Anomalous diffusion in dense QCD matter

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XVth Quark Confinement- August 4th

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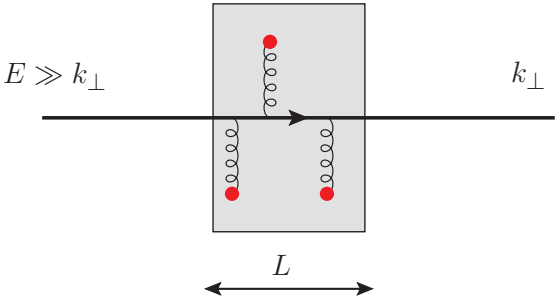
Ref: 2109.12041, 2203.09407

Transverse momentum broadening (TMB) in QCD

- Physical system: a highly energetic parton propagating through a dense QCD medium.
- We compute the transverse momentum distribution $\mathcal{P}(k_\perp)$ of the outgoing parton.



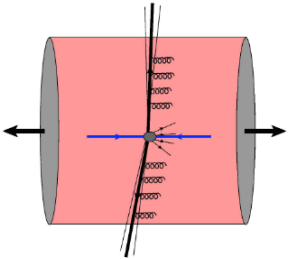
This talk

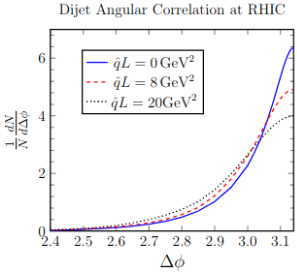
Study the TMB distribution $\mathcal{P}(\boldsymbol{k}_\perp)$ including leading radiative corrections.

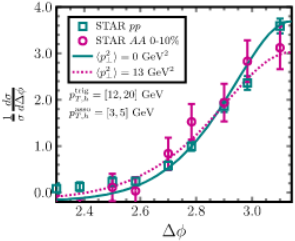
Why is TMB interesting?

Jet quenching

- "Hot QCD": Dijet azimuthal angular distributions in heavy-ion collisions: access to the TMB and the medium properties.
- Ex: studies by Mueller, Wu, Xiao, Yuan 1604.04250 & Chen, Qin, Wei, Xiao, Zhang 1607.01932.







- See also recent measurements of hadron-jet acoplanarity by ALICE and STAR in Jaime Norman's talk.

Why is TMB interesting?

Initial condition to the BK equation in small-x physics

- "Cold QCD": fast probe of gluon distribution in large nuclei $L \propto A^{1/3} \gg 1$ at small-x.
- Beyond LO, the non-local small-x evolution couples fluctuations with $t_f \gg A^{1/3}$ and $t_f \ll A^{1/3}$.

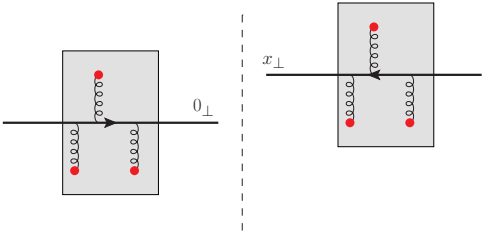
  Cf discussion in Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos 1902.06637
- Understanding TMB in large nuclei including effects of gluon fluctuations inside the target can provide improved initial conditions compared to the McLerran-Venugopalan model.

  McLerran, Venugopalan, 9311205 & 9309289

TMB at "tree level"

- Forward scattering amplitude of an effective dipole with size $\boldsymbol{x}_\perp$,

$$\mathcal{S}(\boldsymbol{x}_\perp) = \frac{1}{N_c}\langle \mathrm{Tr}\, V^\dagger(\boldsymbol{x}_\perp) V(\boldsymbol{0}_\perp)\rangle\,, \qquad \text{with} \quad V(\boldsymbol{x}_\perp) = \mathcal{P} e^{ig\int_{-\infty}^{\infty} dx^+ A^-(x^+,\boldsymbol{x}_\perp)}$$



See Andrey Sadofyev's talk beyond the homogeneous "brick" model.

- Assuming independent multiple interactions,

$$\mathcal{S}(\boldsymbol{x}_\perp) = \exp\left(-\frac{1}{4}\frac{C_R}{N_c}\hat{q}(1/\boldsymbol{x}_\perp^2) L \boldsymbol{x}_\perp^2\right)$$

$\Rightarrow$ LO $\hat{q}$ given by $\hat{q}^{(0)}(1/\boldsymbol{x}_\perp^2) = \hat{q}_0 \ln \frac{1}{\boldsymbol{x}_\perp^2 \mu^2}$, $\mu \sim m_D$ and $\hat{q}_0 = \alpha_s N_c m_D^2 T$

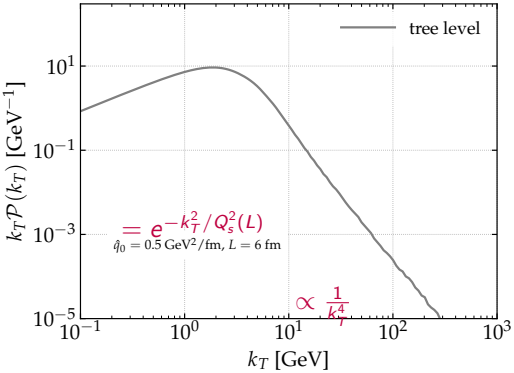
TMB at "tree level" and saturation scale $Q_s(L)$

$\Rightarrow$ Fourier transform of the dipole S-matrix

$$\mathcal{P}^{(0)}(\boldsymbol{k}_\perp) = \int d^2\boldsymbol{x}_\perp e^{-i\boldsymbol{k}_\perp \boldsymbol{x}_\perp} e^{-\frac{1}{4}\hat{q}(1/\boldsymbol{x}_\perp^2)L\boldsymbol{x}_\perp^2}$$

- Q_s emergent momentum scale
- Transition between the unitarity bound $\mathcal{S} \sim 1$ and the dilute regime $\mathcal{S} \ll 1$.
- At tree-level,

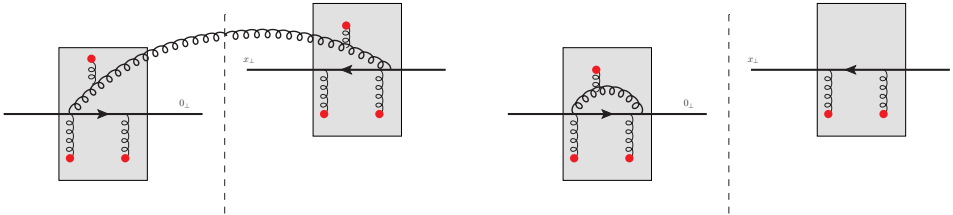
$$Q_s^2(L) \simeq \hat{q}_0 L$$



See also talk by João Barata.

TMB at one loop in a dense QCD medium

- Computation at one-loop in $\alpha_s \ll 1$, but to all-orders in $\alpha_s n$.



- Typical order of magnitude of the NLO correction to $\hat{q}$: Liou, Mueller, Wu, 1304.7677

$$\hat{q}^{(1)}(L, 1/\boldsymbol{x}_\perp^2) \sim \frac{\alpha_s N_c}{\pi} \int_{\tau_0}^{L} \frac{d\tau'}{\tau'} \int_{Q_s^2(\tau)}^{1/\boldsymbol{x}_\perp^2} \frac{d\boldsymbol{k}_\perp'^2}{\boldsymbol{k}_\perp'^2} \times \hat{q}_0$$

- Double log enhancement: $Q_s^2(L) = \hat{q}_0 L \left(1 + \frac{\bar{\alpha}_s}{2} \ln^2(L/\tau_0) + \ldots\right)$ at NLO.
- NB: we ignore classical $\mathcal{O}(g)$ corrections from soft modes in the plasma. Caron-Huot, 0811.1603
  See recent discussions in Ghiglieri, Weitz, 2207.08842

Resummation of the leading radiative corrections

- Resummation to all orders via the evolution equation

$$\frac{\partial \hat{q}(\tau, \boldsymbol{k}_\perp^2)}{\partial \tau} = \int_{Q_s^2(\tau)}^{\boldsymbol{k}_\perp^2} \frac{d\boldsymbol{k}_\perp'^2}{\boldsymbol{k}_\perp'^2} \bar{\alpha}_s(\boldsymbol{k}_\perp'^2)\ \hat{q}(\tau, \boldsymbol{k}_\perp'^2)$$

  with $Q_s^2(\tau) \equiv \hat{q}(\tau, Q_s^2(\tau))\tau$.

- Initial "tree-level" condition at some $\tau = \tau_0$ (can include non-perturbative input and classical $\mathcal{O}(g)$ corrections)

- Exponentiation of the double logarithmic corrections.

$$\mathcal{P}(\boldsymbol{k}_\perp) = \int \mathrm{d}^2\boldsymbol{x}_\perp\, e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}\ \exp\left[-\frac{1}{4}\frac{C_R}{N_c}\hat{q}(L, 1/\boldsymbol{x}_\perp^2)\, L\, \boldsymbol{x}_\perp^2\right]$$

cf Liou, Mueller, Wu, 1304.7677, Blaizot, Mehtar-Tani, 1403.2323, Iancu 1403.1996

Asymptotic limit of TMB at fixed coupling

- Large system size limit of $\hat{q}(L, \boldsymbol{k}_\perp^2)$

$$\frac{\hat{q}(L, \boldsymbol{k}_\perp^2)L}{Q_s^2(L)} \underset{L\to\infty}{=} \begin{cases} e^{2\sqrt{\bar{\alpha}_s}\ln\left(\frac{\boldsymbol{k}_\perp^2}{Q_s^2(L)}\right)} & \text{if } \boldsymbol{k}_\perp^2 \leq Q_s^2(L) \\ e^{\sqrt{\bar{\alpha}_s}\ln\left(\frac{\boldsymbol{k}_\perp^2}{Q_s^2(L)}\right)}\left[1+\sqrt{\bar{\alpha}_s}\ln\left(\frac{\boldsymbol{k}_\perp^2}{Q_s^2(L)}\right)\right] & \text{else} \end{cases}$$

with

$$Q_s^2(L) = \hat{q}_0 L \left(\frac{L}{\tau_0}\right)^{2\sqrt{\bar{\alpha}_s}}$$

$\Longrightarrow$ **geometric scaling** for $\boldsymbol{k}_\perp^2 \ll Q_s^4/\mu^2$ PC, Mehtar-Tani 2109.12041

- Similar to geometric scaling for gluon distribution at small x: $\ln(1/x) \leftrightarrow \ln(L/\tau_0)$.

Lévy flights

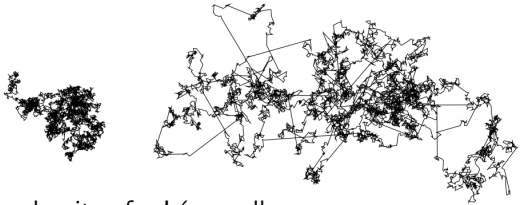
- At large time $L \gg \tau_0$, near the peak

$$\mathcal{S}(\boldsymbol{x}_\perp, L) \approx \exp\left(-\frac{1}{4}(|\boldsymbol{x}_\perp| Q_s(L))^{2-4\sqrt{\bar{\alpha}_s}}\right)$$

- $\Longrightarrow$ the TMB distribution satisfies a fractional Fokker-Planck equation

$$\frac{\partial \mathcal{P}(L, \boldsymbol{k}_\perp)}{\partial L} = \nu \frac{\partial^\gamma \mathcal{P}(L, \boldsymbol{k}_\perp)}{\partial |\boldsymbol{k}_\perp|^\gamma}, \qquad \gamma = 2 - 4\sqrt{\bar{\alpha}_s}$$

Brownian motion



Levy flight

- Equation for the prob. density of a Lévy walker, e.g.
  - $\dot{v} = -\mu v + \eta^\gamma(t)$
  - $\eta^\gamma(t)$ Lévy stable noise ($\gamma = 2$ is the standard white Gaussian noise).

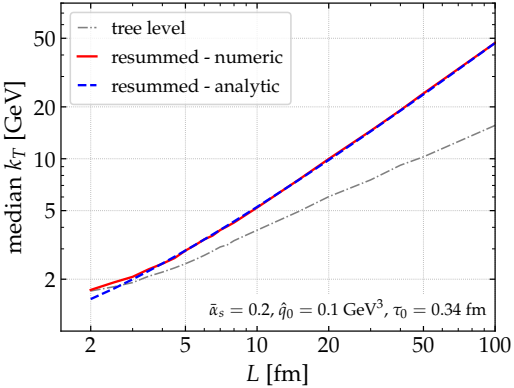
Superdiffusion in momentum space

- $Q_s^2(L) = \hat{q}_0 L \left(\frac{L}{\tau_0}\right)^{2\sqrt{\bar{\alpha}_s}}$
- The median of the distribution scales like

$$\mathcal{M} \sim L^{1/2+\sqrt{\bar{\alpha}_s}}$$

- $\Rightarrow$ super-diffusive behaviour.

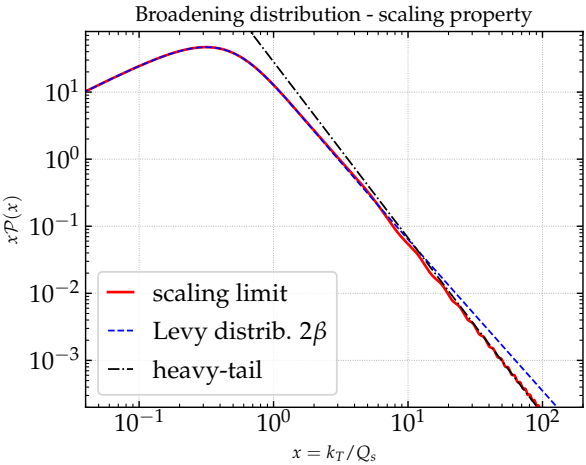
NLO corrections yield super-diffusion in momentum space.



Heavy-tailed distribution

- $\hat{q} \simeq e^{\sqrt{\bar{\alpha}_s}\ln(\boldsymbol{k}_\perp^2/Q_s^2)}$ at large k_T.
- Fourier transform of the "stretched" exponential $\exp(-[...]\boldsymbol{x}_\perp^\gamma)$ with $\gamma \simeq 2 + 2\sqrt{\bar{\alpha}_s} > 2$
- Heavy tailed distribution

$$\mathcal{P}(\boldsymbol{k}_\perp) \propto \frac{1}{k_T^{4-2\sqrt{\bar{\alpha}_s}}}$$



PC, Mehtar-Tani 2109.12041

Beyond the asymptotic limit

- We have determined the limit $L \to \infty$ of the TMB distribution.
- What about the sub-asymptotic corrections?
- Are they universal = independent of the initial conditions?
- Can they be used down to realistic values of L?

Wave front propagation into unstable state

- Non-linear evolution equation in the same universality class as FKPP equation (reaction-diffusion processes):

$$\partial_t \phi = \partial_x^2 \phi + \phi - \phi^k$$

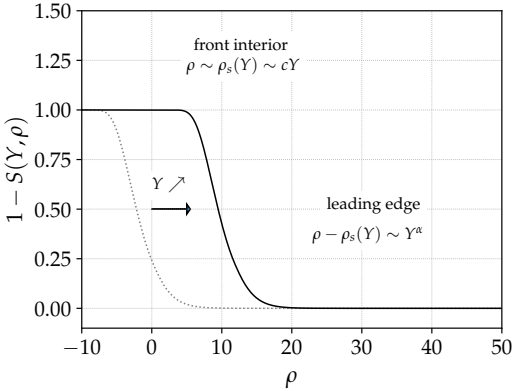
Ebert, van Saarloos, 0003181, Brunet, Derrida, 0005362

- Traveling wave interpretation (similar to the BK case)

Munier, Peschanski, 0310357 - Beuf, 1008.0498

- Location of the wave front $\Leftrightarrow$ $\rho_s = \ln(Q_s^2/\mu^2)$, time $\Leftrightarrow$ $Y = \ln(L/\tau_0)$.
- Universality of the wave-front velocity $\dot{\rho}_s$:

$$\frac{d\ln(Q_s^2(L))}{d\ln L} = c + \frac{\delta_1}{Y} + \frac{\delta_2}{Y^{3/2}}$$



Leading edge expansion

- Diffusive deviation from the asymptotic limit, with we consider.

$$\hat{q}(Y = \ln(L/\tau_0), \rho = \ln(\boldsymbol{k}_\perp^2/\mu^2)) = \hat{q}_0 e^{\rho_s(Y)-Y} e^{\beta x}\left[\, Y^\alpha G\left(\frac{x}{Y^\alpha}\right) + \ldots\right]$$

$$\dot{\rho}_s(Y) = c + \delta\dot{\rho}_s(Y)$$

- Diffusion power characteristics of the universality class of the evolution equation.
- Homogeneity conditions fix the power α.
- $\alpha = 1/2$ for fixed coupling, $\alpha = 1/6$ for running coupling (cf last slide).

Results for fixed coupling

- For fixed coupling, we find the pre-asymptotic behaviour

$$\frac{\hat{q}(L,\boldsymbol{k}_\perp^2)L}{Q_s^2(L)} = \begin{cases} \exp\left(\beta x - \frac{\beta x^2}{4cY}\right)\left[1+\beta x - \frac{3x}{c(1+c)Y}\left(1+\frac{\beta(c+4)x}{6}\right) + \mathcal{O}\left(\frac{1}{Y^2}\right)\right] & \text{if } x \geq 0 \\ \exp\left(2\beta x - \frac{3}{c(1+c)}\frac{x}{Y} + \mathcal{O}\left(\frac{1}{Y^2}\right)\right) & \text{if } x < 0\,. \end{cases}$$

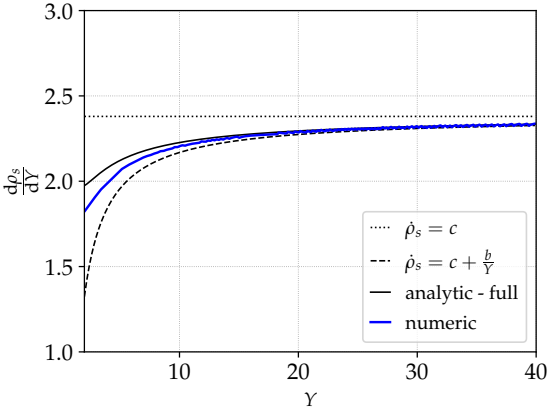
with

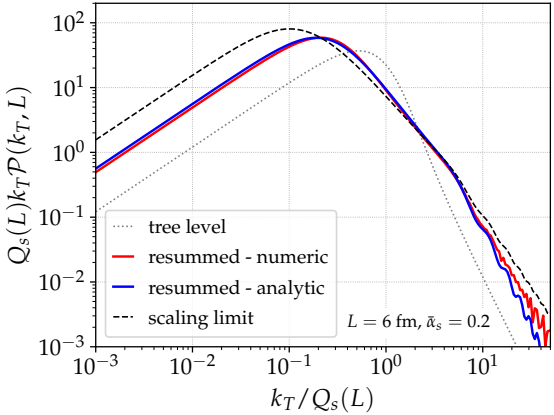
$$\ln(Q_s^2(L)/\mu^2) = (1+2\sqrt{\bar{\alpha}_s})Y - \frac{3}{2}(1+\sqrt{\bar{\alpha}_s})\ln(Y) + \frac{3\sqrt{\pi}\bar{\alpha}_s^{-1/4}}{2}\frac{1}{\sqrt{Y}} + \mathcal{O}(Y^{-1})$$

PC, Mehtar-Tani 2109.12041

- $x = \ln(\boldsymbol{k}_\perp^2/Q_s^2(L))$, $Y = \ln(L/\tau_0)$.

Some plots





- Sub-asymptotic corrections enable one to have a good agreement with the numeric.
- Analytic results can be systematically improved.

Beyond the double logarithmic approximation

- Our study relies on two pillars:
  - The evolution of Q_s is dominated by the double log regime of QCD since $\rho_s \sim Y$ $\Rightarrow$ one can use BFKL or DGLAP with a non-linear saturation boundary to get the universal behaviour of Q_s.
  - Universal terms of the asymptotic series of $\ln(Q_s^2(L)/\mu^2)$ at large L are mainly controlled by the linear behaviour of the evolution equation.
- Using NLO, N^{2}LO BFKL equation one can systematically compute the α_s expansion of c, δ_1, δ_2.

Example: saturation scale in weakly-coupled $\mathcal{N} = 4$ SYM plasma

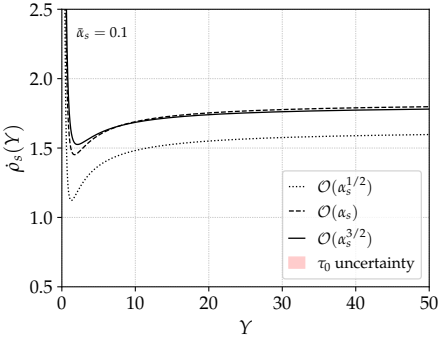
- BK/BFKL equation known at three loops in planar $\mathcal{N} = 4$ SYM theory.

  Velizhanin 1508.02857, Caron-Huot, Herranen 1604.07417

- One can get the coefficients c, δ_1 and δ_2 up to order $\alpha_s^{3/2}$. Ex:

$$c = 1 + 2\sqrt{\bar{\alpha}_s} + 2\bar{\alpha}_s + \left(1 - \frac{\pi^2}{6}\right)\bar{\alpha}_s^{3/2} + \mathcal{O}(\alpha_s^2)$$

PC, Mehtar-Tani, in prep.

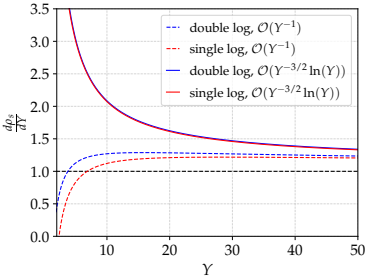


Running coupling and single logarithmic corrections in the $L \to \infty$ limit

- Final result, exact to all orders in pQCD (but for $Y = \ln(L/\tau_0) \to \infty$...)
- All universal terms in the asymptotic expansion of $Q_s^2(L)$

$$\frac{d\ln(Q_s^2(L))}{d\ln L} = 1 + \frac{4b_0}{(4b_0Y)^{1/2}} + \frac{2\xi_1 b_0}{(4b_0Y)^{5/6}} + (1 - 8b_0 + 4b_0 B_g)\frac{1}{4Y} - \frac{7\xi_1^2 b_0}{270}\frac{1}{(4b_0Y)^{7/6}}$$
$$- (5 + 1944b_0)\frac{\xi_1 b_0}{81}\frac{1}{(4b_0Y)^{4/3}} - b_0^2(2 - 16b_0 + 8b_0 B_g)\frac{\ln(Y)}{(4b_0Y)^{3/2}} + \mathcal{O}\left(\frac{1}{Y^{3/2}}\right)$$

- In agreement with Iancu, Triantafyllopoulos 1405.3525 for the linearized equation.



PC, Mehtar-Tani, 2203.09407 & in prep.

Summary

- Study of the effect of radiative corrections on transverse momentum broadening in a dense QCD medium for large system sizes.
- TMB satisfies extended geometric scaling.
- Radiative corrections yield super-diffusive behaviour in momentum space, and a heavy tail with power index smaller than the typical Rutherford behaviour.
- The DLA non-linear evolution equations share similar mathematical properties as equations for wave front propagation into unstable states.
- Enable to compute the universal behaviour of the TMB distribution in pQCD, independent of non-perturbative aspects.