Twisting with a Flip

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In this talk I will explore 4d $\mathcal{N} = 2$ susy field theories on euclidean 4-manifolds $\mathcal{M}$. (and 2d $\mathcal{N} = (2, 2)$ theories on S^2)

There are two seemingly distinct classes of examples from which to draw inspiration:

- Topologically twisted $\mathcal{N} = 2$ theories on $\mathcal{M} \rightarrow$ Donaldson invariants Witten...
- Untwisted $\mathcal{N} = 2$ theories on $S^4_b \rightarrow$ Partition function/ Susy Wilson loops Pestun....

Window on the dynamics of strongly interacting QFT.

Interplay with geometry of $\mathcal{M}_4$.

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Interplay with geometry of $\mathcal{M}_4$.

What is the most general setting to explore? Klare, Zaffaroni; Butter, Inverso, Lodato

Motivation 1: Donaldson-Witten

Twisting of $\mathcal{N} = 2$ SYM results in topological theory Witten
This theory localizes on instantons on four manifold $\mathcal{M}_4$.

$$F^+ = 0 \qquad\qquad \text{Elliptic problem}$$

Schematically:

$$\int_{\mathcal{M}} \sqrt{g} F^2 = \int_{\mathcal{M}} \sqrt{g}\,(F^+)^2 + \int F \wedge F$$

Adding fermions and SUSY

$$\delta^2 = 0\ , \qquad S_{SYM} = \delta(...) + \text{topological term}$$

Adding equivariance

$\mathcal{M}_4$ can admit a torus action $T^2 \times \mathcal{M} \to \mathcal{M}$
e.g. $\mathbb{R}^4 = \mathbb{C}^2$ and $(z_1, z_2) \to (z_1 e^{i\theta_1}, z_2 e^{i\theta_2})$

Deformation of DW Losev, Moore, Nekrasov, Shatashvili

$$\delta^2 = \mathcal{L}_v \qquad \text{where} \qquad v = \epsilon_1 \frac{\partial}{\partial_{\theta_1}} + \epsilon_2 \frac{\partial}{\partial_{\theta_2}}$$

Instanton partition function on $\mathbb{R}^4_{\epsilon_1, \epsilon_2}$ Nekrasov

$$Z^{\text{inst}}_{\epsilon_1, \epsilon_2}(a, q) = Z_{1-\text{loop}}(a) \sum_n q^n \text{vol}_n(\epsilon_1, \epsilon_2, a)$$

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For other non-compact toric $\mathcal{M}_4$ Nekrasov; Gottsche, Nakajima, Yoshioka; Gasparim, Liu Bershtein; Bonelli, Ronzani, Tanzini

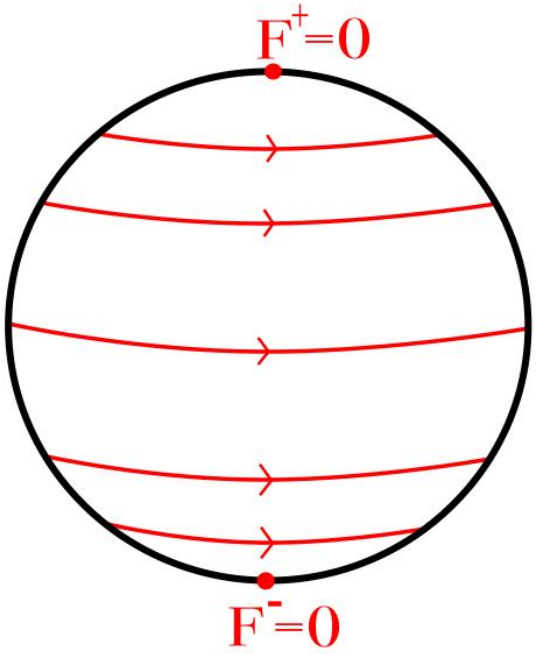
- The vector field can have multiple fixed points
- Flux sectors $H^2(\mathcal{M})$

$$Z(a, \epsilon_1, \epsilon_2) = \sum_{k \in \text{flux}} \prod_{i \in \text{fixed}} Z^{\text{inst}}_{\epsilon^i_1,\epsilon^i_2}(a + \Delta(k, i, \epsilon^i_{1,2}), q)$$

Pestun's sphere

Pestun uncovered a different way of placing $\mathcal{N} = 2$ SYM on the four-sphere S^4 preserving SUSY.

The theory localizes to instantons on the north pole and anti-instantons on the south pole
Pestun; Hama, Hosomichi.



$$Z = \int da Z^{\mathrm{inst}}_{\epsilon_1,\epsilon_2}(ia, q) Z^{\mathrm{antinst}}_{\epsilon_1,-\epsilon_2}(ia, \bar{q})$$

Localization involves transversely elliptic operators

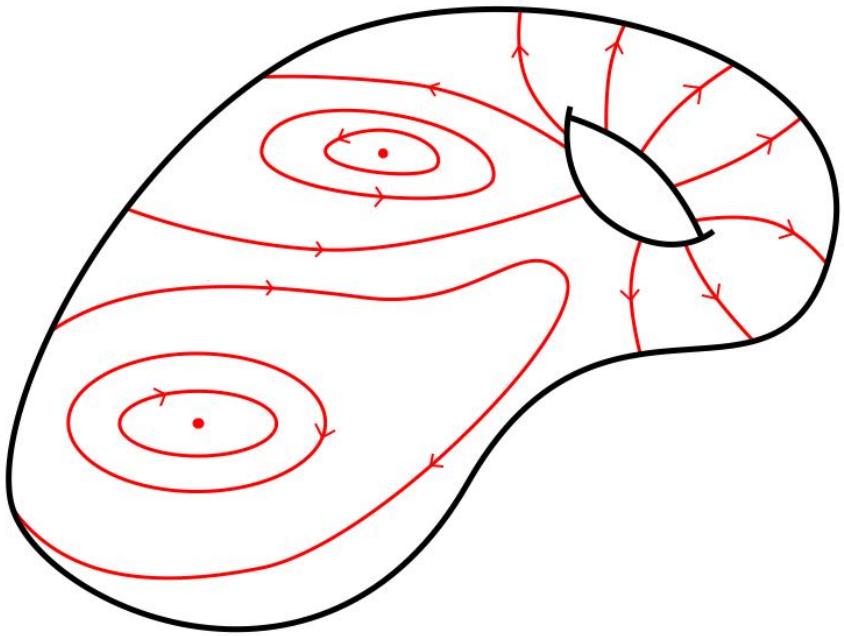
Is there a relation with topological twisting?

Summary of results

We construct a wide class of SUSY field theories on $\mathcal{M}_4$ that admit an isometry (even better a T^2 action)

Equivariant topological twisting and Pestun's theory are special cases.

Localizes to instantons or anti-instantons at different fixed points of the isometry



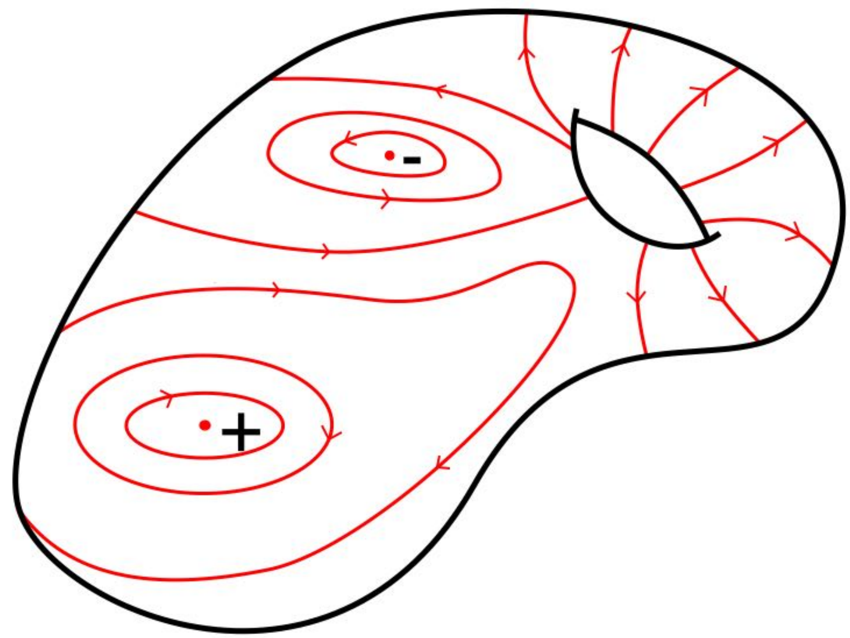
Susy observables are not topological but depend only weakly on metric.

Interplay between susy and transverse ellipticity.

Setting:

- $\mathcal{M}$ with smooth metric g.
- v real Killing vector field with (at most) isolated fixed points.

  Generically the orbits of v are not compact. There are then two isometries generating a torus action on $\mathcal{M}$.
- Equip fixed points with binary $\pm$ label.



These data can be used to specify a smooth $\mathcal{N} = 2$ supergravity background to which to couple $\mathcal{N} = 2$ SYM preserving susy.

The supercharge squares to a motion along v.

Today I will present a twisted description of the theory.

Cohomological theory: the DW complex

Equivariant Donaldson Witten theory is written using twisted fields:

$$A\ ,\psi,\phi\ ,\qquad \varphi\ ,\eta\ ,\qquad \chi^+,\ H^+$$

On which Susy acts as follows

$$\delta A = i\psi\ ,\qquad \delta\psi = \iota_v F + i d_A\phi\ ,\qquad \delta\phi = \iota_v\psi$$

$$\delta\chi^+ = H^+\ ,\qquad \delta H^+ = i\mathcal{L}_v^A\chi^+ - i[\phi,\chi^+]$$

$$\delta\varphi = i\eta\ ,\qquad \delta\eta = \iota_v d_A\varphi - [\phi,\varphi]$$

The algebra closes off shell

$$\delta^2 = \mathcal{L}_v + \text{gauge transf}$$

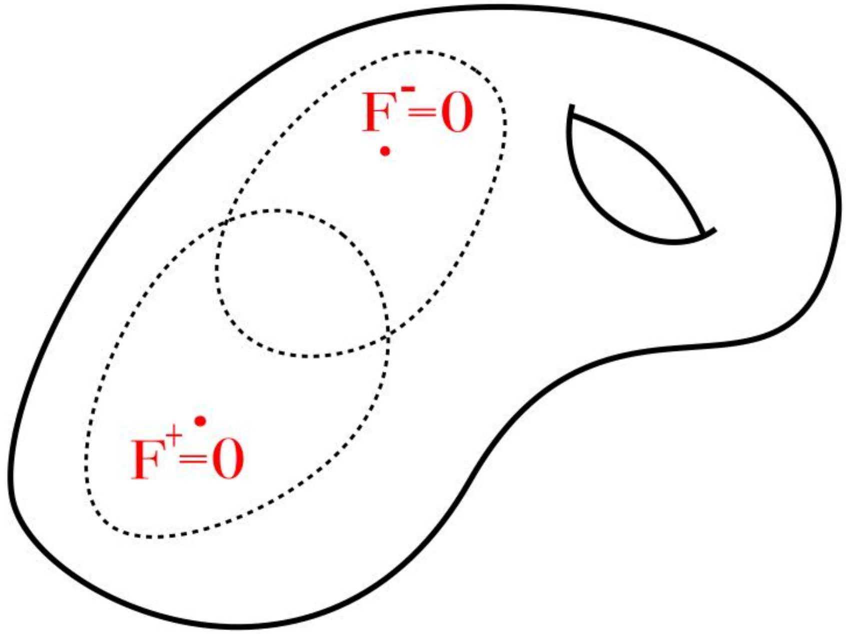
$\star$ (hence the metric) enters in defining $\chi^+,\ H^+$.

We want to replace self duality on χ, H with a different condition.

Self-duality near $-$ fixed points and anti-self duality near $+$ fixed points.

The vector field v gives a map between self-dual and anti sef-dual two forms ($\kappa = g(v)$)

$$B \to -B + \frac{2}{|v|^2} \kappa \wedge i_v B$$



It is well defined away from the fixed points. We can use it to glue together Ω^+ and Ω^- via transition functions away from $v = 0$.

This establishes a bundle of two forms with the desired property.

We can define a projector on two forms which depends on a function $0 < \omega < \pi$

$$P^+_\omega B = \frac{1}{1+\cos^2(\omega)}\left(B + \cos\omega \star B - \frac{\sin(\omega)^2}{|v|^2}\kappa \wedge i_v B\right)$$

- $(P^+_\omega)^2 = P^+_\omega$ and its eigenspaces are orthogonal.

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- $(P^+_\omega)^2 = P^+_\omega$ and its eigenspaces are orthogonal.
- it is well defined provided that $\sin(\omega) \to 0$ at fixed points
- Choose $\omega = 0$ at $+$ fixed points and $\omega = \pi$ at $-$ fixed points

$$P^+_\omega \to \frac{1}{2}(1+\star) \qquad \text{approaching} + \text{fixed points}$$

$$P^+_\omega \to \frac{1}{2}(1-\star) \qquad \text{approaching} - \text{fixed points}$$

New Twisted variables

We can "twist" ordinary fields in $\mathcal{N} = 2$ vector multiplet

$$X,\ \bar{X},\ A_\mu,\ \lambda^i_\alpha,\ \tilde{\lambda}^i_{\dot{\alpha}},\ D_{ij}$$

The twisted fields include two Grassmann even and one Grassmann odd scalars

$$\sigma\ ,\quad \phi\ ,\quad \eta$$

The connection and a Grassmann odd one form

$$A_\mu\ ,\quad \psi_\mu$$

Finally one even and one odd two forms

$$\chi_{\mu\nu}\ ,\quad H_{\mu\nu}\ .$$

These two forms satisfy

$$P^+_\omega \chi = 0\ ,\quad P^+_\omega H = 0\ .$$

These split into multiplets under the action of supersymmetry

$$\delta A = i\psi \ , \qquad \delta\psi = \iota_v F + i d_A \phi \ , \qquad \delta\phi = \iota_v \psi$$

$$\delta\chi = H \ , \qquad \delta H = i\mathcal{L}_v^A \chi - i[\phi, \chi]$$

$$\delta\varphi = i\eta \ , \qquad \delta\eta = \iota_v d_A \varphi - [\phi, \varphi]$$

Comment: the definition of σ and ϕ involves both X and $\bar{X}$. As a consequence the notion of holomorphy is changed.

$$\sigma = -i(X - \bar{X}) \ , \quad \phi = (X + \bar{X}) + \cos(\omega)(X - \bar{X})$$

Note however that ϕ approaches either X or $\bar{X}$ at the fixed points of v.

The action

$$\mathrm{Tr}\left[F \wedge \star F\right] = \mathrm{Tr}\left[(1+\cos^2\omega)(P^+_\omega F) \wedge \star F + \frac{\sin^2\omega}{||v||^2}\iota_v F \wedge \star \iota_v F\right]$$
$$- \cos\omega \; \mathrm{Tr}\left[F \wedge F\right] \; .$$

We use this identity to write a supersymmetric action

$$\mathcal{L} \; \mathrm{Vol}_M = \mathcal{O} + \delta\{\ldots\} \; .$$

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$$\mathcal{L}\ \mathrm{Vol}_M = \mathcal{O} + \delta\{\ldots\} \; .$$

$$\mathcal{O} = \frac{i}{4\pi}\int_{\mathcal{M}} \Big(\Omega_0 + \Omega_2 + \Omega_4\Big) \wedge \mathrm{Tr}\Big(\phi + \psi + \mathrm{F}\Big)^2$$
$$= \int_{\mathcal{M}} \Big(\Omega_0 \mathrm{Tr}(\mathrm{F}^2) + 2\Omega_2 \wedge \mathrm{Tr}(\phi \mathrm{F}) + \mathrm{Tr}(\phi^2)\Omega_4 + \Omega_2 \wedge \mathrm{Tr}(\psi^2)\Big) \; ,$$

$$\Omega_0 = \tau \sin^2\frac{\omega}{2} + \bar{\tau}\cos^2\frac{\omega}{2}$$

Ω_2 and Ω_4 are forms written explicitly in terms of ω and v_μ.

Why is $\mathcal{O}$ supersymmetric?

$$\mathcal{O} = \int_{\mathcal{M}} \Big(\cos(\omega) + \Omega_2 + \Omega_4 \Big) \wedge \mathrm{Tr}\Big(\phi + \Psi + \mathrm{F}\Big)^2$$

One can check that

$$\delta \mathrm{Tr}\Big(\phi + \Psi + \mathrm{F}\Big)^{\mathrm{k}} = (\mathrm{id}_{\mathrm{A}} + \iota_{\mathrm{v}}) \mathrm{Tr}\Big(\phi + \Psi + \mathrm{F}\Big)^{\mathrm{k}} .$$

Moreover $\Omega = \cos(\omega) + \Omega_2 + \Omega_4$ is equivariantly closed

$$(id + \iota_v)(\cos(\omega) + \Omega_2 + \Omega_4) = 0$$

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Moreover $\Omega = \cos(\omega) + \Omega_2 + \Omega_4$ is equivariantly closed

$$(id + \iota_v)(\cos(\omega) + \Omega_2 + \Omega_4) = 0$$

Note that shifting the equivariantly closed multi-form Ω by a equivariantly exact term leads to a Q-exact deformation of $\mathcal{O}$.

$$\Omega + (id_A + \iota_v)(\ldots) \Rightarrow \mathcal{O} + \delta(\ldots)$$

Deformations

Our theory could in principle depend on

- The function $\cos(\omega)$ entering the projector and Ω
- the metric on $\mathcal{M}_4$
- The choice of vector field v (that is ϵ_1 and ϵ_2 in toric case)

Deformations

Our theory could in principle depend on

- The function $\cos(\omega)$ entering the projector and Ω
- the metric on $\mathcal{M}_4$
- The choice of vector field v (that is ϵ_1 and ϵ_2 in toric case)

If we change $\cos(\omega)$ we need to modify $H_{\mu\nu}$ and $\chi_{\mu\nu}$. This only affects Q-exact terms. (provided that $\iota_v d\cos(\omega) = 0$).

The observable $\mathcal{O}$ changes as well because of Ω.

Formally the change in Ω is exact:

$$\Delta\Omega = (id + \iota_V)\left(\frac{\kappa \wedge \Delta\Omega}{\iota_V\kappa} - i\frac{\kappa \wedge d\kappa \wedge \Delta\Omega}{(\iota_V\kappa)^2}\right) \ .$$

For generic $\Delta\Omega$ this only makes sense away from the fixed points

It is well defined if $\Delta\cos(\omega)$ vanishes at the fixed points.

We conclude that $\cos(\omega)$ can be changed with a Q-exact deformation as long as the distribution of $\pm$ at the fixed points remains fixed.

Changing the metric while keeping v fixed and Killing can be analyzed similarly.

Again P^+_ω is changed which requires a redefinition of $H_{\mu\nu}$ and $\chi_{\mu\nu}$

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As long as the change in the metric is smooth and compatible with v it results in a Q-exact deformation.

Finally changing v is not a Q-exact deformation.

Hence susy observables depend on $\epsilon_{1,2}$.

We can take $\epsilon_{1,2}$ to be complex. Because Ω depends only on v and not v^* the dependence on $\epsilon_{1,2}$ is holomorphic.

Localization

General idea: add supersymmetric Q-exact terms that are positive definite to action: $t\,\delta(V) > 0$ where $t \in \mathbb{R}^+$.

Susy observables are unchanged. As $t \to +\infty$ path integral localizes on configurations such that $\delta(V) = 0$.

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- Need to choose reality conditions for bosonic fields. Can use those inherited from original theory:

$$(X)^* = \bar{X} \ , \qquad F^a_{\mu\nu} \in \mathbb{R} \ .$$

- Insure that action has positive real part. Dependence on $\mathcal{M}$.
- Choose localizing terms. Simplest choice also depends on $\mathcal{M}$.

For simply connected $\mathcal{M}$ end up with the following localization locus (for $SU(N)$ SYM):

$$\phi = \mathrm{diag}(\phi^i)\ , \qquad \varphi = \mathrm{diag}(\varphi^i)\ , \qquad i = 1, ..., N-1\ .$$

and

$$\phi^i = a^i - i\cos(\omega)\varphi^i \qquad a^i \in \mathbb{R}\ \mathrm{constant}$$

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The gauge group is broken to its Cartan H. One gets to integrate over H bundles satisfying.

$$\iota_v \varphi^i = 0 \ , \quad \iota_v F^i - d(\cos(\omega)\varphi^i) = 0 \ , \quad P^+_\omega \Omega^i = 0 \ .$$

where

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where

$$\Omega^i = F^i - \star(\kappa \wedge d\varphi^i) - \dots$$

Discrete solutions are distinguished by integer fluxes

$$\frac{1}{2\pi}\int_{C^n} F^a = k^a_n \ .$$

Partition function in simply connected case

The general answer for $Z_{\mathcal{M}_{\epsilon_1,\epsilon_2}}$ with p fixed points labelled by $+$ points and $(l-p)$ points labelled by $-$ will be given by

$$\sum_{k_i} \int_{\mathbf{h}} da\; e^{-S_{cl}} \prod_{i=1}^{p} Z^{\text{inst}}_{\epsilon_1^i,\epsilon_2^i}\left(ia + k_i(\epsilon_1^i,\epsilon_2^i), q\right) \prod_{i=p+1}^{l} Z^{\text{ainst}}_{\epsilon_1^i,\epsilon_2^i}\left(ia + k_i(\epsilon_1^i,\epsilon_2^i), \bar{q}\right) .$$

The parameters $(\epsilon_1^i, \epsilon_2^i)$ can be obtained from T^2-action around the fixed point x_i.

The precise form for the shifts in each flux sector can be worked out for specific cases e.g. $\mathbb{C}P^2$. Lundin, Ruggeri

The structure and regularization of the perturbative contributions can be worked out in general Mauch, Ruggeri

Dimensional Reduction [Bershadsky,Johansen, Sadov, Vafa]

Consider a $\mathcal{N} = 2$ theory on a product of Riemann surfaces $C \times \Sigma$.

Performing a partial topological twist on C and making C small we obtain a $\mathcal{N} = (2,2)$ σ-model on Σ whose target is the moduli space of flat connections on C.

The topologically twisted $\mathcal{N} = 2$ gives rise to the A twist of the σ-model.

For our framework we need a Killing vector with fixed points. For instance we can start with Σ being S^2.

What do we get upon reduction?

Dimensional Reduction

If both fixed points on $C \times S^2$ are of the same kind we get an (equivariant) A twist or $\bar{A}$ twist of the σ-model.

With fixed points of different kinds we should get an interpolation between the A twist and the $\bar{A}$ twist at the two poles of S^2.

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For the (equivariant) A model we have

$$
\begin{aligned}
&\delta X^{\mu} = \Psi^{\mu} \;, \quad \delta \Psi^{\mu} = \mathcal{L}_{V} X^{\mu} \;, \\
&\delta \chi^{\mu} = H^{\mu} - \Gamma^{\mu}_{\nu\rho} \Psi^{\rho} \chi^{\nu} \;, \\
&\delta H^{\mu} = \mathcal{L}^{\Gamma}_{\nu} \chi^{\nu} - \Gamma^{\mu}_{\nu\rho} \Psi^{\nu} H^{\rho} + \frac{1}{2} R^{\mu}_{\nu\rho\sigma} \chi^{\nu} \Psi^{\rho} \Psi^{\sigma} \;.
\end{aligned}
$$

The one forms χ^{μ} and H^{μ} are in the kernel of $\frac{1}{2}(1 + \star J)$ where J is the complex structure of the target space.

Dimensional Reduction

In the "exotic" theory the projector is $(\kappa = g(v))$

$$P^{+} = \frac{1}{1+\cos^2\theta}\big(1 - \cos(\theta) \star J - \kappa \wedge \iota_v\big)$$

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Up to δ exact terms the action is:

$$\int (\Omega_0 + \Omega_2)(\omega_{\mu\nu} dX^\mu \wedge dX^\nu + \omega_{\mu\nu}\Psi^\mu\Psi^n u)$$

where ω is the target space Käler form while $(\Omega_0 + \Omega_2)$ is equivariantly closed but not exact e.g.

$$(\Omega_0 + \Omega_2) = \cos\theta + \sin\theta d\phi \wedge d\theta$$

Dimensional Reduction

This theory is a cohomological rewriting of the $\mathcal{N} = (2,2)$ theories on S^2 studied by Benini, Cremonesi; Closset, Cremonesi; Jia, Sharpe; Doroud, Gomis, Le Floch, Lee

In particular one can consider GLSM flowing in the IR to $\mathcal{N} = (2,2)$ non-linear σ-models with Calabi-Yau target spaces. The corresponding partition functions on S^2 computes the quantum corrected Kähler potential for the Kähler moduli space of the Calabi-Yau. Jockers, Kumar, Lapan, Morrison, Romo; Gomis, Lee; Gerchkovitz, Gomis, Komargodski; Hsin, Komargodski, Schwimmer, Seiberg, Theisen

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Hence BPS configurations need to be constant except for defects at the poles. These singular configurations are hard to control.

BPS localization locus in the A model: holomorphic maps $X^i(z)$.

In the equivariant model one gets in addition that $\mathcal{L}_v X^\mu = 0$.

Hence BPS configurations need to be constant except for defects at the poles. These singular configurations are hard to control.

This problem persists in the "exotic" theory. However we can start by localizing around constant maps X^μ. We thus recover results of Halverson, Jockers, Lapan, Morrison; Hori, Romo directly from the σ-model.

Some directions to explore

- Study line/surface operators.
- Study cases with a lot of symmetry in detail e.g. $S^2 \times S^2$
- Understand the contribution of fluxes in general.
- Study how to include instanton corrections in 2d.
- Theory interpolating between B and $\bar{B}$ model?
- Generalization of AGT correspondence?

Thank You !