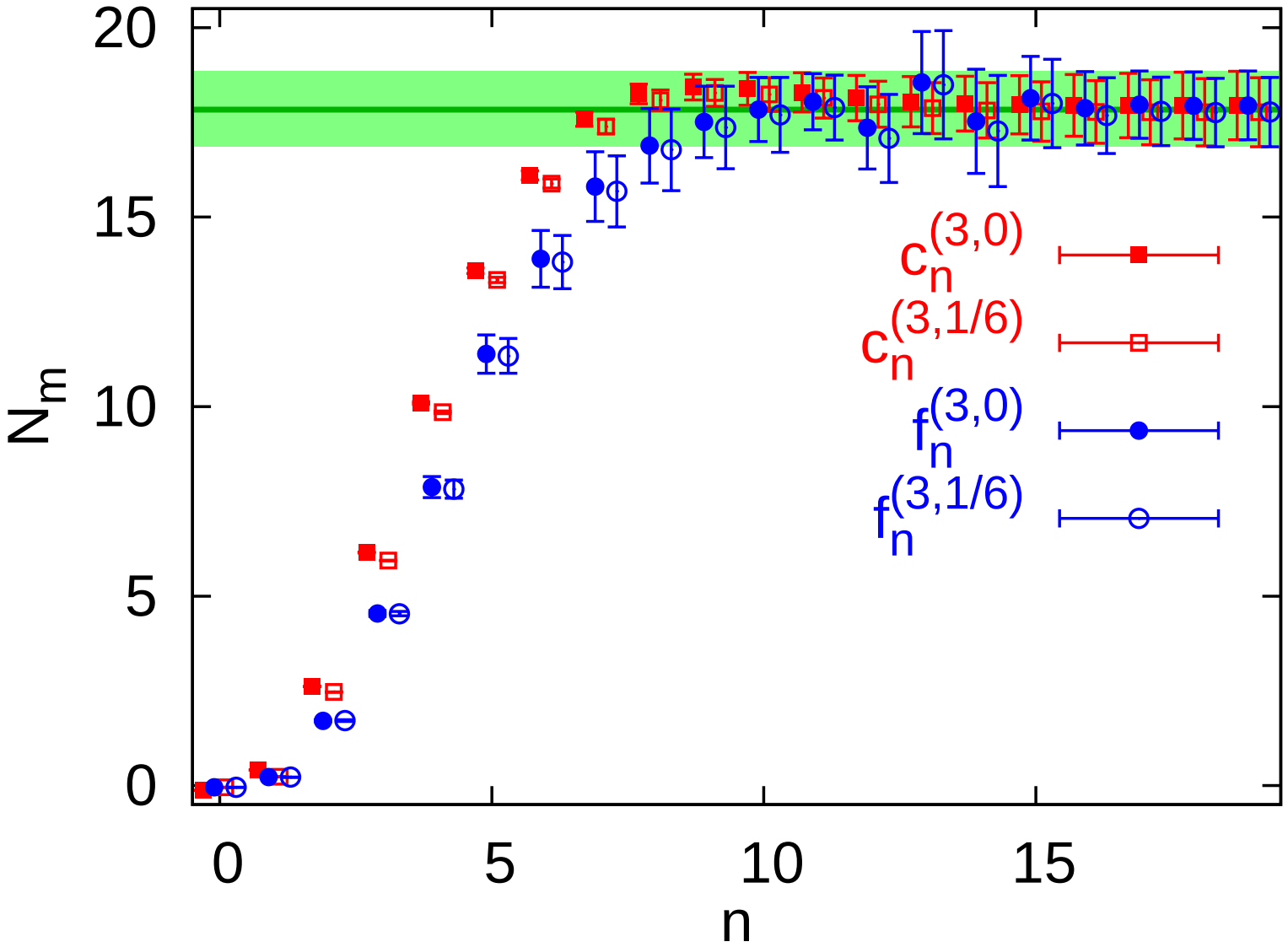

Figure: N_m^{latt} determined via r_n truncated at NNNLO, from the coefficients $c_n^{(3,0)}$, $c_n^{(3,1/6)}$, $f_n^{(3,0)}$ and $f_n^{(3,1/6)}$. The horizontal band is our final result: $N_m = 17.9(1.0)$