High-order corrections to the pressure of cold QED

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In collaboration with: T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho and A. Vuorinen

Based on: [2204.11279], [2204.11893]

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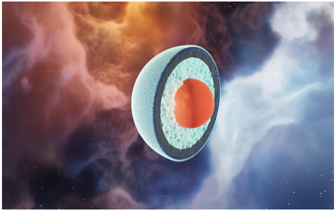
QCD thermodynamics at high density

- Aim: Understand bulk thermodynamic properties of cold ($T = 0$) and dense ($\mu_B \neq 0$) QCD using perturbation theory
  - Accessing μ-axis of the phase diagram from first-principles
- Applications in neutron stars:
  - Constraining neutron-star matter equation of state [Annala et al., Nature Phys. 16 (2020)] [Komoltsev, Kurkela, PRL 128 (2022)]



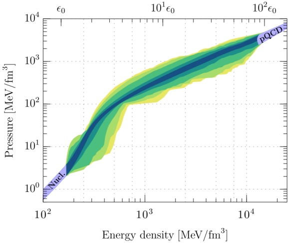
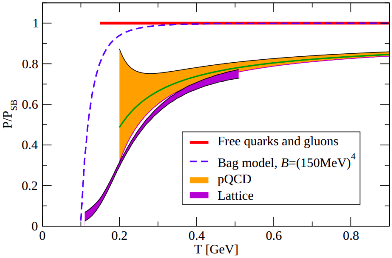


Image: Jyrki Hokkanen, CSC; Annala et al., PRX 12 (2022)

Calculating the pressure in cold and dense pQCD

Tools to study QCD pressure



$T > 0, \mu = 0$

- pQCD at high T
- Lattice QCD applicable
- Pressure well understood throughout the T-range

Image: Fraga, Kurkela, Vuorinen, EPJA 52 (2016)

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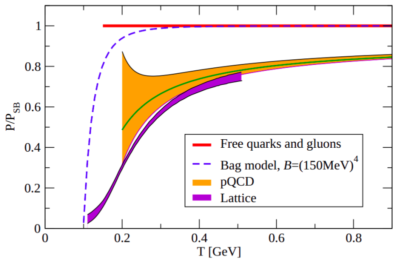

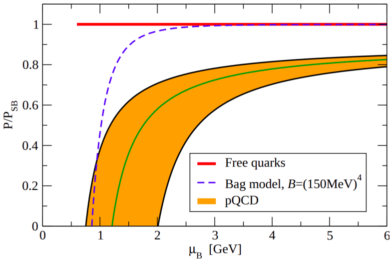

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- pQCD at high T
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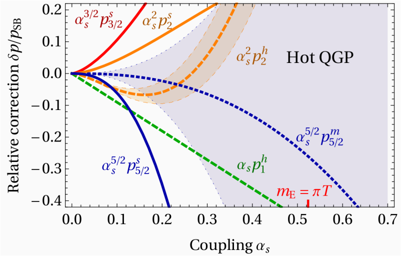
$T = 0, \mu > 0$

- pQCD at high μ
- Sign Problem $\Rightarrow$ lattice QCD not applicable

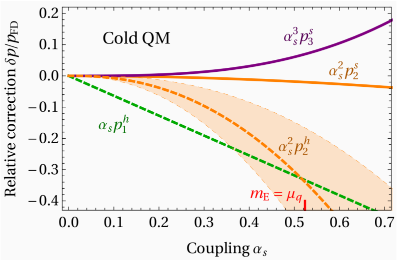
$\Rightarrow$ pQCD only reliable first-principles method!

Image: Fraga, Kurkela, Vuorinen, EPJA 52 (2016)

Finite μ pressure behaves calmly



High T pressure converges poorly



Finite μ, $T = 0$ pressure shows much better convergence
$\Rightarrow$ perturbation theory useful

Image: Gorda, Kurkela, Paatelainen, Säppi, Vuorinen, PRL 127 (2021)

Framework for calculating dense pQCD pressure

1. Generate Feynman diagrams from partition function:

$$p(\mu) \sim \ln Z = \ln \int \mathcal{D}\overline{\psi}\psi\overline{c}cAe^{-S_{\mathrm{QCD}}}$$

$$\overset{\mathrm{pQCD}}{=} \text{ sum of connected vacuum diagrams (no ext. legs)}$$

- Imaginary-time formalism
- Euclidean Feynman rules: Fermionic 4-momenta: $P^\alpha = (p^0 - i\mu, \mathbf{p})$

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Cancellation of IR divergences not so "simple" due to medium induced scales!

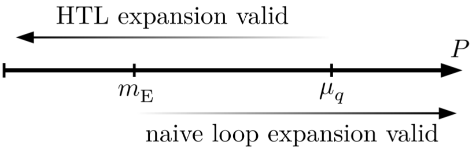
IR divergences handled by HTL effective theory

Only two scales at finite μ and zero T:

- Hard scale ($\sim \mu$): diagrams in full theory
- Soft scale ($\sim g\mu$): hard thermal loop (HTL) effective theory for gluons

No non-perturbative ultrasoft scale ($\sim g^2\mu$) since gluons not thermally excited at $T = 0$

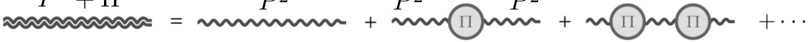
$\Rightarrow$ pQCD applicable to arbitrarily high orders! (no Linde problem)



HTL resummation for soft gluons

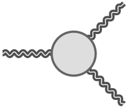
HTL effective theory [Braaten, Pisarski, NPB 337 (1989)]

$$\frac{1}{P^2+\Pi} = \frac{1}{P^2} + \frac{1}{P^2}\, g^2\mu^2\, \frac{1}{P^2} + \cdots$$



Π = LO HTL gluon self-energy Π acts as a mass $\sim g^2\mu^2$

- HTL self-energy = dominant contribution to a self-energy when external momenta are soft ($P \sim g\mu \ll \mu$)
- Soft gluons with momenta $P \sim g\mu$ must be resummed
- HTL vertex functions must be used as well



$\Rightarrow$ Perturbative series contain terms non-analytic in g (i.e. $\ln g$)

Structure of 3-loop pressure at finite μ

[Freedman, McLerran, PRD 16 (1977)]:

$$p(\mu) = a_0 + a_1 g^2 + a_{2,0} g^4 \ln g + a_{2,1} g^4 + O(g^6)$$

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 = fermion,  = hard gluon,  = soft gluon

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  - IR safe:  $\Rightarrow a_{2,1}$

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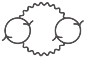
  - 

IR div.

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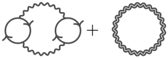
  -  $1/\varepsilon$'s cancel $\Rightarrow \ln g$ from ε/ε terms $\Rightarrow a_{2,0}$ & $a_{2,1}$

    IR div. UV div.

Cancellation of IR divergences at 3-loop order

- (Renormalized) soft and hard contributions:

$$\alpha_s^2 p_2^s = m_E^4 \left(\frac{m_E}{\Lambda_h}\right)^{-2\varepsilon} \left(\frac{p_{-1}^s}{2\varepsilon} + p_0^s\right)$$

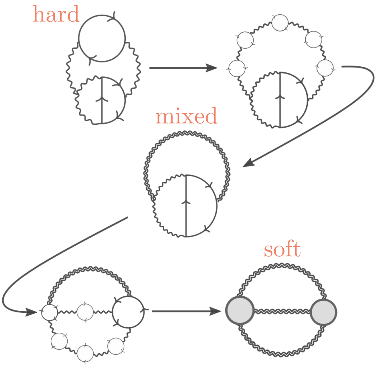
$$\alpha_s^2 p_2^h = m_E^4 \left(\frac{\mu}{\Lambda_h}\right)^{-2\varepsilon} \left(\frac{p_{-1}^h}{2\varepsilon} + p_0^h\right)$$

- In-medium effective mass scale: $m_E \propto g\mu$
- UV divergence of HTL theory matches to IR divergence of naive theory: $p_{-1}^s = -p_{-1}^h$

$$\alpha_s^2 (p_2^s + p_2^h) = m_E^4 \left(p_{-1}^s \ln \frac{\mu}{m_E} + p_0^s + p_0^h\right)$$

- Factorization scale Λ_h cancels, left with $\ln(\mu/m_E) \sim \ln g$

HTL resummation at 4-loop order (N^{3}LO)

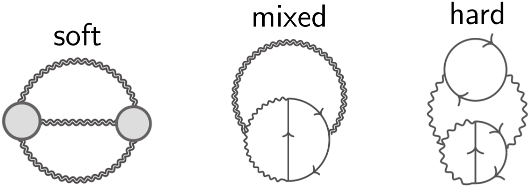


Moving between sectors by HTL-resumming soft gluons

Soft contributions organized into sectors based on the number of soft loop momenta

- Hard sector: No soft momenta, four-loop diagrams in naive theory
- Mixed sector: One soft momentum, HTL propagator with NLO HTL self-energies
- Soft sector: Two soft momenta, two-loop diagrams in HTL theory

State-of-the-art pQCD pressure



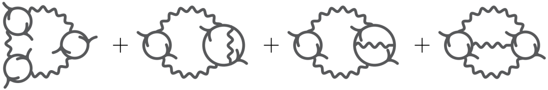
- Current state-of-the-art result by [Gorda, Kurkela, Paatelainen, Säppi, Vuorinen, PRL 127 (2021)]: fully soft contributions at order g^6
  - Determines the coefficient of the leading logarithm $g^6 \ln^2 g$
- Missing from g^6 result: mixed and hard contributions

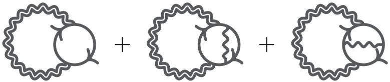
Quantum electrodynamics: testbed for QCD

- Rest of the talk: QED
- N^3LO correction (e^6) to QED pressure with soft/mixed/hard organization:
$$\alpha^3 p_3 = \alpha^3 (p_3^s + p_3^m + p_3^h)$$
- Divergences cancel between the sectors resulting in $\ln \alpha$ terms
$$\alpha^3 p_3 = \alpha^3 (a_0 + a_1 \ln \alpha + a_2 \ln^2 \alpha)$$
- a_2 fully determined by soft sector
- HTL vertex functions vanish in QED
$\Rightarrow$ no fully soft parts, $p_3^s = 0 \Rightarrow a_2 = 0$
- Complete result given by p_3^m and p_3^h

Contributions to N^{3}LO QED pressure

$\alpha^3 p_3^h$ (IR safe): 

$\alpha^3 p_3^h$ (IR div.): 

$\alpha^3 p_3^m$ (UV div.): 

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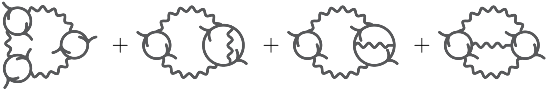
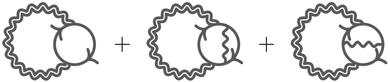
$\alpha^3 p_3^h$ (IR div.): 

$\alpha^3 p_3^m$ (UV div.): 

- Divergences cancel between 2nd and 3rd rows
  $\Rightarrow$ coefficient for $\alpha^3 \ln \alpha$
- Mixed diagrams contain NLO HTL self-energy insertions (of order $O(e^2 K^2)$ and $O(e^4 \mu^2)$)
  - Computed in Minkowski space using real-time formalism: tailored for computing n-point functions

Results

2-loop part of NLO results of order $e^4\mu^2$ in [K. Seppänen et al., 2204.11279] generalizes finite T result from [Carignano et al. Phys.Lett.B 801 (2020)] to finite μ:

$$\Pi_{\mathrm{T}}^{\mathrm{2loop}}(K) = -\frac{e^4\mu^2}{8\pi^4}\frac{k^0}{2k}\log\frac{k^0+k}{k^0-k}$$

$$\Pi_{\mathrm{L}}^{\mathrm{2loop}}(K) = -\frac{e^4\mu^2}{8\pi^4}\left\{1+2\left(1-\frac{k_0^2}{k^2}\right)\left[1-\frac{k^0}{2k}\log\frac{k^0+k}{k^0-k}\right]^2\right\}$$

$+O(\varepsilon)$ terms for finite ε/ε contributions in the pressure

- Result finite and gauge independent

Also need so-called power corrections to 1-loop result $O(e^2K^2)$

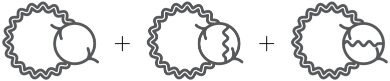
- Generalized from [Carignano et al. Phys.Lett.B 780 (2018)]

Calculation of N^{3}LO QED pressure

Done in companion paper [K. Seppänen et al., 2204.11893]:

- Mixed diagrams: NLO self-energy $\times$ HTL-resummed prop.

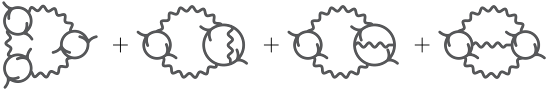
$$\alpha^3 p_3^m =$$



  - Computed fully

- IR sensitive hard 4-loop diagrams:

$$\alpha^3 p_3^{h,\text{IR div.}} =$$



  - Divergences computed, cancel with mixed diagrams
  - Explicit logarithms of renormalization scale $\bar{\Lambda}$ computed
  - First one (leading large-N_f) computed fully

- IR safe hard 4-loop diagrams:

$$\alpha^3 p_3^{h,\text{IR safe}} =$$  $+$  $+$  $+$  $+$ 

  - Explicit logarithms of $\bar{\Lambda}$ computed

$\Rightarrow$ Computed almost complete N^{3}LO result

- Missing: a pure number from hard diagrams (subleading in large-N_f)

Results: N^{3}LO correction to QED pressure

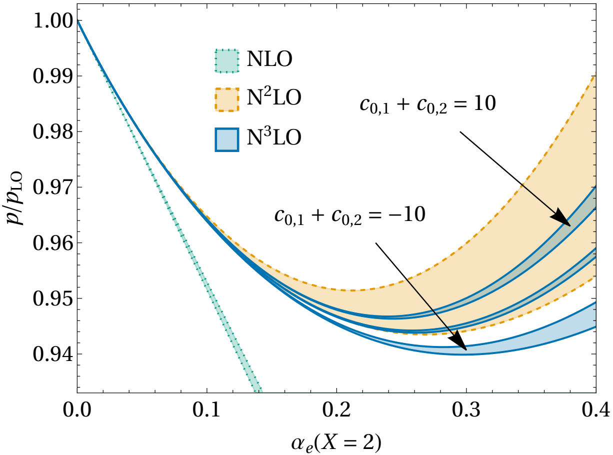
[K. Seppänen et al., 2204.11893]:

$$\frac{\alpha^3 p_3}{p_{\mathrm{LO}}} = N_f^2 \left(\frac{\alpha}{\pi}\right)^3 \left[a_{3,1} \ln^2 \left(N_f \frac{\alpha}{\pi}\right) + a_{3,2} \ln \left(N_f \frac{\alpha}{\pi}\right) + a_{3,3} \ln \left(N_f \frac{\alpha}{\pi}\right) \ln \frac{\bar{\Lambda}}{2\mu} + a_{3,4} \ln^2 \frac{\bar{\Lambda}}{2\mu} + a_{3,5} \ln \frac{\bar{\Lambda}}{2\mu} + a_{3,6}\right]$$

$a_{3,1}$	0
$a_{3,2}$	$-\frac{5}{4} + \frac{33}{2} N_f^{-1} + \frac{1}{48}\left(7 - 60 N_f^{-1}\right)\pi^2$
$a_{3,3}$	2
$a_{3,4}$	$-\frac{2}{3}$
$a_{3,5}$	$-\frac{79}{9} + \frac{2}{3}\pi^2 + \frac{2}{3}(13 - 8\ln 2)\ln 2 + \delta - \frac{31}{4} N_f^{-1}$
$a_{3,6}$	$1.02270(2) + \left(2.70082 + \frac{1}{2} c_{0,1}\right) N_f^{-1} + \frac{1}{2} c_{0,2} N_f^{-2}$

- $\delta \simeq -0.8563832$ [Vuorinen, PRD 68 (2003)]
- $c_{0,1}$ and $c_{0,2}$ remain unknown (pure numbers from hard diagrams)

Cold and dense QED pressure up to N^{3}LO



- Physical QED: $N_f = 1$
- Sum of unknown constants varied between -10 and 10 (quite extreme)

Summary and outlook

- Cold and dense pQCD applicable to arbitrary order
- Pressure seems to converge much better than in high T case
- Multiloop integrals split into sectors based on # of soft gluons
- HTL theory describes the soft gluons
- Almost complete N^{3}LO pressure computed in QED

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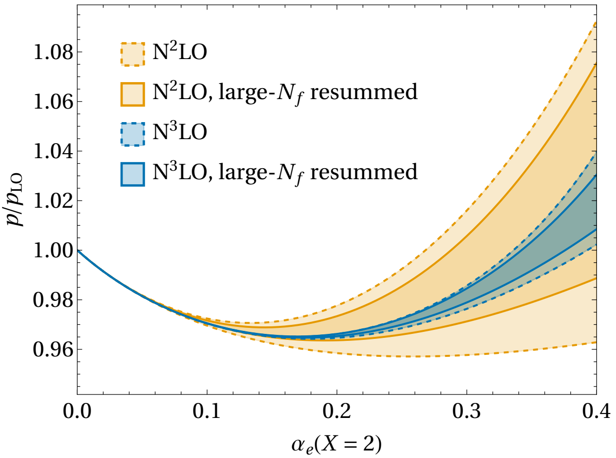
Next steps:

1. Generalize NLO photon self-energy to QCD
   - 3 more 2-loop diagrams, more to compute but same tools apply
   - In the progress: preliminary results seem similar to QED case
2. Generalize N^{3}LO QED pressure to QCD
   - Soft sector does not vanish but has already been computed
   - Mixed and hard sectors have to be computed

Thanks! Questions?

[2204.11279]
[2204.11893]

Extra: Large-N_f-resummed pressure



Here we have chosen $N_f = 3$ and $c_{0,1} = c_{0,2} = 3$

Extra: Speed of sound in QED matter