Heavy quark diffusion from the lattice

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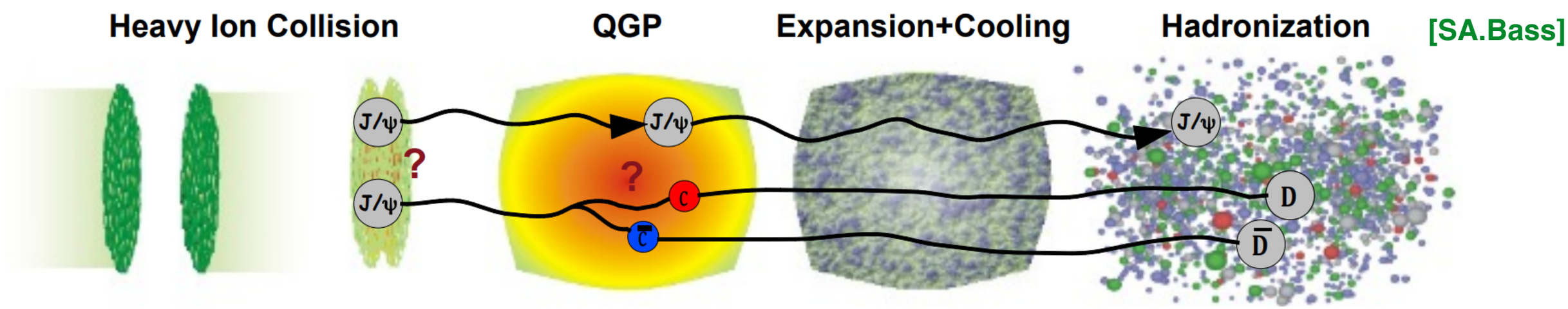
XVth Quark Confinement and the Hadron Spectrum

Stavanger, Norway, 1-6 Aug 2022

Heavy quarks in heavy ion collisions

Heavy quarkonia are produced only in the early stage of collisions



- Some remain as bound state in the whole evolution
- Some release constituents and travel through QGP, **thermalize via diffusion**

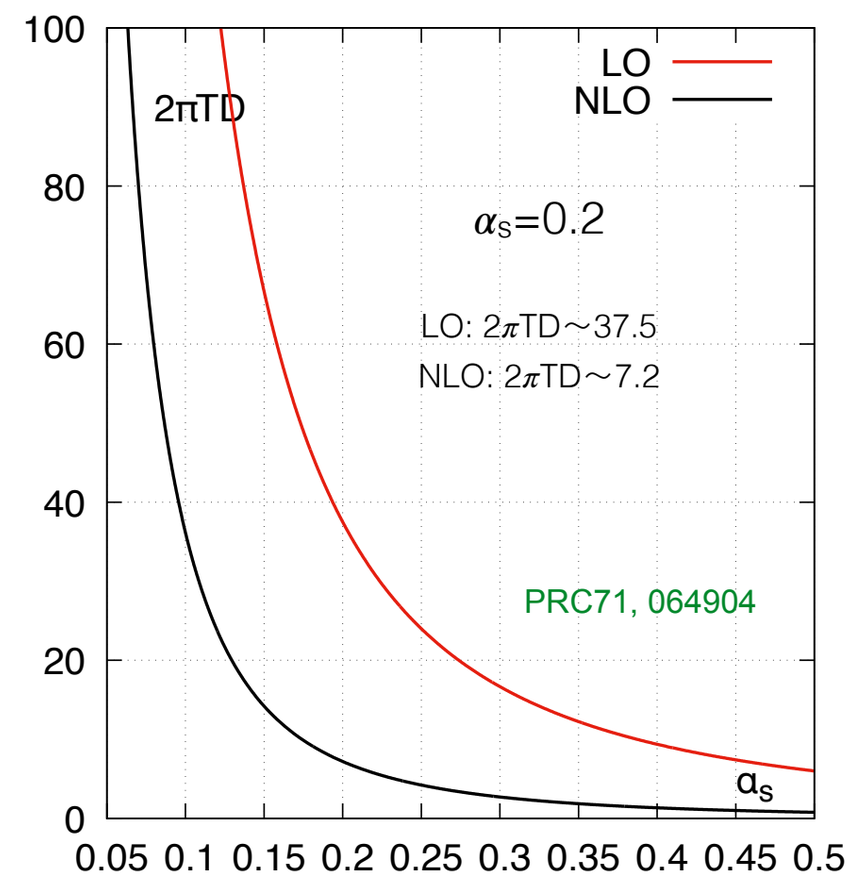
Q 1: At what temperature do quarkonia dissociate?

see J-I. Skullerud's talk on Mon. at 14:40-15:00 in Track D

Q 2: How fast do heavy quarks thermalize in QGP?

$$\tau_{\rm kin} = \eta_D^{-1} \qquad \eta_D = \frac{T}{M_{\rm kin} D}$$

- Heavy quark diffusion can be calculated perturbatively
- But the convergence at interested coupling is poor



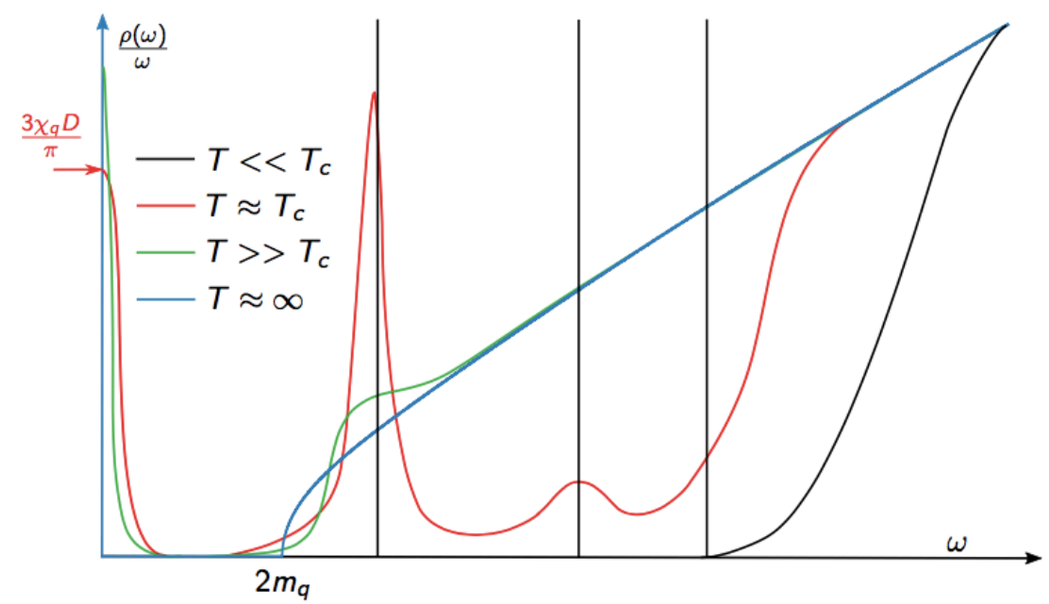
Heavy quark diffusion from lattice QCD

- Lattice QCD provides non-perturbative determination from first principle
- Heavy quark diffusion determination requires spectra extracted from lattice correlators

  - ✳ Deformation of SPF
    —> dissociation temperature

  - ✳ Transport peak:
    —> heavy quark diffusion coefficient

$$D = \frac{1}{3\chi_q} \lim_{\omega \to 0} \frac{\rho_{ii}^{trans}(\omega)}{\omega}$$



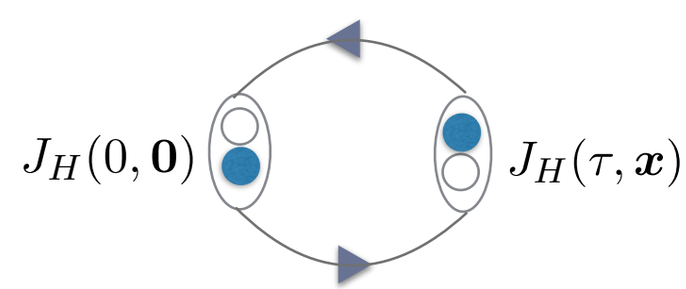
- Analytic continuation and spectral reconstruction

see A. Rothkophf's talk on Mon. at 9:30-10:00

$$G_H(\tau, \vec{p}) = \sum_{x,y,z} exp(-i\,\vec{p}\cdot\vec{x})\langle J_H(0,\vec{0}) J_H^\dagger(\tau,\vec{x}) = \int \frac{d\omega}{2\pi} \frac{\cosh(\omega(\tau T - 1/2)/T)}{\sinh(\omega/2T)} \rho_H(\omega, \vec{p}, T)$$

inversion methods with prior information

- ✳ Backus-Gilbert Method — B. B. Brandt, et al., PRD93, 054510(2016)
- ✳ Maximum Entropy Method — M. Asakawa, et al., PPNP. 46(2001) 445-508
- ✳ New Bayesian Method — Y. Burnier and A. Rothkopf, PRL 111,18,182003
- ✳ Stochastic Approaches — HTS, et al., PRD97, 094503
- ✳ Machine learning — H.-T. Ding, et al., 2110.13521
- ✳ …

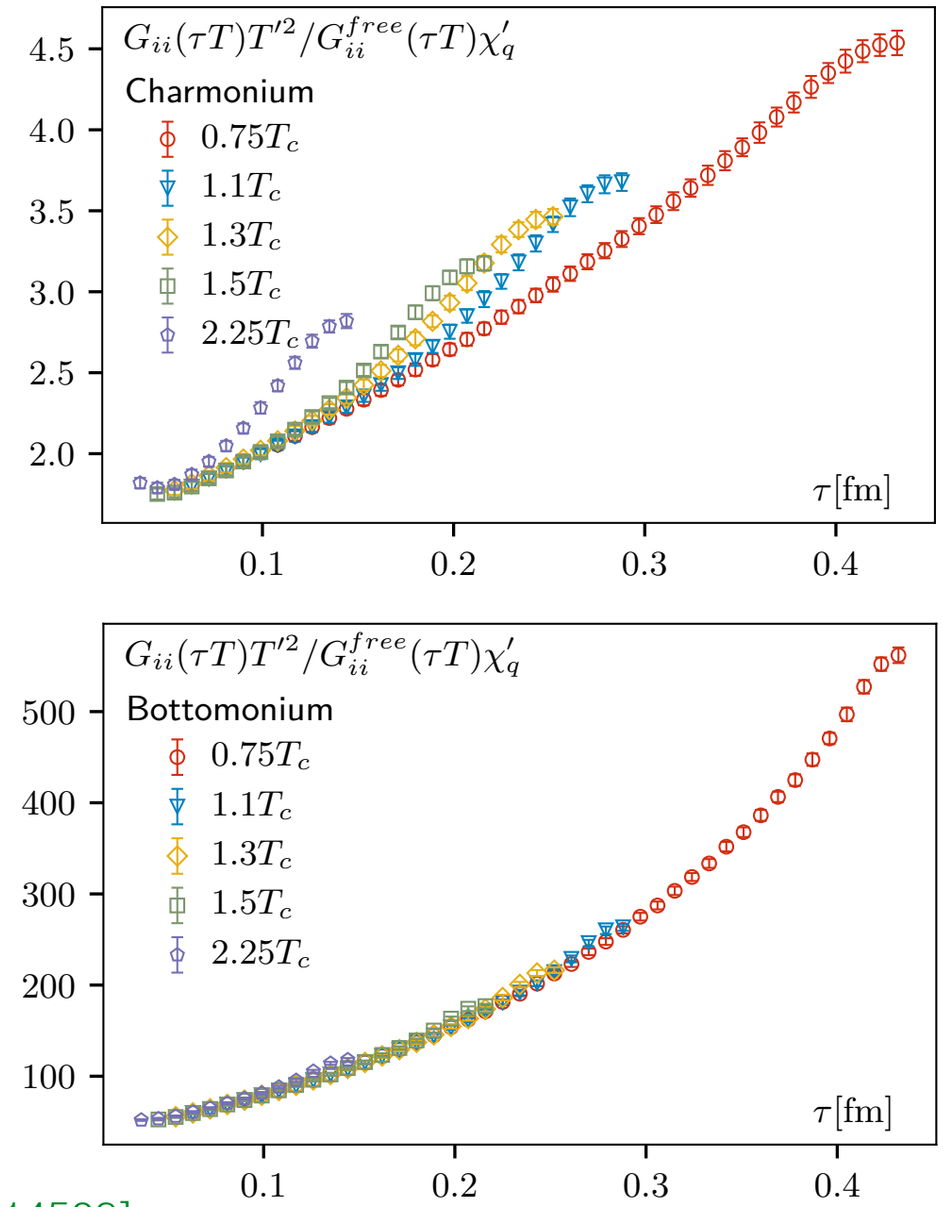


Outline

- Heavy quark diffusion coefficient *D*
  - ✳ Charm quark diffusion from charmonium correlators
  - ✳ Bottom quark diffusion from bottomonium correlators
- Heavy quark momentum diffusion coefficient *kappa*
  - ✳ Leading contributions *kappa_E* from color-electric field correlators
  - ✳ Subleading contributions *kappa_B* from color-magnetic field correlators

Charmonium and bottomonium correlators

β	r_0/a	a[fm](a^{-1}[GeV])	N_σ	N_τ	T/T_c	# confs
7.192	26.6	0.018(11.19)	96	48	0.75	237
				32	1.1	476
				28	1.3	336
				24	1.5	336
				16	2.25	237
7.394	33.8	0.014(14.24)	120	60	0.75	171
				40	1.1	141
				30	1.5	247
				20	2.25	226
7.544	40.4	0.012(17.01)	144	72	0.75	221
				48	1.1	462
				42	1.3	660
				36	1.5	288
				24	2.25	237
7.793	54.1	0.009(22.78)	192	96	0.75	224
				64	1.1	291
				56	1.3	291
				48	1.5	348
				32	2.25	235



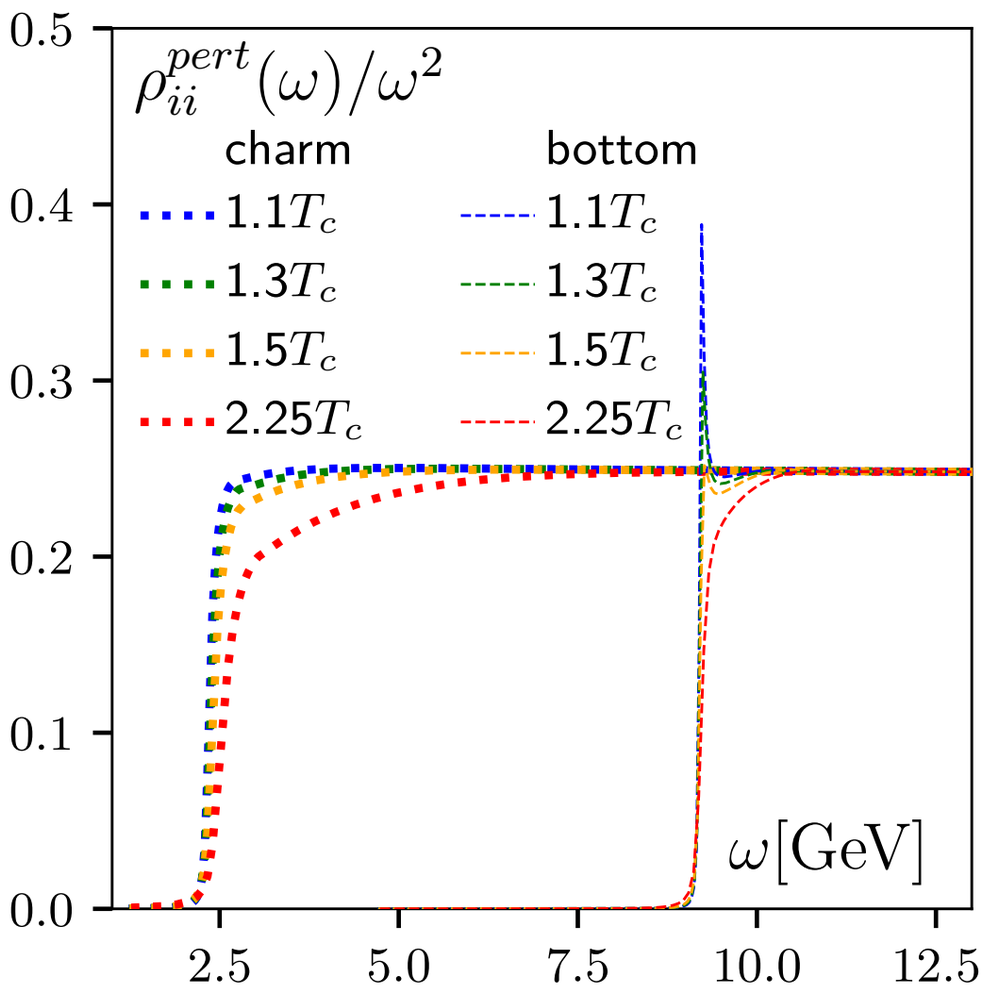
[HTS et al., PRD104(2021) 11, 114508]

- Charmonium and bottomonium correlators at physical mass in quenched approximation
- Extrapolation to continuum limit based on large&fine lattices
- Temperature from $1.1T_c$ - $2.25T_c$

Perturbative spectral function

- pNRQCD calculations applicable around the threshold [M. Laine, JHEP05(2007)028]
- Ultraviolet asymptotics valid well above the threshold [Y. Burnier et al., EPJC72, 1902(2012)]
- Combine two parts by interpolation: [Y. Burnier et al., JHEP11(2017)206]

$$\rho_V^{pert}(\omega) = A^{match}\Phi(\omega)\rho_V^{\mathrm{pNRQCD}}(\omega)\theta(\omega^{match}-\omega) + \rho_V^{vac}(\omega)\theta(\omega-\omega^{match})$$



[HTS et al., PRD104(2021) 11, 114508]

- Subtract the perturbative contribution

$$\rho_{ii}^{mod}(\omega) = A\rho_V^{pert}(\omega - B)$$

$$\rho_{ii}(\omega) = \rho_{ii}^{trans}(\omega) + \rho_{ii}^{mod}(\omega)$$

- Modeling the transport peak:

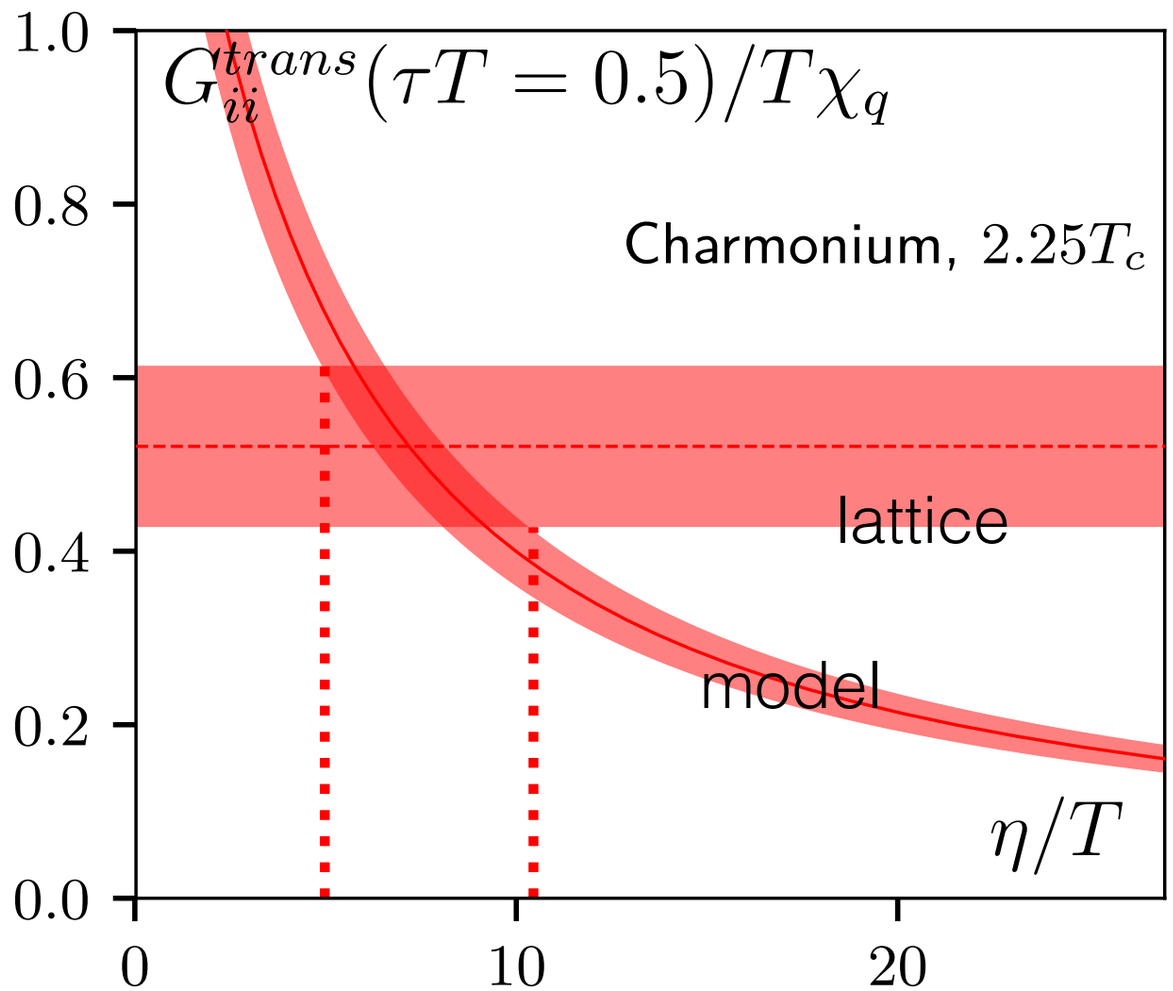
$$G_{trans}(\tau) = \int \frac{d\omega}{\pi} K(\omega,\tau)\rho_{ii}^{trans}(\omega)$$

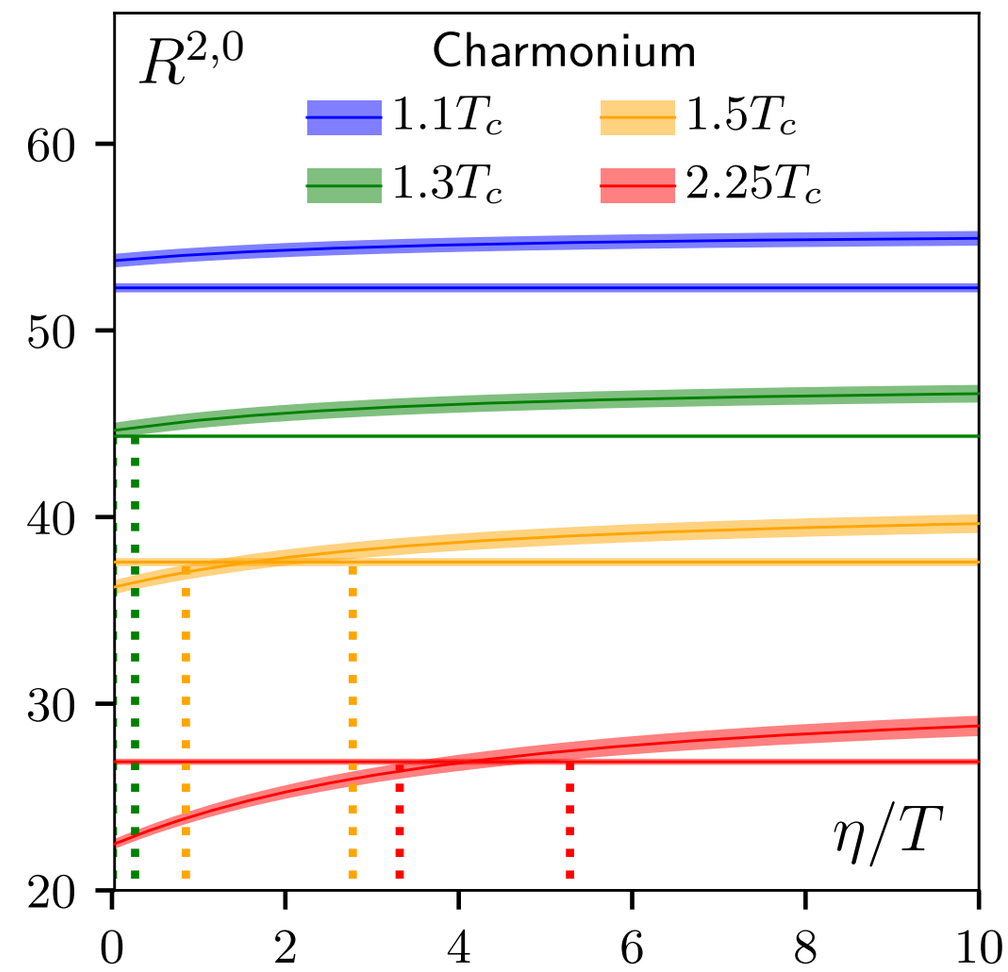
$$\rho_{ii}^{trans}(\omega) = 3\chi_q D \frac{\omega\eta^2}{\omega^2+\eta^2}$$

Fix the transport peak

- Transport peak plays its most significant role at midpoint
- Calculate midpoint correlator by integrating Lorentzian ansatz with varying eta
- Compare with lattice data and find range for eta from intersections $\eta_D = \frac{T}{M_{\rm kin} D}$

[HTS et al., PRD104(2021) 11, 114508]

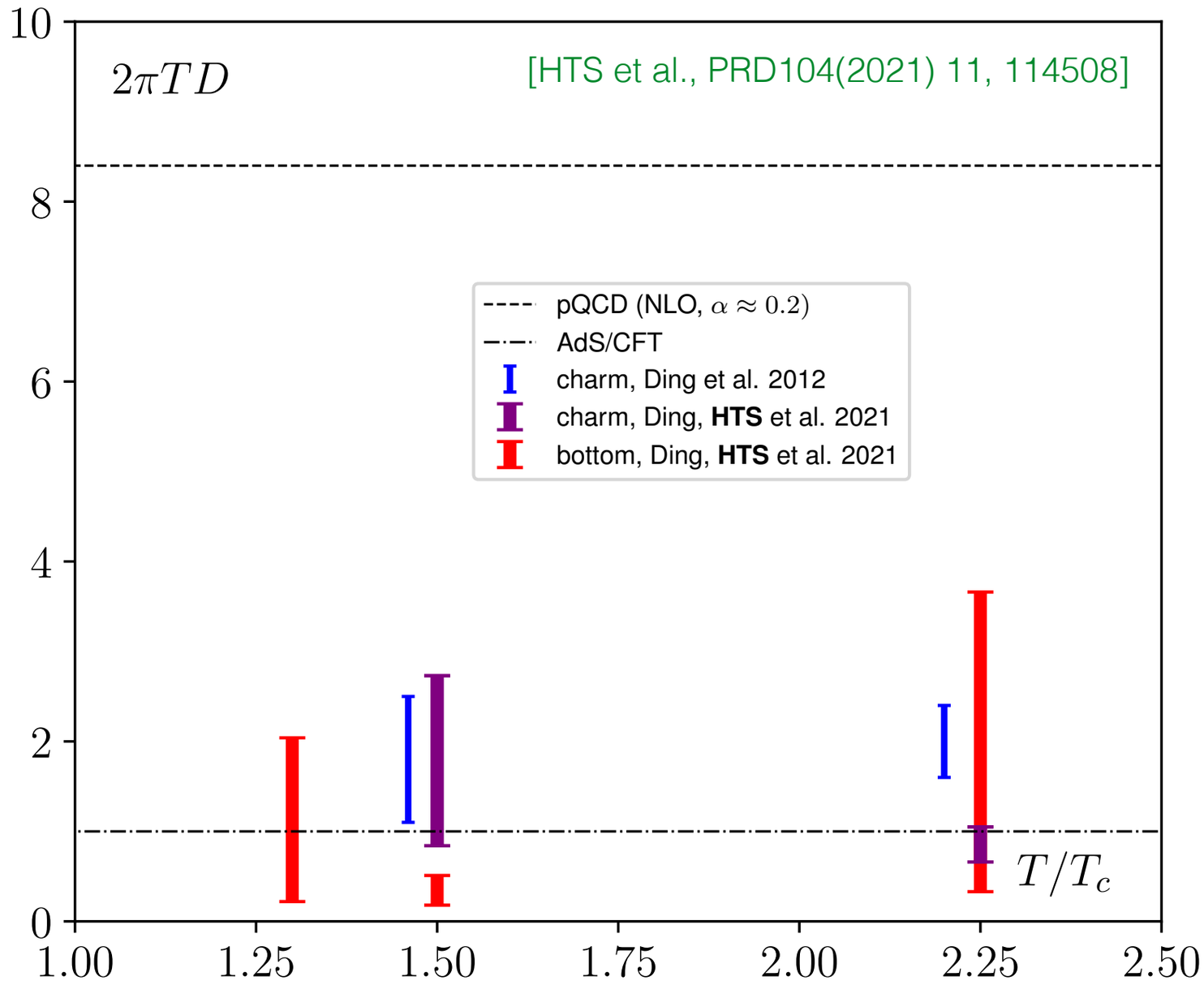




$$G_{trans}(\tau) = \int \frac{d\omega}{\pi} K(\omega, \tau) \rho_{ii}^{trans}(\omega)$$

$$R^{2,0}(A, B, \eta) = \frac{G_{mod}^{(2)}(A, B) + G_{trans}^{(2)}(\eta)}{G_{ii}^{mod}(\tau T = 0.5) + G_{ii}^{trans}(\tau T = 0.5)}$$

Charm&bottom quark diffusion coefficient



- Consistent with results from AdS/CFT and previous lattice study at close-to-charm quark mass at finite lattice spacing
- Lattice determinations favor the AdS/CFT calculations

Heavy quark momentum diffusion

- Construct a kinetic mass dependent momentum diffusion coefficient

$$\kappa^{(M)} \equiv \frac{M^2\omega^2}{3T\chi_q}\sum_i \frac{2T\rho_V^{ii}(\omega)}{\omega}\Big|^{\eta\ll|\omega|\lesssim\omega_{\rm UV}} \quad + \quad D = T/(\eta M) \quad \Longrightarrow \quad D = \frac{2T^2}{\kappa^{(M)}}$$

- Large quark mass limit in effective field theory [S. Caron-Huot et al., JHEP 0904 (2009) 053]

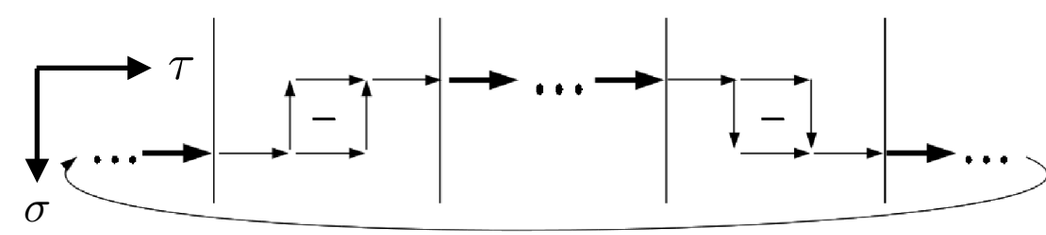
$$\kappa \equiv \frac{\beta}{3}\sum_{i=1}^{3}\lim_{\omega\to 0}\left[\lim_{M\to\infty}\frac{M^2}{\chi_q}\int_{-\infty}^{\infty}\mathrm{d}t\; e^{i\omega(t-t')}\int \mathrm{d}^3\vec{x}\,\left\langle\frac{1}{2}\{\mathcal{F}^i(t,\vec{x}),\mathcal{F}^i(0,\vec{0})\}\right\rangle\right] \qquad \mathcal{F}^i \equiv M\frac{\mathrm{d}\mathcal{J}^i}{\mathrm{d}t}$$

- Carry out large quark mass limit for the operators

$$\kappa \equiv \frac{\beta}{3}\sum_{i=1}^{3}\lim_{\omega\to 0}\left[\lim_{M\to\infty}\frac{M^2}{\chi_q}\int_{-\infty}^{\infty}\mathrm{d}t\int \mathrm{d}^3\vec{x}\,\left\langle\frac{1}{2}\{\left[\hat{\phi}^\dagger gE^i\hat{\phi}-\hat{\theta}^\dagger gE^i\hat{\theta}\right](t,\vec{x}),\left[\hat{\phi}^\dagger gE^i\hat{\phi}-\hat{\theta}^\dagger gE^i\hat{\theta}\right](0,\vec{0})\}\right\rangle\right]$$

- Perform analytic continuation and discretize the operator on the lattice

$$G_{EE}(\tau) = -\frac{1}{3}\sum_{ii=1}^{3}\frac{\langle \mathrm{Re}\ \mathrm{Tr}[U(\beta,\tau)gE_i(\tau,\vec{0})U(\tau,0)gE_i(0,\vec{0})]\rangle}{\langle \mathrm{Re}\ \mathrm{Tr}[U(\beta,0)]\rangle}$$

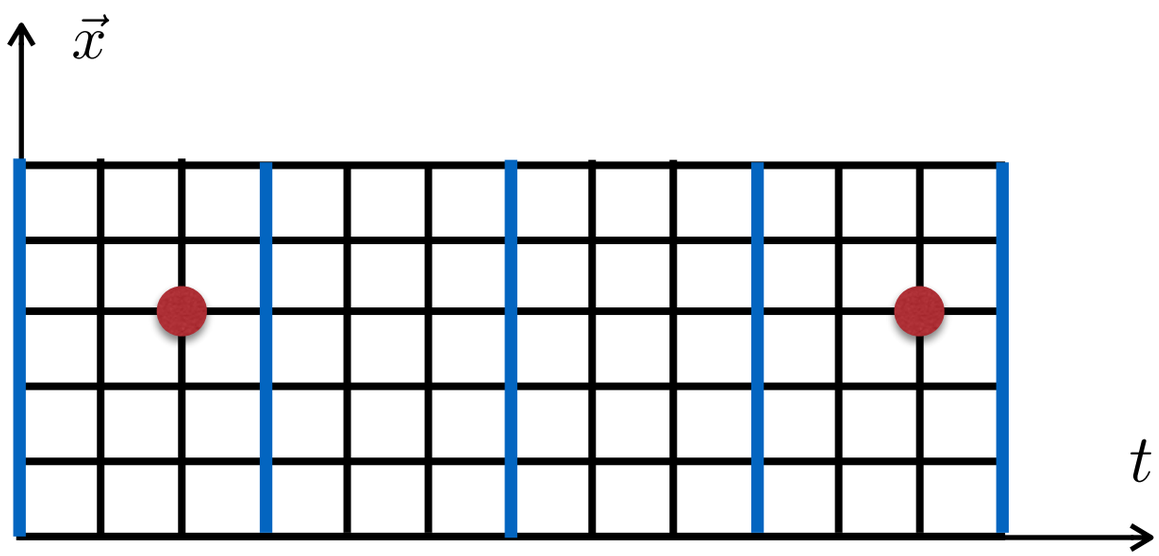


$$\Rightarrow \quad \kappa = \lim_{\omega\to 0} 2T\frac{\rho(\omega)}{\omega}$$

- Correlators cheap to measure on the lattice
- Less structure in spectral functions (no transport peak and resonance peak)

Multi-level algorithm

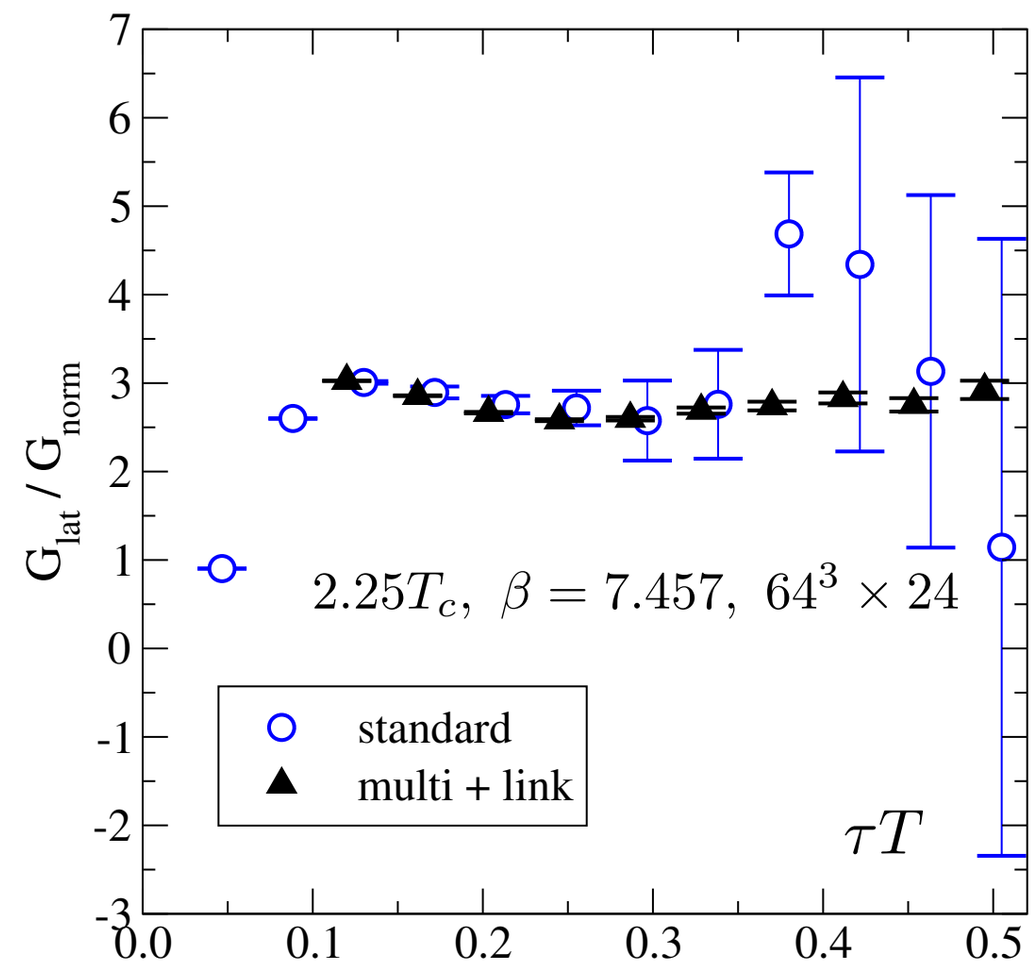
A sketch of multi-level algorithm



Independent updates in each sub-lattice followed by a measuring of operator

[M. Luscher and P. Weisz, JHEP 09 (2001) 010]

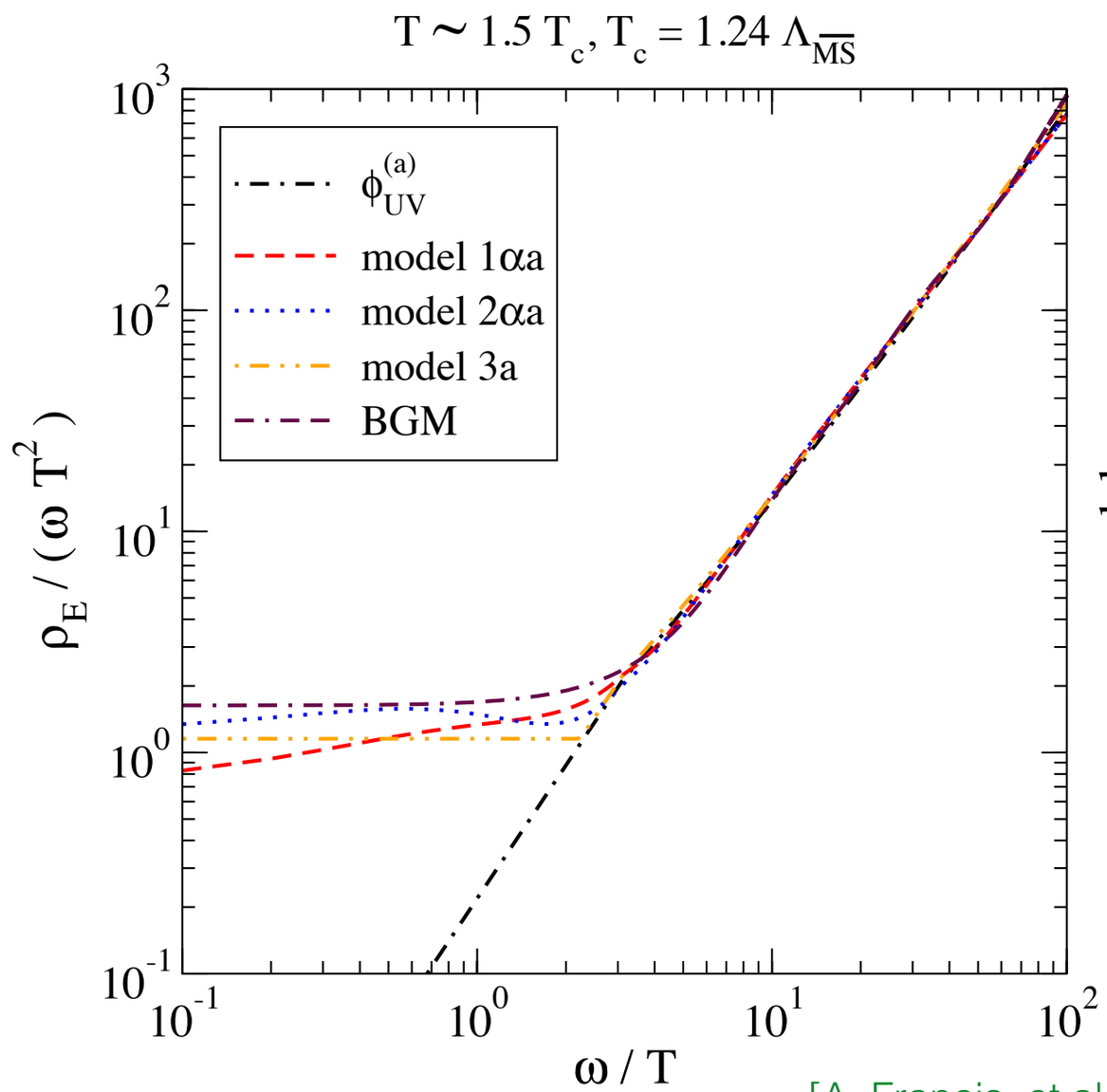
G_{EE} from multi-level and link-integration:

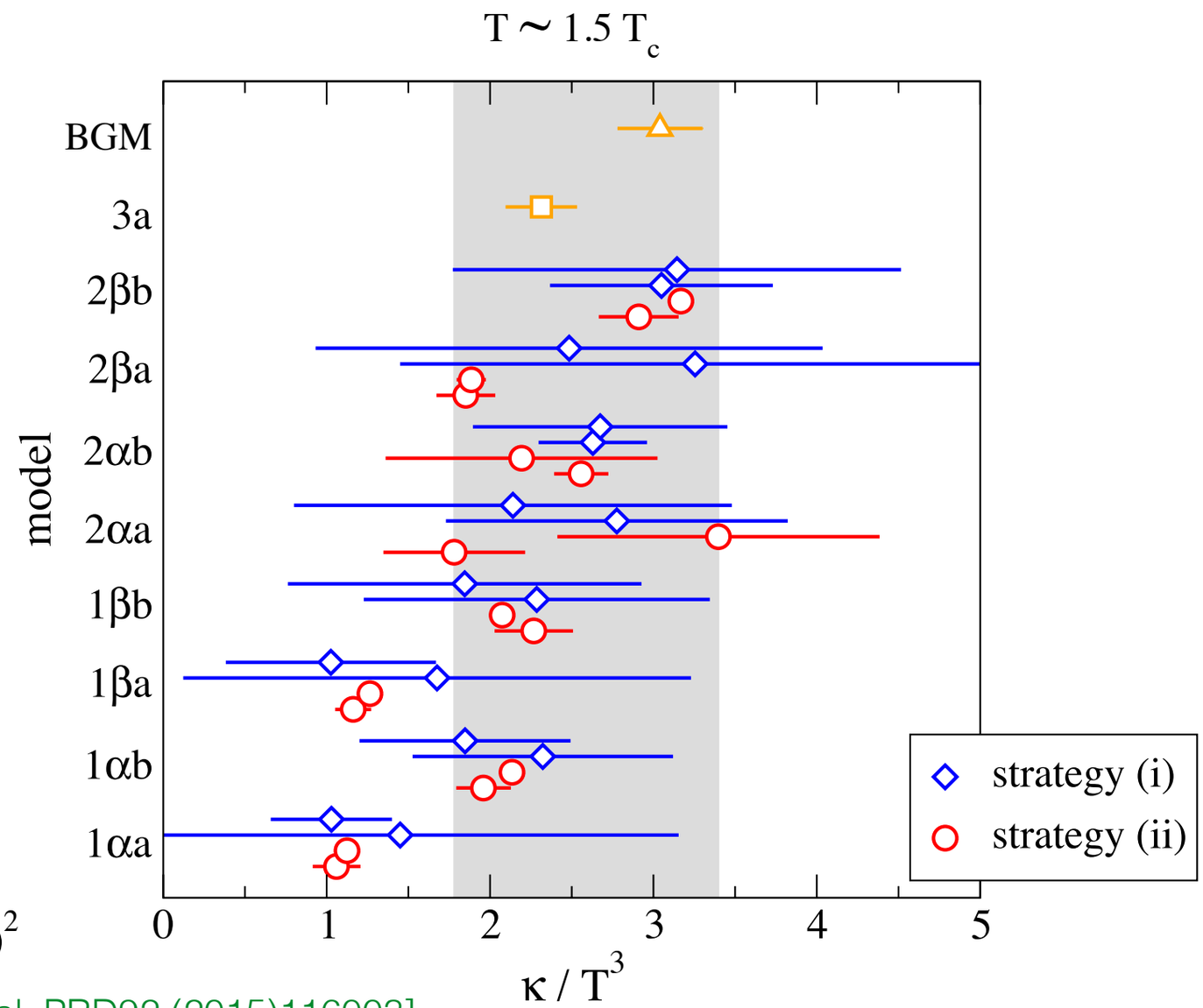


[A. Francis, et al, PRD92 (2015)116003]

- Multi-level method reduces noise in correlators
- Multi-level is only applicable in quenched approximation

kappa from multilevel method





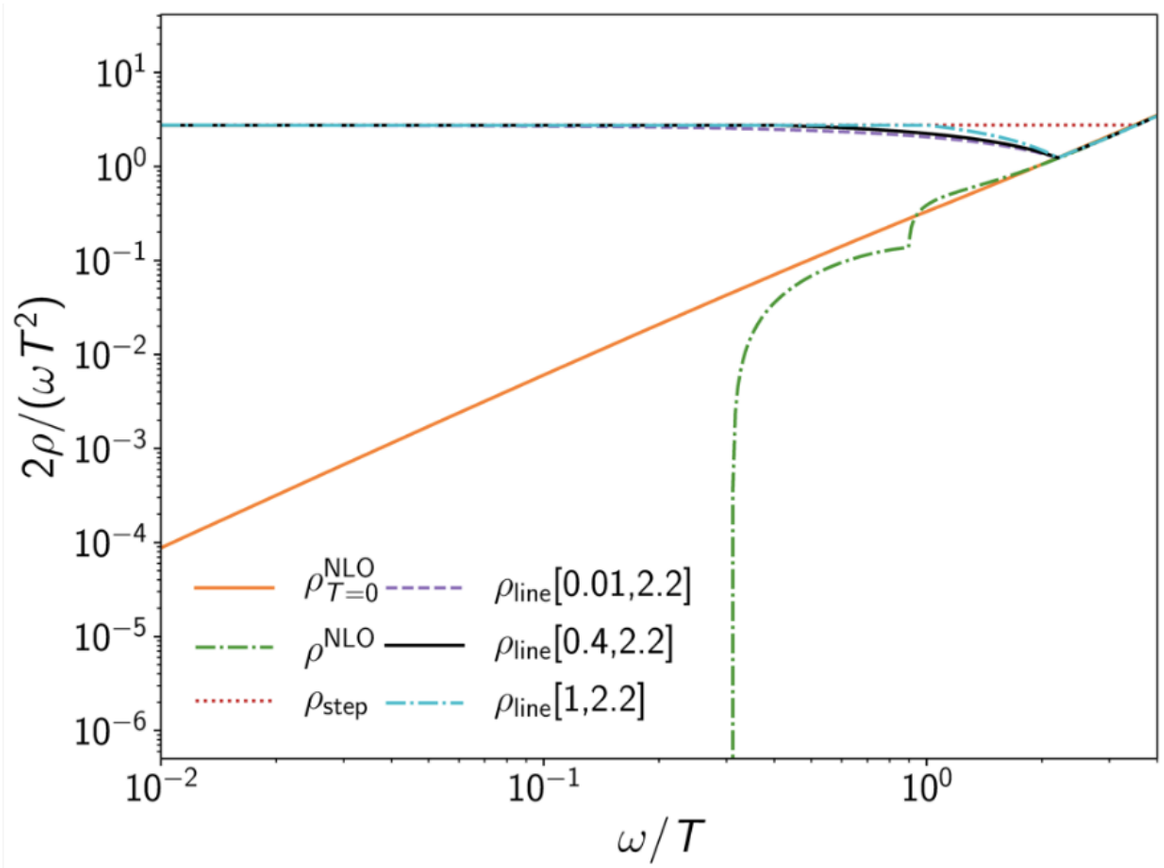
[A. Francis, et al, PRD92 (2015)116003]

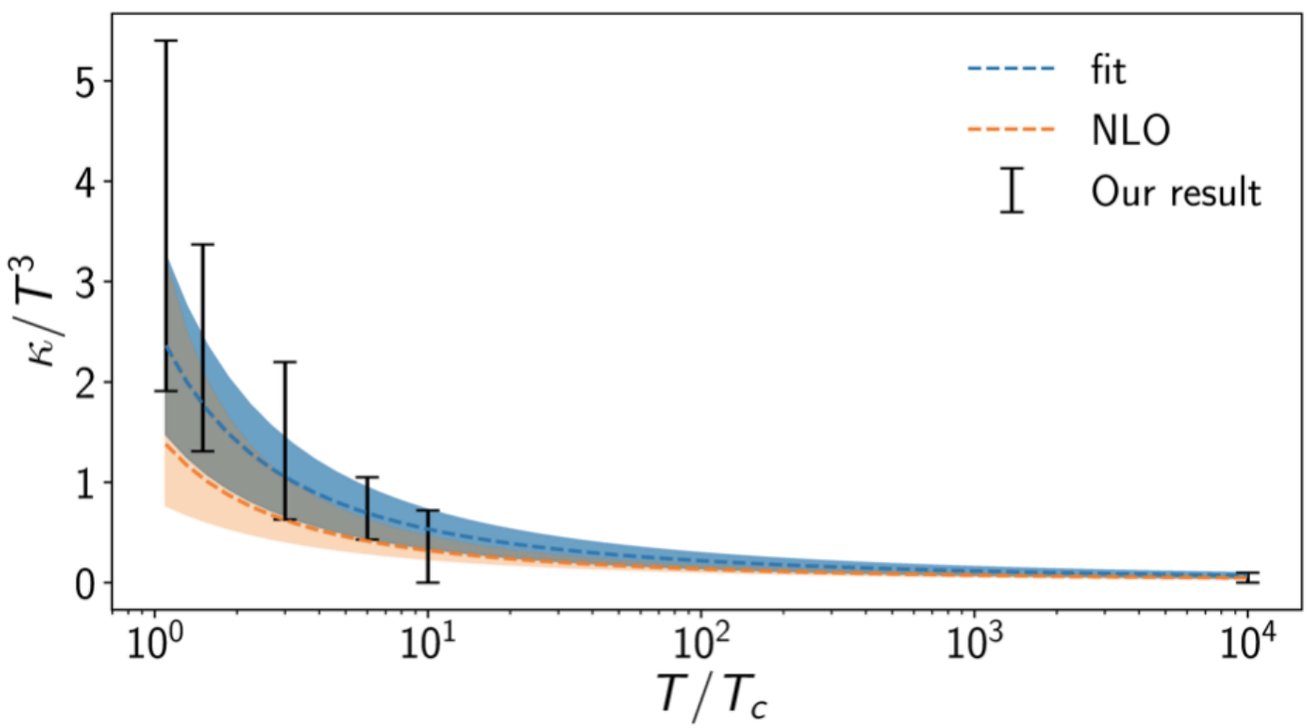
- Models with correct asymptotic behavior
- Modeling corrections to IR part by a power series in frequency

$$\phi_{UV}(\omega) \equiv \frac{g^2(\bar{\mu}_\omega) C_F \omega^3}{6\pi}\left[1 + \mathcal{O}(g^2(\bar{\mu}_\omega)\right]$$

$$\phi_{IR}(\omega) \equiv \frac{\kappa\omega}{2T}$$

Temperature dependence of kappa





[N. Brambilla, et al, PRD102 (2020) 7, 074503]

$$\rho_{\rm line}(\omega,T) = \rho^{\rm IR}(\omega,T)\theta(\omega^{\rm IR}-\omega) + \left[\frac{\rho^{\rm IR}(\omega,T)-\rho^{\rm UV}(\omega,T)}{\omega^{\rm IR}-\omega^{\rm UV}}\left(\omega-\omega^{\rm IR}\right)+\rho^{\rm IR}(\omega,T)\right]\times$$
$$\theta(\omega-\omega^{\rm IR})\theta(\omega^{\rm UV}-\omega)+\rho^{\rm UV}(\omega,T)\theta(\omega-\omega^{\rm UV})$$

$$\rho_{\rm step}(\omega,T) = \rho^{\rm IR}(\omega,T)\,\theta(\Lambda-\omega)+\rho^{\rm UV}_{T=0}(\omega,T)\,\theta(\omega-\Lambda)$$

similar study by
[D. Banerjee, et al, arXiv: 2204.14075]

- Wide temperature range from 1.1T_c to 1e4T_c
- Model of linear connection of IR and UV part
- Model of hard switch between IR and UV part

Gradient flow

- A diffusion equation along flow time *t* towards the stationary point of the action:

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} \sim -\frac{\delta S_G[B_\mu(x,t)]}{\delta B_\mu(x,t)} \sim D_\nu G_{\nu\mu}(x,t) \qquad B_\nu(x,t)|_{t=0} = A_\nu(x)$$

- LO solution shows a "smearing radius":

[Luscher & Weisz, JHEP1102(2011)051]
[Narayanan & Neuberger, JHEP0603(2006)064]

$$B_\nu^{\mathrm{LO}}(x,t) = \int \mathrm{d}y (\sqrt{2\pi}\sqrt{8t}/2)^{-4} \exp\left(\frac{-(x-y)^2}{\sqrt{8t}^2/2}\right) B_\nu(y)$$

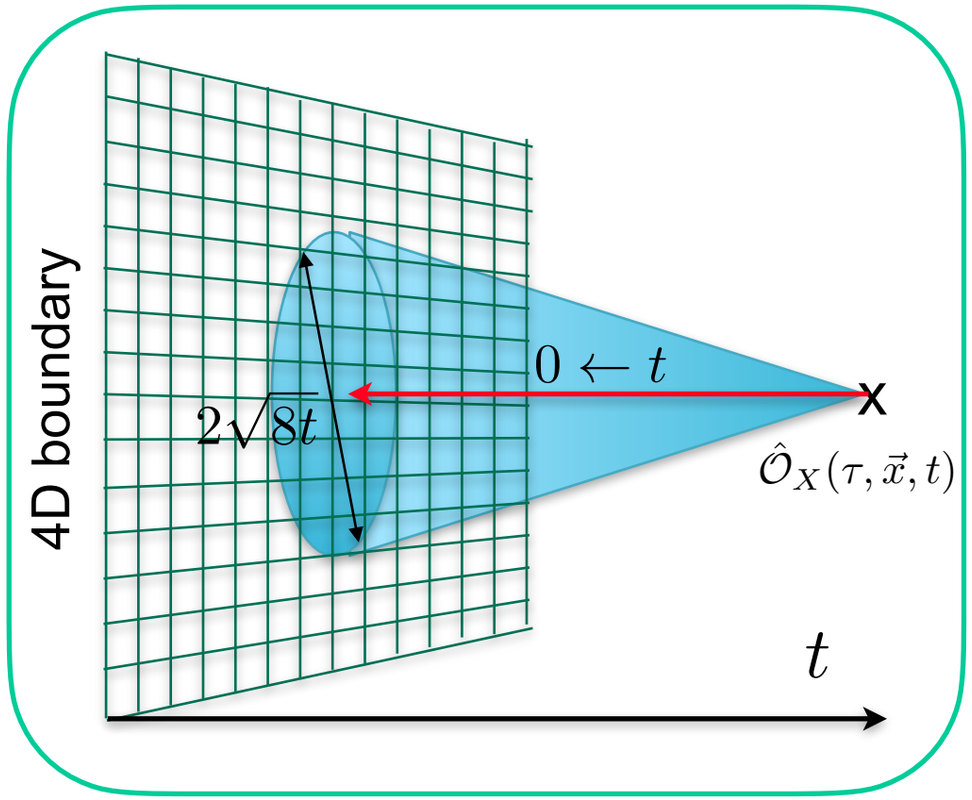
- Small flow time expansion of operators:

$$\mathcal{O}(x,t) \xrightarrow{t\to 0} \sum_k c_k(t)\mathcal{O}_k^R(x)$$

- Continuum extrapolation followed by flow-time-to-zero extrapolation to obtain "correct" physics

- Applications:

  running coupling / defining operators / scale setting / noise reduction / topo. charge / ...



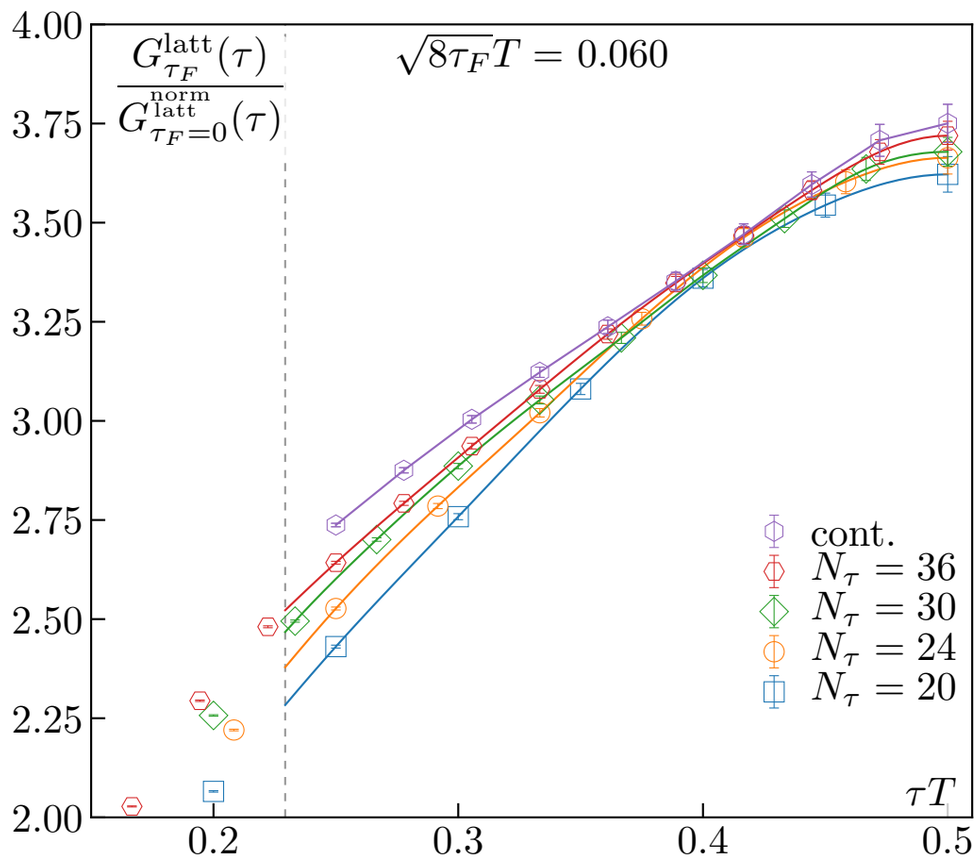
For introduction of gradient flow, see A. Shindler's talk on Mon. at 10:00-10:30

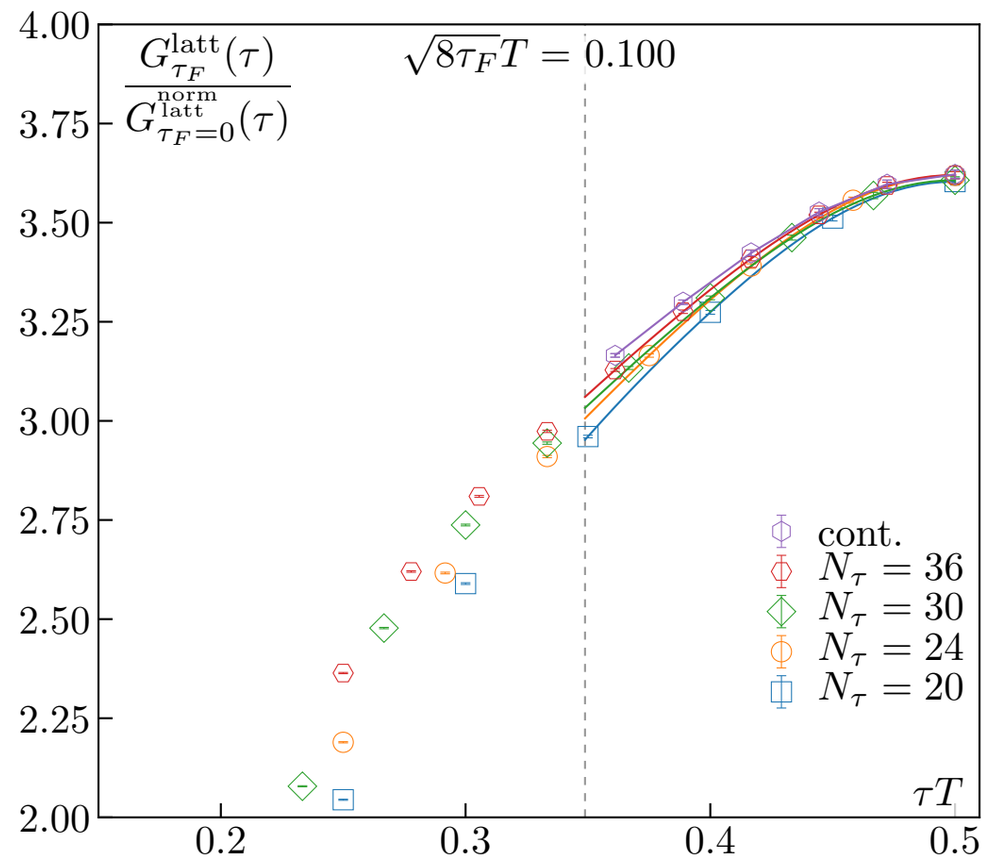
kappa from gradient flow: quenched case

β	$a[\mathrm{fm}](a^{-1}[\mathrm{GeV}])$	N_σ	N_τ	T/T_c	#confs.
6.8736	0.026 (7.496)	64	16	1.50	10000
7.0350	0.022 (9.119)	80	20	1.50	10000
7.1920	0.018 (11.19)	96	24	1.50	10000
7.3940	0.014 (14.21)	120	30	1.50	10000
7.5440	0.012 (17.01)	144	36	1.50	10000

- Quenched approximation on large, fine, isotropic lattices
- High precision for reliable $a \to 0$ extrapolation and $t \to 0$ extrapolation

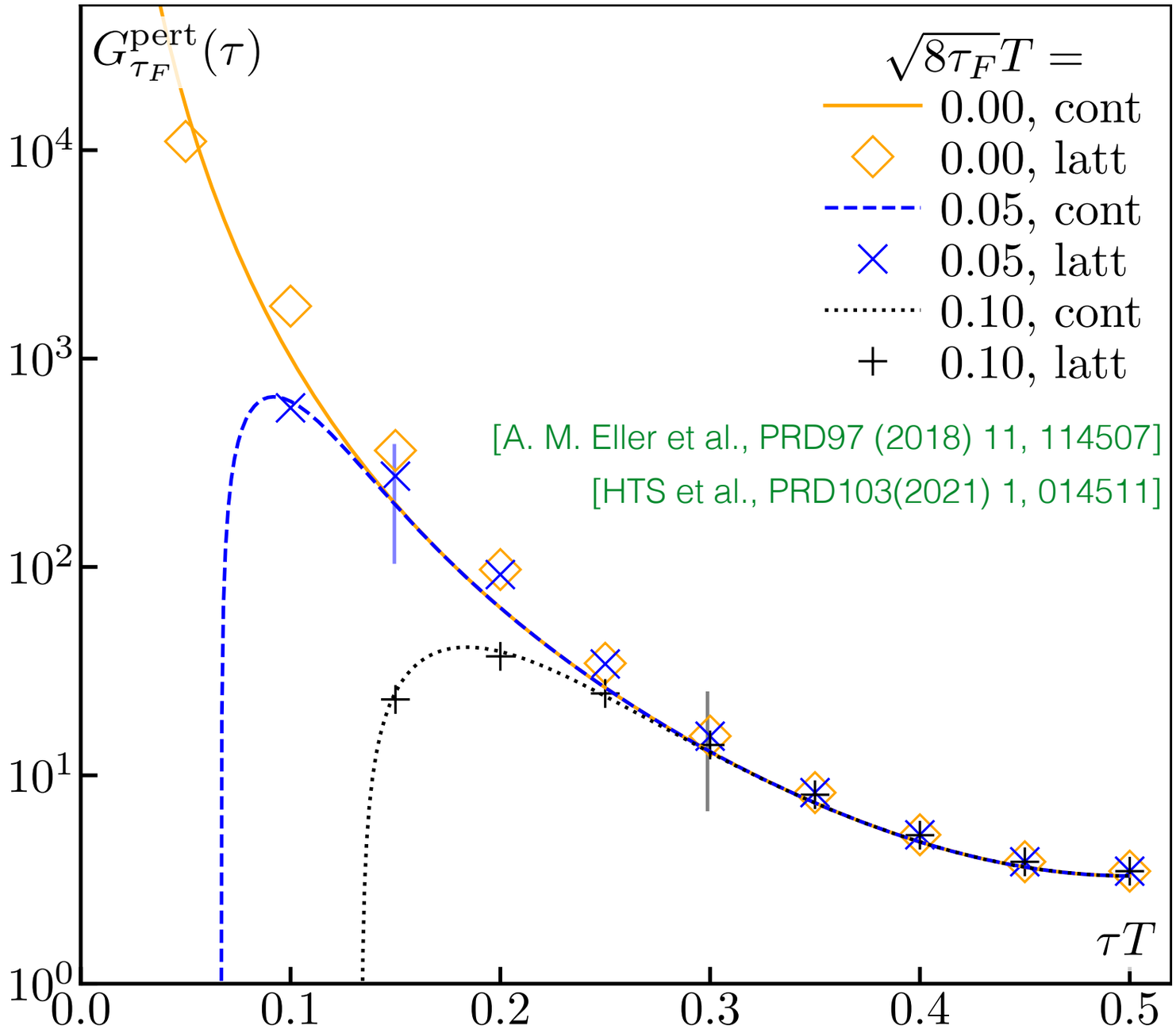
[HTS et al., PRD103(2021) 1, 014511]





- Gradient flow reduces the noise in correlators
- Gradient flow removes the lattice effects (disordering)
- Need proper flow time range

kappa from gradient flow: flow time limit



- Flow destroys the signal at small distances
- Large distance parts are not affected
- More points are destroyed at larger flow times
- At most 1% deviation of flowed correlators from unflowed ones determines maximum flow time:

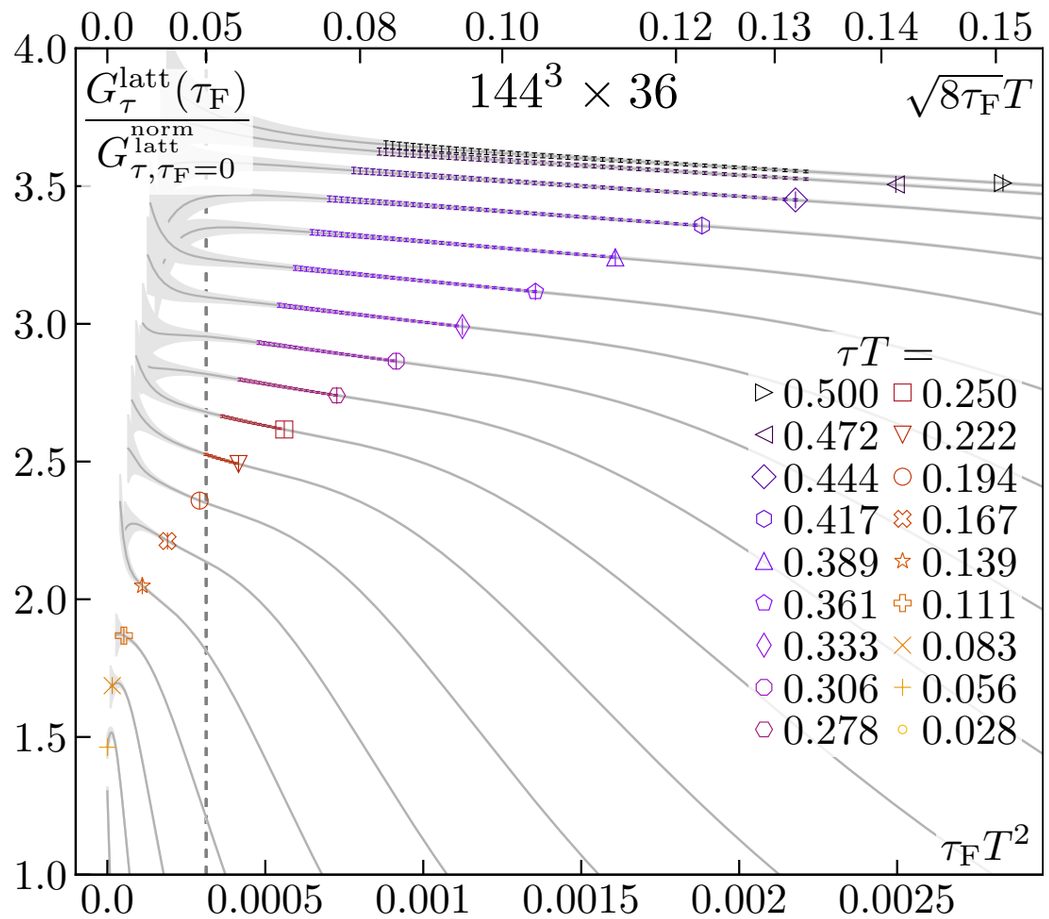
$$a \lesssim \sqrt{8t} \lesssim \frac{\tau - a}{3}$$

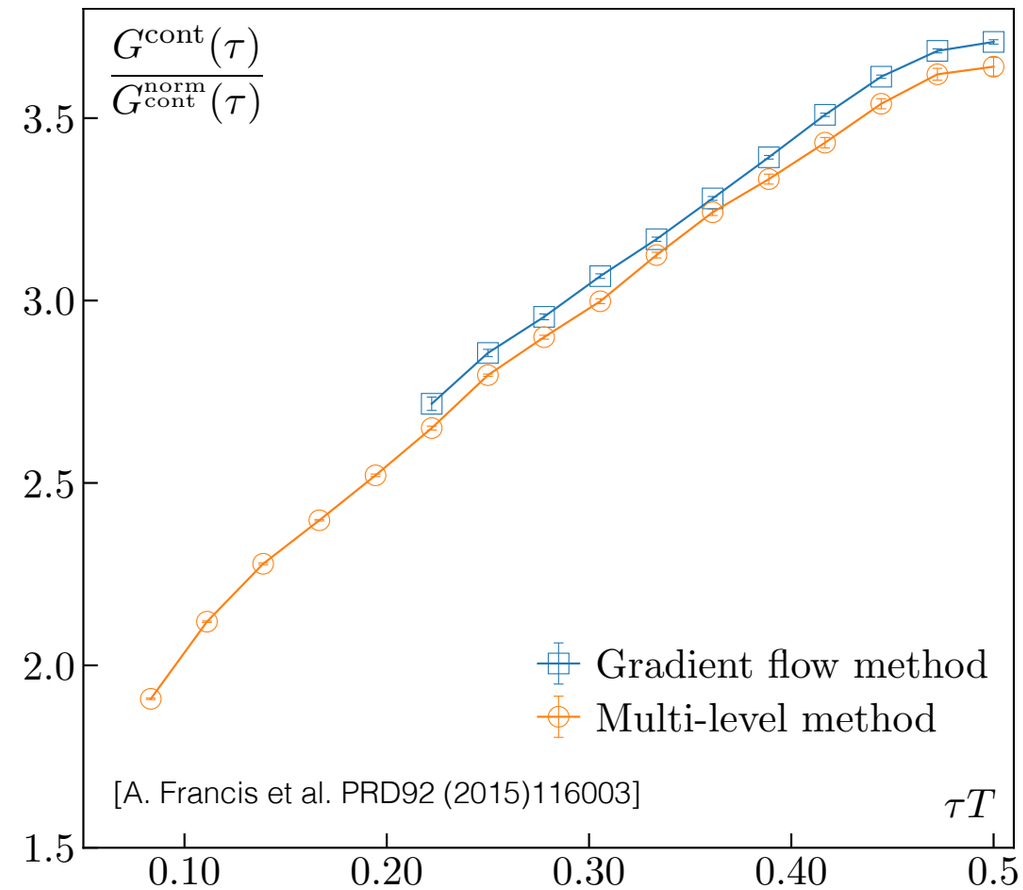
a : to suppress lattice effects

a : lattice version of the perturbative flow time limit

kappa from gradient flow: double extrapolation

[HTS et al., PRD103(2021) 1, 014511]





- continuum extrapolation: $G_{\tau,\tau_F}(N_\tau) = \frac{m}{N_\tau^2} + G^{\rm cont}_{\tau,\tau_F}$
- flow-time-to-zero extrapolation: $G(\tau_F) = a \cdot \tau_F + b$

similar study by
[N. Brambilla, et al, arXiv: 2206.02861]

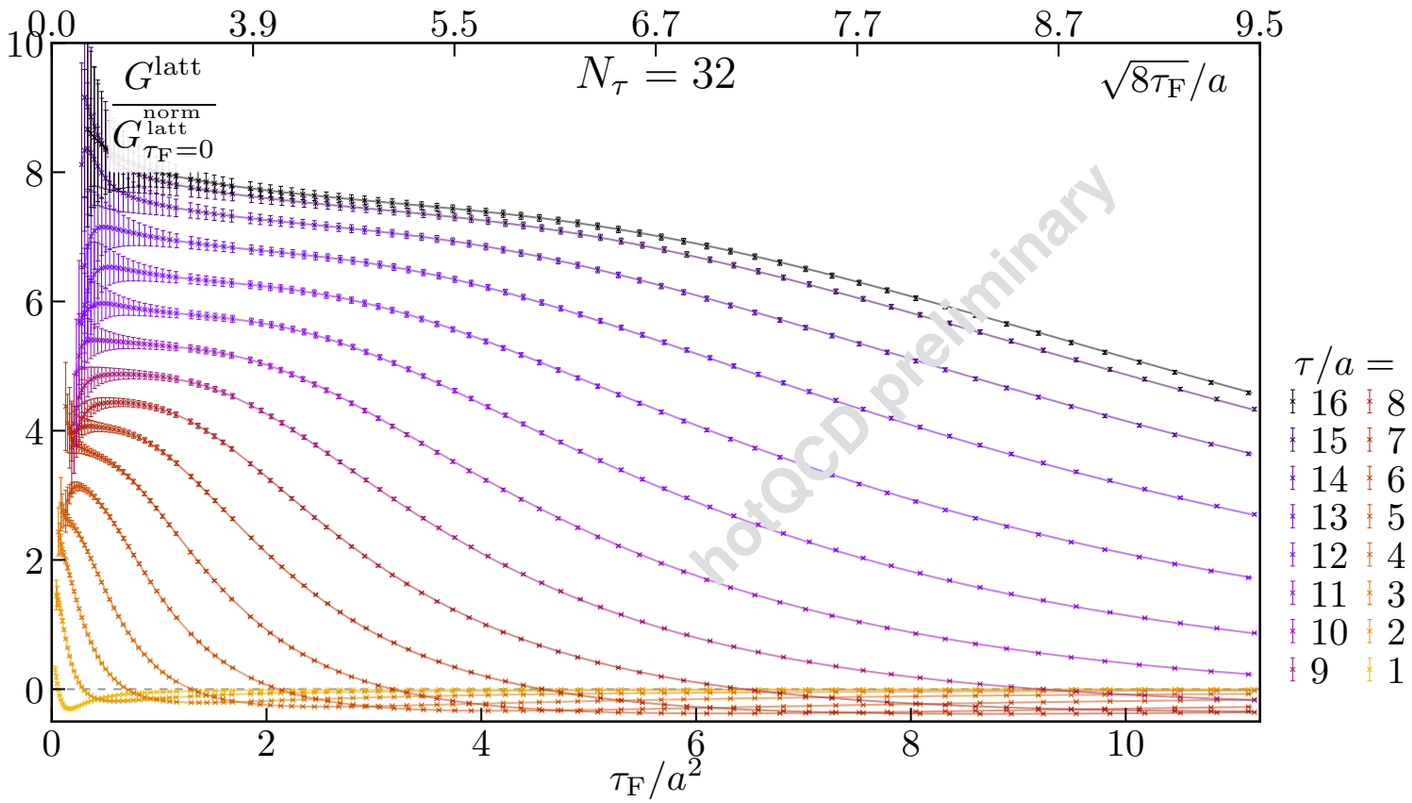
- Perturbative flow time range is applicable on the lattice
- Correlators within flow time limits are linear in flow time
- Beyond the upper bound correlators are destroyed soon
- An overall shift between correlators from gradient flow and multi-level
- Spectral reconstruction on double-extrapolated correlators

kappa from gradient flow: full QCD (I)

$N_f = 2+1$, HISQ, $m_\pi = 320$ MeV

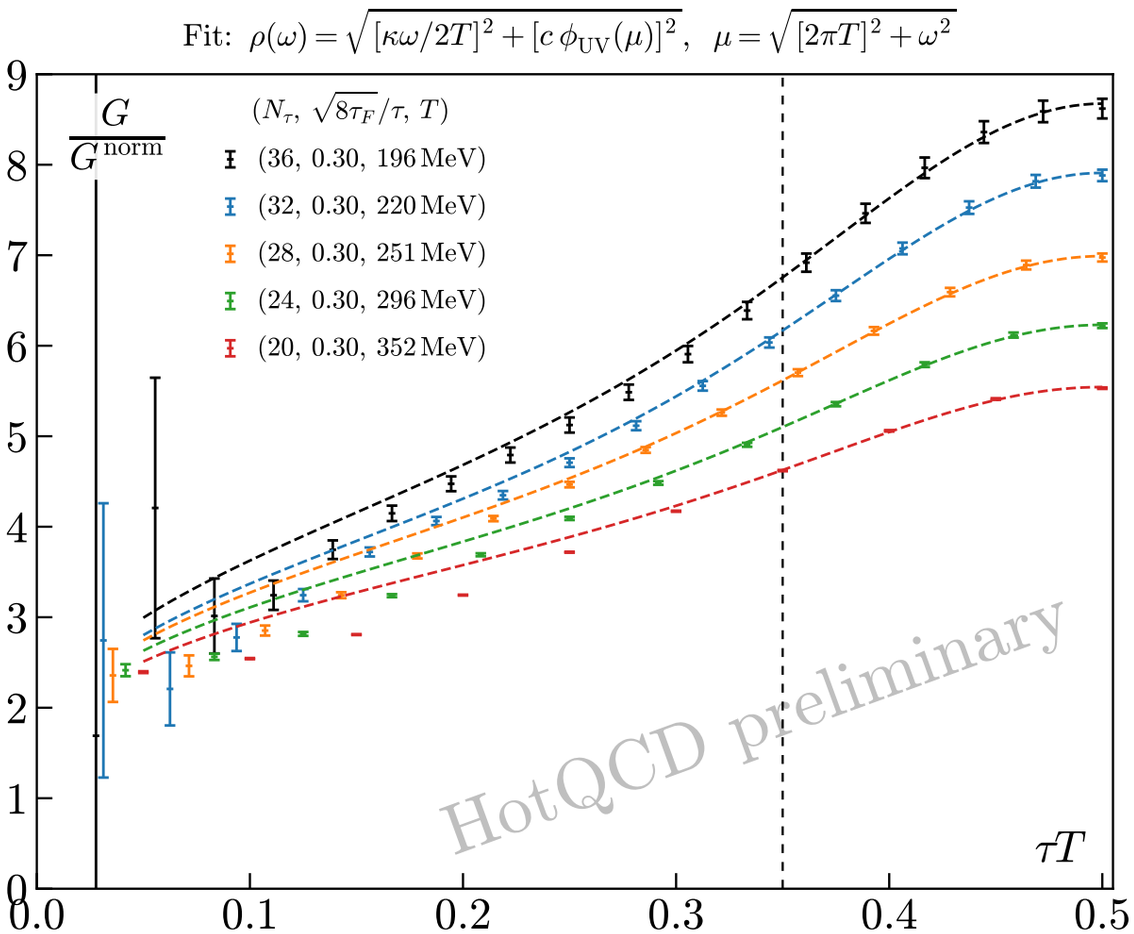
T [MeV]	$N_\sigma^3 \times N_\tau$	a [fm]	#
196	$96^3 \times 36$	0.028	1800
220	$96^3 \times 32$	0.028	1700
251	$96^3 \times 28$	0.028	1600
296	$96^3 \times 24$	0.028	1600
352	$96^3 \times 20$	0.028	4600

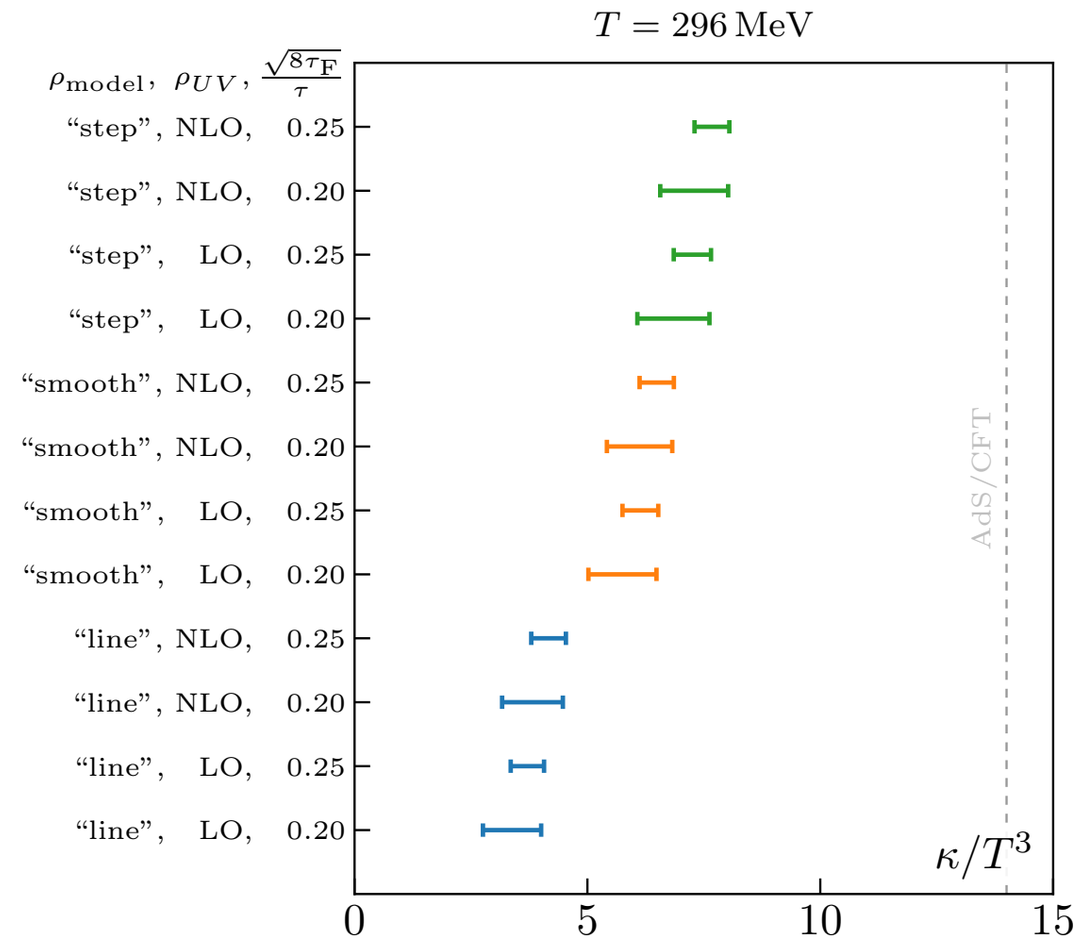
[L. Altenkort, HTS et al., in preparation]



- First study on heavy quark momentum diffusion with dynamic quarks
- Simplified gradient flow applied (only flow gauge fields)
- Similar behavior for color-electric field correlators as in the quenched case

kappa from gradient flow: full QCD (II)

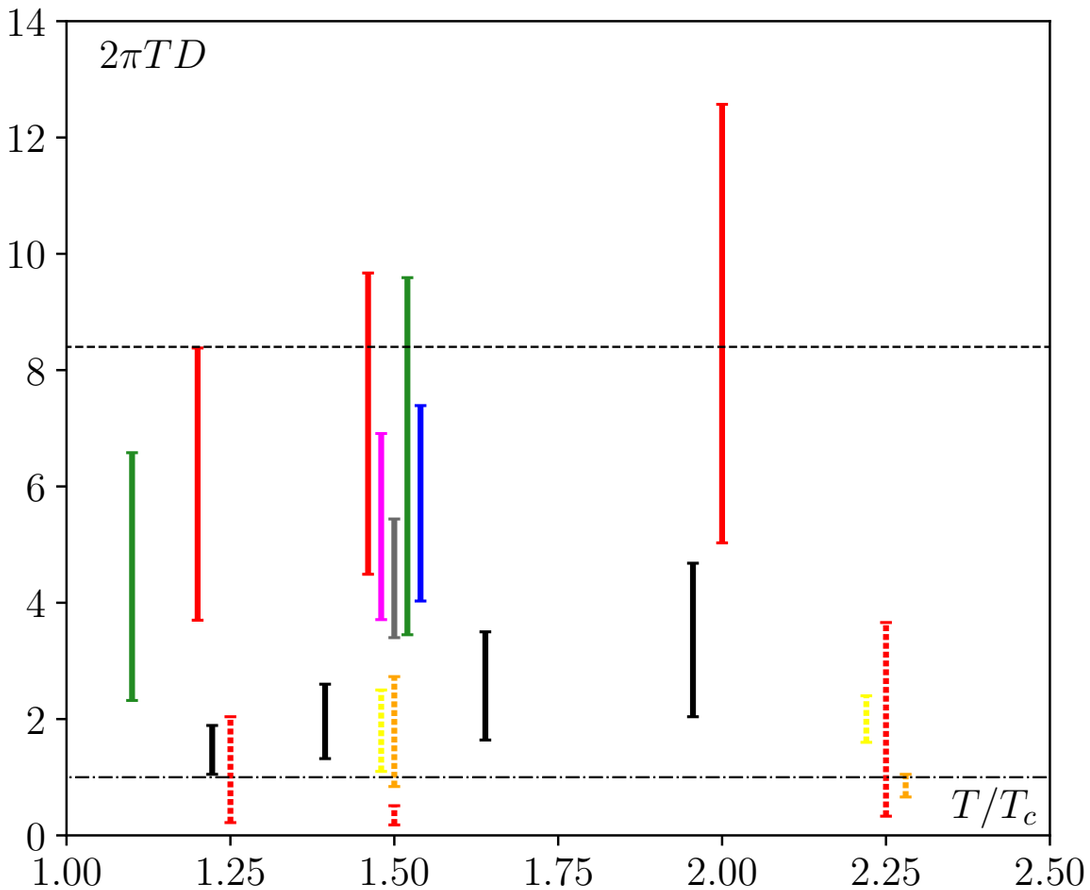


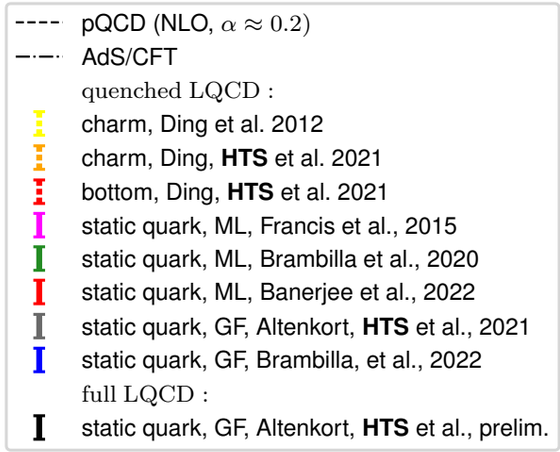


[L. Altenkort, HTS et al., in preparation]

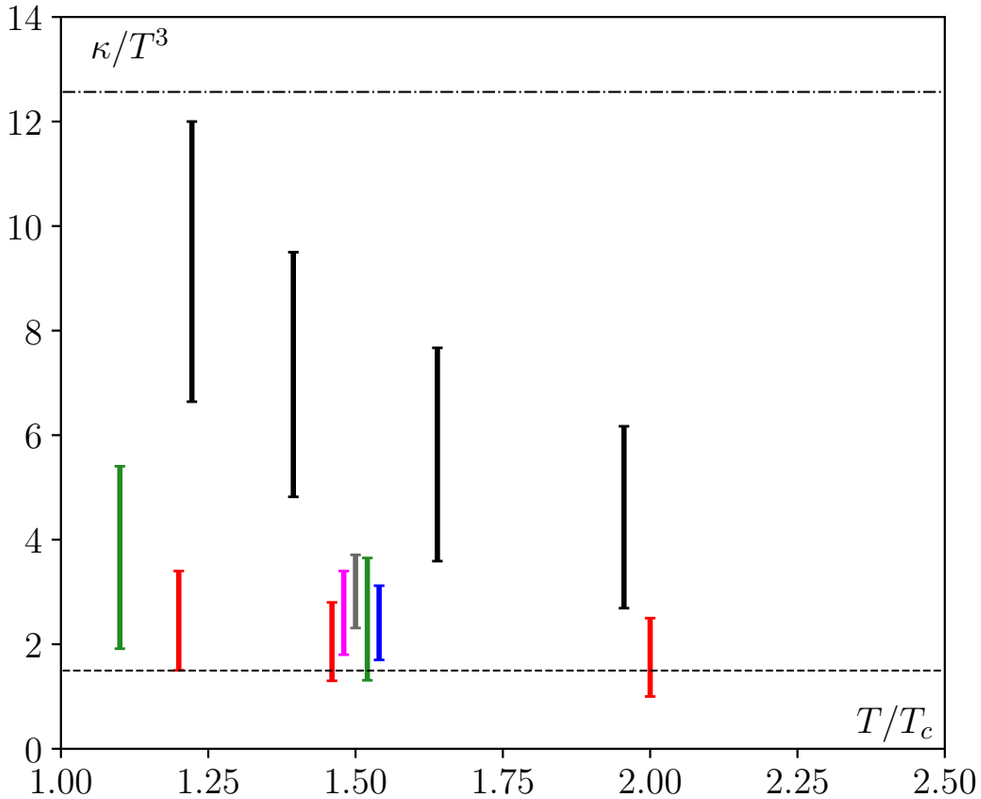
- Spectral reconstruction at justified finite flow time $a \lesssim \sqrt{8t} \lesssim \frac{\tau - a}{3}$
- Well-controlled systematic uncertainty in spectral reconstruction
- Continuum extrapolation on the way

A summary of kappa_E





$$2\pi TD = \frac{4\pi}{\kappa/T^3}$$



- Consistent results among studies with static quark in quenched approximation
- Significant decreasing of 2piTD when dynamical quarks are turned on
- 2piTD from static quarks in full QCD are close to those from quenched simulations at physical charm&bottom quark mass
- Subleading contributions?

[A. Bouttefeux and M. Laine, JHEP12(2020)150]

Corrections from color-magnetic correlators

- Full kappa is defined on full force-force correlator

[S. Caron-Huot et al., JHEP 0904 (2009) 053]
[A. Bouttefeux and M. Laine, JHEP12(2020)150]
[M. Laine, JHEP06(2021)139]

$$\kappa \equiv \frac{\beta}{3}\sum_{i=1}^{3}\lim_{\omega\to 0}\left[\lim_{M\to\infty}\frac{M^2}{\chi_q}\int_{-\infty}^{\infty}\mathrm{d}t\; e^{i\omega(t-t')}\int \mathrm{d}^3\vec{x}\,\left\langle\frac{1}{2}\{\mathcal{F}^i(t,\vec{x}),\mathcal{F}^i(0,\vec{0})\}\right\rangle\right]$$

$$\langle F_i(t')F_j(t)\rangle = q^2\left\{\langle E_i(t')E_j(t)\rangle + \frac{1}{3}\langle \mathbf{v}^2\rangle\langle \delta_{ij}B_k(t')B_k(t) - B_j(t')B_i(t)\rangle\right\} \qquad \kappa = \kappa_E + \frac{2}{3}\langle \mathbf{v}^2\rangle\kappa_B$$

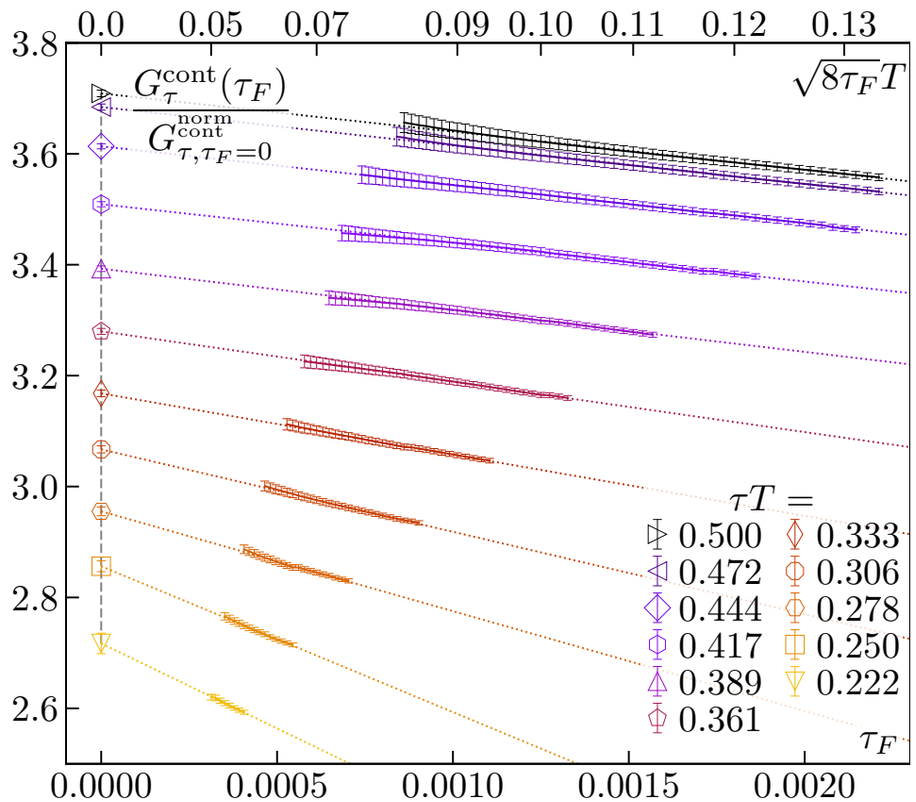
- Subleading contribution from G_{BB}

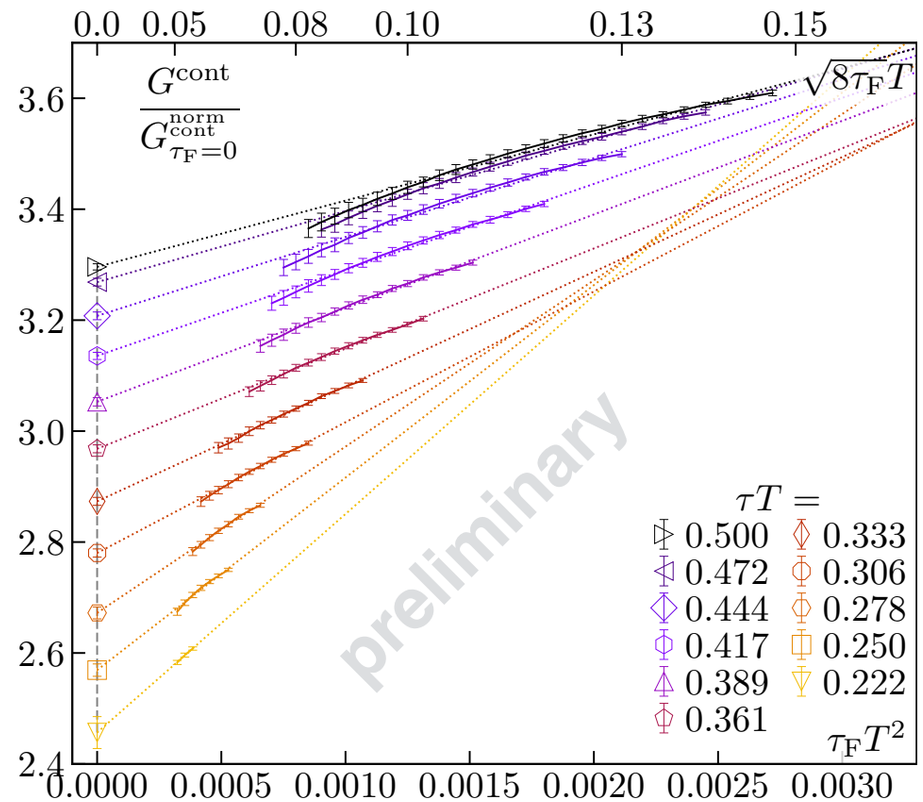
$$G_{BB}(\tau) = \frac{1}{3}\sum_{ii=1}^{3}\frac{\langle \mathrm{Re}\ \mathrm{Tr}[U(\beta,\tau)gB_i(\tau,\vec{0})U(\tau,0)gB_i(0,\vec{0})]\rangle}{\langle \mathrm{Re}\ \mathrm{Tr}[U(\beta,0)]\rangle}$$

- Different renormalization for G_{EE} and G_{BB}

$$Z_E = 1+\delta Z_E = 1+\mathcal{O}(g^4)$$

$$Z_B = 1+\delta Z_B = 1+\frac{g^2 C_A}{(4\pi)^2}\left[\frac{1}{\epsilon}+2\ln\left(\frac{\bar{\mu}e^{\gamma_E}}{4\pi T}\right)-2\right]+\mathcal{O}(g^4)$$





A non-zero anomalous dimension for G_{BB} !

[HTS et al., PRD103(2021)1,014511]
[L. Altenkort, HTS et al., in preparation]

Renormalization of color-magnetic field correlators

- Renormalization of G_{BB} obtained from multilevel method

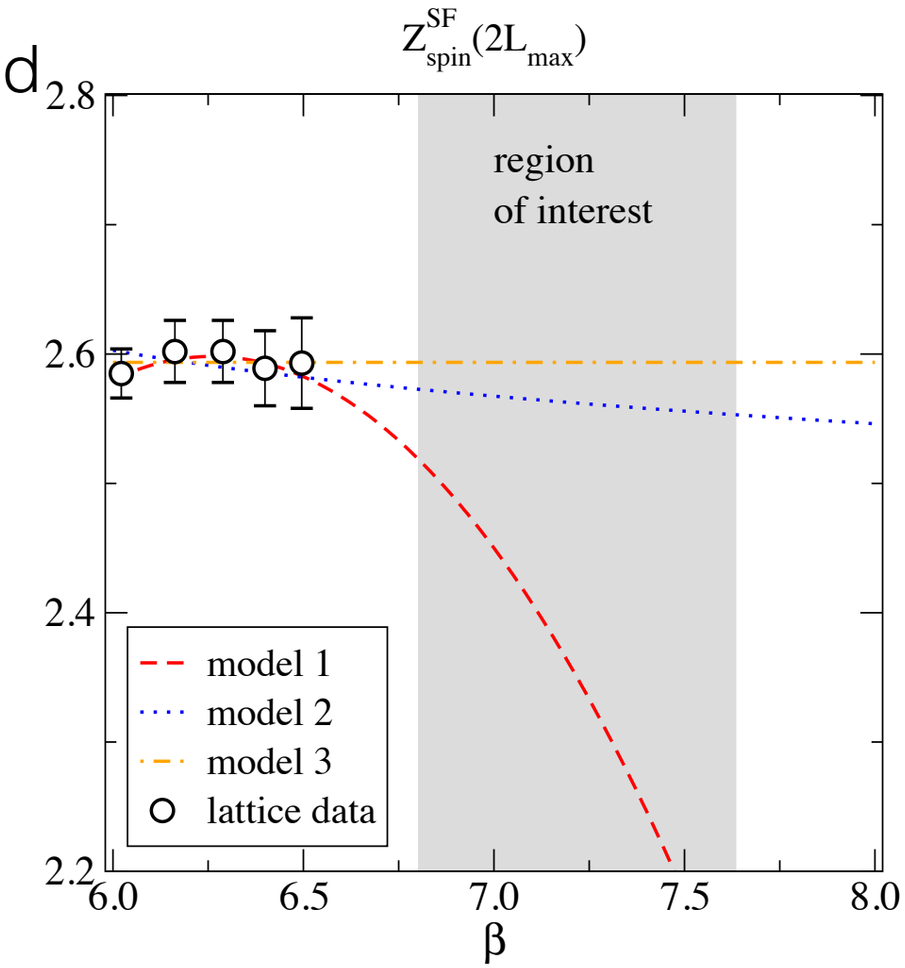
$$Z_B = 1 + \delta Z_B \;=\; 1 + \frac{g^2 C_A}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2\ln\left(\frac{\bar{\mu} e^{\gamma_E}}{4\pi T}\right) - 2\right] + \mathcal{O}(g^4)$$

$$\frac{[G_B(\tau)]_{\text{physical}}}{[G_B(\tau)]_{\text{bare,L}}} = \left\{\frac{\Phi_{\overline{\text{ms}}}(\bar{\mu} = 19.179T)}{\Phi_{\text{RGI}}} \times \frac{\Phi_{\text{RGI}}}{\Phi_{\text{SF}}(\frac{1}{2L_{\max}})} \times Z^{\text{SF}}_{\text{spin}}(2L_{\max})\right\}^2$$

- first two terms are RGI (running) constants
- last term suffers from large uncertainty

$$Z^{\text{SF}}_{\text{spin}}(2L_{\max}) \overset{\text{model 1}}{\approx} 2.58 + 0.14\,(\beta - 6) - 0.27(\beta - 6)^2$$

[D. Banerjee et al., arXiv: 2204.14075]



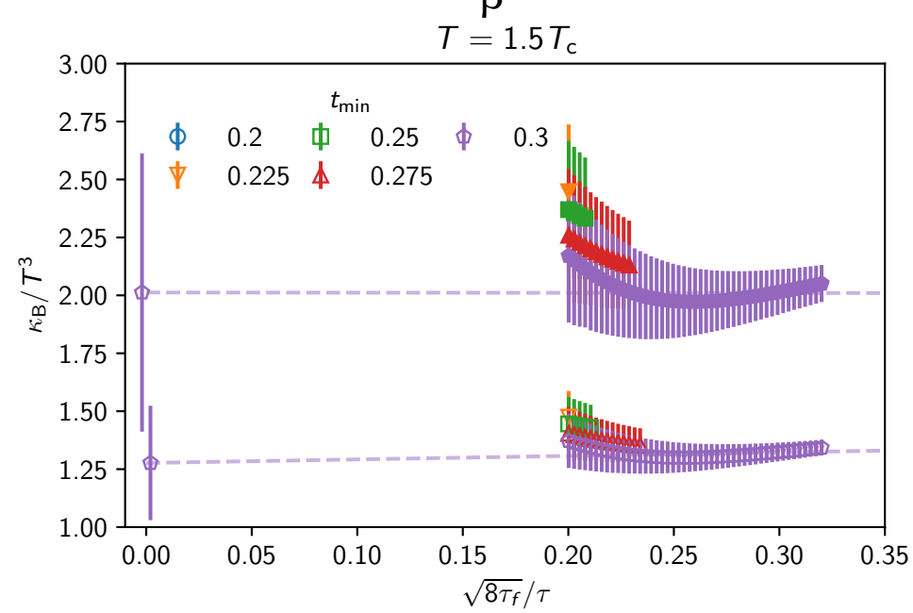
- Renormalization of G_{BB} obtained from gradient flow

[N. Brambilla et al., arXiv: 2206.02861]

$$G_B^{\text{flow,UV}}(\tau, \tau_F) = (1 + \gamma_0 g^2 \ln(\mu\sqrt{8\tau_F}))^2 Z_{\text{flow}} G_B^{\overline{\text{MS}},\text{UV}}(\tau, \mu) + h_0 \cdot (\tau_F/\tau)$$

$$\rho_B^{\text{UV}}(\omega, \tau_F) = Z_{\text{flow}} \frac{g^2(\mu)\omega^3}{6\pi}(1 + g^2(\mu)(\beta_0 - \gamma_0)\ln(\mu^2/(A\omega^2)) + g^2(\mu)\gamma_0 \ln(8\tau_F\mu^2)$$

- treat the renormalization constant as a fit parameter
- spectral reconstruction at finite flow time
- flow-to-zero-extrapolation of kappa_B



Determine the anomalous dimension?

- Define the problem in gradient flow-scheme $Z_B = 1 + \frac{g^2 C_A}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2\ln\left(\frac{\bar{\mu}e^{\gamma_E}}{4\pi T}\right) - 2\right] + \mathcal{O}(g^4)$
- Match MSbar-scheme and gradient flow-scheme

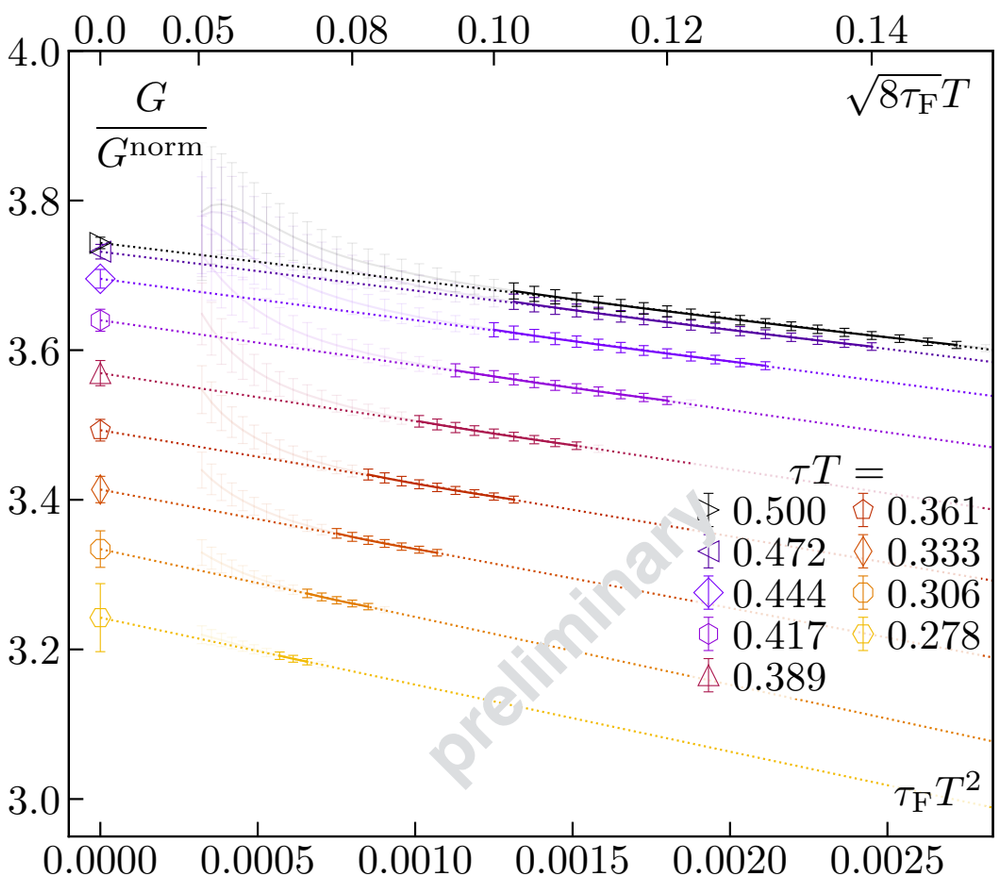
$$\mathrm{WeWant} = Z_B^2 \langle BB\rangle_{\mathrm{MS}}$$
$$\langle BB\rangle_{\tau_F} \equiv Z^{-2}\langle BB\rangle_{\mathrm{MS}}$$
$$\mathrm{WeWant} = Z_B^2 Z^2 \langle BB\rangle_{\tau_F}$$

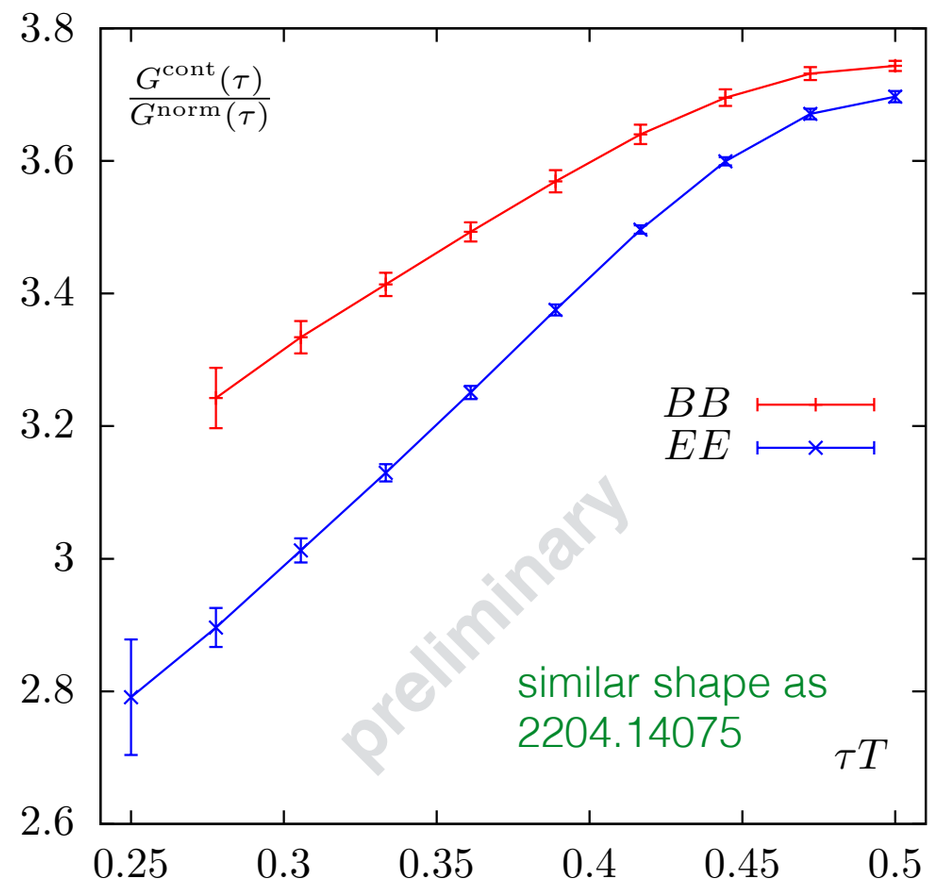
- Scale dependence must go for "WeWant" and $\langle BB\rangle_{\tau_F}$

$$Z^2 = \left(1 - 2\frac{g^2 C_A}{16\pi^2}\ln(\mu^2\tau_F)\right)\left(1 + 2K\frac{g^2 C_A}{16\pi^2}\right) \equiv Z_f^2 Z_K^2$$

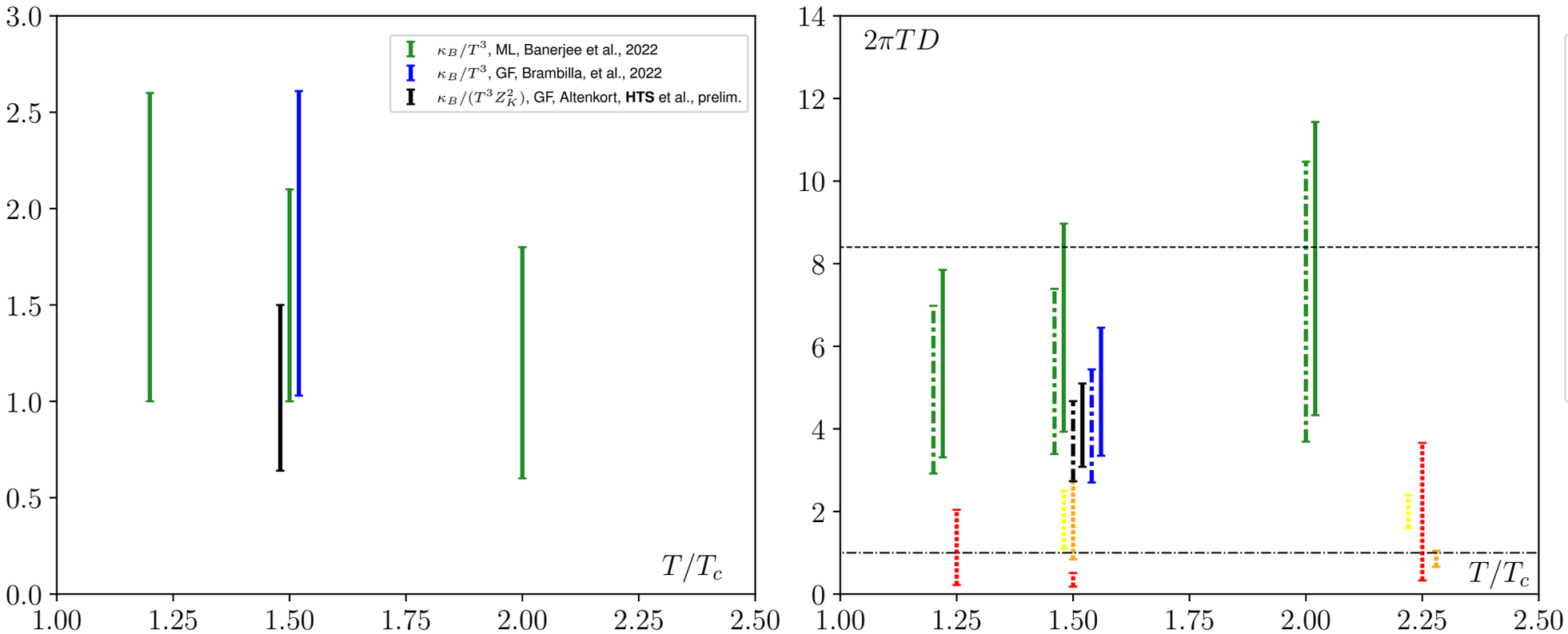
- Solve *Zf(tauF)* using RGE
- "K" remains to be determined

- Simple fix of running scale $Z_f^2(\tau_F T^2 = 0.002719) = 1.0$ $Z_B^2(\bar{\mu} = 4\pi e^{(1-\gamma_E)}\, T = 19.18\, T) = 1.0$





A summary of quenched results for kappa_B



$$\kappa = \kappa_E + \frac{2}{3}\langle \mathbf{v}^2 \rangle \kappa_B$$

$\langle \mathbf{v}^2 \rangle$ from 2204.14075

- Consistent estimates for kappa_B among different studies
- kappa_B is smaller than kappa_E
- kappa_B brings 2piTD down, closer to the AdS/CFT line
- kappa_B in full QCD is on the way
- charm & bottom quark diffusion in full QCD is on the way

Concluding remarks

- The charm & bottom quark diffusion have been investigated at physical mass in the continuum limit from quenched LQCD
- The color-electric contribution to heavy quark momentum diffusion has been calculated in quenched & full LQCD
- The color-magnetic contribution to heavy quark momentum diffusion has been calculated from quenched LQCD
- Consistent results are obtained among various LQCD studies on heavy quark momentum diffusion
- Heavy quark diffusion and heavy quark momentum diffusion approach closer when taking subleading contribution into account
- Dynamic quarks are crucial in the study of heavy quark diffusion

Backup

- Solve Z_f using RGE

$$Z_f^2(\tau_F) = \exp \int_{\tau_F{}'=.002719/T^2}^{\tau_F{}'=\tau_F} \frac{d\tau_F{}'}{\tau_F{}'} \left(\frac{-3g^2(\tau_F{}')}{8\pi^2} \right)$$

$$g^2(\tau_F) = \frac{128\pi^2}{24} \tau_F{}^2 \langle E \rangle_{\tau_F}$$