Target space geometry for topological strings

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String geometry in Stavanger

10th August 2022

Based on arxiv: 2108.09310

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Plan of talk:

1.) Instanton complexes

2.) BRST complex & $Z_{1\text{-loop}}$ for topological strings

3.) Generalised geometry for sugra & strings

4.) $G_L \times G_R$ Dolbeault (double) complex
& recovering topological string quantities

5.) Target space action for G_2 string

6.) other points

7.) Summary & Outlook

Instanton complex I

•) Gauge theory w/ gauge group G

•) Connection $A \in \Omega^1(\mathfrak{g})$ s.t. $F = *F \in \Omega^2(\mathfrak{g})$ ← Instanton

(locally)

I.e. $F_\pm \sim F \pm *F$

$F_- = 0$

$(P^- \circ d)$

•) Consider : $0 \to \Omega^0(\mathfrak{g}) \xrightarrow{d_A} \Omega^1(\mathfrak{g}) \xrightarrow{d_A} \Omega^2(\mathfrak{g})_- \to 0$

•) Moduli space $\dfrac{\{\delta A : \delta F_- = 0\}}{\delta A \sim \delta A + d_A \alpha} = H^1(\ \ )$

What happened there?

•) $so(4) \sim so(3)_+ \oplus so(3)_-$

$\| ?$

$\Lambda^2 T^*_p = (\Lambda^2 T^*_p)_+ \oplus (\Lambda^2 T^*_p)_-$ at each $p \in M$

•) $F_- = 0 \rightsquigarrow \not{D}\psi^- \sim \not{F}\epsilon^- = 0$

$\hookrightarrow F \in so(3)_+$

$SO(3)_+$ structure on M

Complex:

$$0 \to \Omega^0(\mathcal{Y}) \xrightarrow{d_A} \Omega^1(\mathcal{Y}) \xrightarrow{\check{d}_A} \Omega^2(\mathcal{Y})_- \to 0$$

$\Omega^2(\mathcal{Y}) / \Omega^2(\mathcal{Y})_+$

Instanton complex 2 [Reyes Carrion '95]

- de Rham complex $\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$
- G-structure for $G \subset SO(d)$
- (Really $N(G)$-structure for $N(G) = \{ g \in SO(d) : g G g^{-1} = G \}$)
- At $x \in M$, Lie algebra $\mathfrak{g} \subset \Lambda^2 T^*_x M \sim \mathfrak{so}(d)$
- Quotient $\left(\Omega^\bullet / _{\mathfrak{g} \wedge \Omega^\bullet}, \check{d} = P \circ d \right)$ (P: projector)
- Turns out $\check{d}^2 = 0$ on quotient
  if G structure integrable i.e. G-holonomy

Slight modification :

•) In case $su(N) \subset so(2N)$

$$\not{F}\epsilon = 0 \quad \Longleftrightarrow \quad \begin{cases} F^{(2,0)} = F^{(0,2)} = 0 \\ \omega \lrcorner F = 0 \end{cases}$$

•) Want : only $F^{(2,0)} = F^{(0,2)} = 0$ (holomorphic)

↖ matches holomorphic Chern-Simons / open topological string.

•) ∴ Quotient by $u(N) \wedge \Omega^\bullet = N(\mathfrak{g}) \wedge \Omega^\bullet$ instead

I.e. $SU(N) \subset SO(2N)$ structure

$\widetilde{G} = U(N)$ $\qquad$ $\widetilde{\mathcal{J}} \sim \Omega^{1,1} \subset \Omega^2$

$\Omega^{0,0}$

$\Omega^{1,0}$ $\quad$ $\Omega^{0,1}$

$\Omega^{2,0}$ $\quad$ $\Omega^{1,1}$ $\quad$ $\Omega^{0,2}$

$\Omega^{3,0}$ $\quad$ $\Omega^{2,1}$ $\quad$ $\Omega^{1,2}$ $\quad$ $\Omega^{0,3}$

$\vdots$ $\quad$ $\vdots$ $\quad$ $\ddots$ $\quad$ $\ddots$

$(H^1 \sim$ infinitesimal moduli of holomorphic connection$)$

so quotient $\longrightarrow$ Dolbeault complexes

$(\Omega^{\bullet,0}, \partial)$ $\qquad$ $(\Omega^{0,\bullet}, \bar{\partial})$

E.g. G_2 structure

$\varphi \in \Omega^3 \quad *\varphi \in \Omega^4$

$d\varphi = d*\varphi = 0$

$\tilde{G} = G_2$

$\tilde{\gamma} = \Omega^2_{(14)} \subset \Omega^2$

↑ (14 representation of G_2)

Quotient complex:

$$0 \to \Omega^0 \to \Omega^1 \to \Omega^2_{(7)} \to \Omega^3_{(1)} \to 0$$

$(\omega_{mn} = \varphi_{mnp} v^p)$ ↑ $(\omega_{mnp} = \lambda \varphi_{mnp})$

This is the G_2 Dolbeault complex.

($H^1 \sim$ moduli of "G_2-instantons" $\quad *\varphi \wedge F = 0$)

G_2 Hodge theory

Isomorphisms $\theta_1 : \Omega^0_1 \to \Omega^3_1$ $\qquad \theta_7 : \Omega^1 \to \Omega^2_7$

$$\begin{array}{ccccccc}
\Omega^0 & \xrightarrow{\check{d}} & \Omega^1 & \xrightarrow{\check{d}} & \Omega^2_7 & \xrightarrow{\check{d}} & \Omega^3_1 \\
\theta_1 \downarrow & & \theta_7 \downarrow & & \downarrow \theta_7^{-1} & & \downarrow \theta_1^{-1} \\
\Omega^3_1 & \xrightarrow{\check{d}^\dagger} & \Omega^2_7 & \xrightarrow{\check{d}^\dagger} & \Omega^1 & \xrightarrow{\check{d}^\dagger} & \Omega^0
\end{array}$$

$\rightsquigarrow$ Inner product $(\alpha_p, \beta_p) = \kappa_p \int \alpha \wedge * \beta$ $\qquad \begin{cases} 1 & p = 0,1 \\ 3 & p = 2 \\ 7 & p = 3 \end{cases}$

$\rightsquigarrow$ $\check{d}^\dagger \sim \theta \check{d} \theta$ $\quad$ (e.g. $\check{d}^\dagger_2 = \theta_7^{-1} \check{d} \theta_7^{-1}$)

$\rightsquigarrow$ $\check{\Delta}^2 = \theta_7 \check{\Delta}^1 \theta_7^{-1}$ $\qquad \check{\Delta}^3 = \theta_1 \check{\Delta}^0 \theta_1^{-1}$

Laplacians depend only on G_2 rep^n (not form degree)

Topological strings : worldsheet on CY_3 $N=(2,2)$ SCFT

Idea : Define new Lorentz symmetry s.t. $\exists$ scalar Q

with $Q^2 = 0$ & $\mathcal{L} = Q$-exact

Restrict theory to "physical states/operators"

$$\{Q, \mathcal{O}_i\} = 0$$

If $Q|0\rangle = 0$ then $\mathcal{O}_i \sim \mathcal{O}_i + \{Q, \Lambda\}$

as give same $\langle 0| \mathcal{O}_1 \cdots \mathcal{O}_k |0\rangle$

$\rightsquigarrow$ Physical ops form complex w/ differential Q

$$\cdots \longrightarrow \boxed{H}^k \xrightarrow{Q} \boxed{H}^{k+1} \xrightarrow{Q} \cdots$$

Observables $\longleftrightarrow$ cohomology

$N=(2,2)$ has fields: $(x^\mu, \psi_\pm^a, \bar\psi_\pm^{\bar a})$

& supercharges $(Q_\pm, \bar Q_\pm)$

A-model: Redefine Lorentz s.t. $Q = Q_- + \bar Q_+$

$\psi_-^a, \bar\psi_+^{\bar a}$ scalar

$\rightsquigarrow$ local dimensionless physical ops

$$\mathcal{O}_A = A_{a_1\dots a_p \bar b_1 \dots \bar b_q}\, \psi_-^{a_1}\cdots\psi_-^{a_p}\, \bar\psi_+^{\bar b_1}\cdots\bar\psi_+^{\bar b_q}$$

$\longleftrightarrow \Omega^{p,q}$ via $dz^a \leftrightarrow \psi_-^a$, $d\bar z^{\bar a}\leftrightarrow \bar\psi_+^{\bar a}$

Action of Q $\longleftrightarrow$ de Rham differential $d = \partial + \bar\partial$

$\therefore$ Observables $\longleftrightarrow$ de Rham cohomology $H^\bullet(X)$

B-model : $Q = \overline{Q}_+ + \overline{Q}_-$, $\eta^{\bar{a}} = \overline{\psi}_+^{\bar{a}} + \overline{\psi}_-^{\bar{a}}$

twist s.t. scalar → $\theta_a = g_{a\bar{b}} \left(\overline{\psi}_+^{\bar{b}} - \overline{\psi}_-^{\bar{b}} \right)$

Physical ops :

$$\mathcal{O}_B = B_{\bar{a}_1 \cdots \bar{a}_p}^{\quad b_1 \cdots b_q} \eta^{\bar{a}_1} \cdots \eta^{\bar{a}_p} \theta_{b_1} \cdots \theta_{b_q}$$

$\hookrightarrow \Omega^{0,p}(\Lambda^{q,0} T)$ $\quad \eta \mapsto d\bar{z}$, $\theta \leftrightarrow \frac{\partial}{\partial z}$

$Q \sim \bar{\partial}$ & observables $\longleftrightarrow H_{\bar{\partial}}^{0,\bullet}(\Lambda^{\bullet,0} T)$

Isomorphism $\Omega^{0,q}(\Lambda^{p,0} T) \overset{\Omega^{(n,0)}}{\cong} \Omega^{n-p,q}$

$\rightsquigarrow$ $Q_+ \sim \bar{\partial} + \partial^+$ $\quad Q_- \sim \bar{\partial} - \partial^+$

← Improved version Later !

Partition fns:

$$Z_1 = e^{-F_1} \qquad F_1 = \delta(H_L - H_R)\,\frac{1}{2}\log\left[\prod_{F_L,F_R} \det'(H_L+H_R)^{(-1)^{F_L F_R}}\right]$$

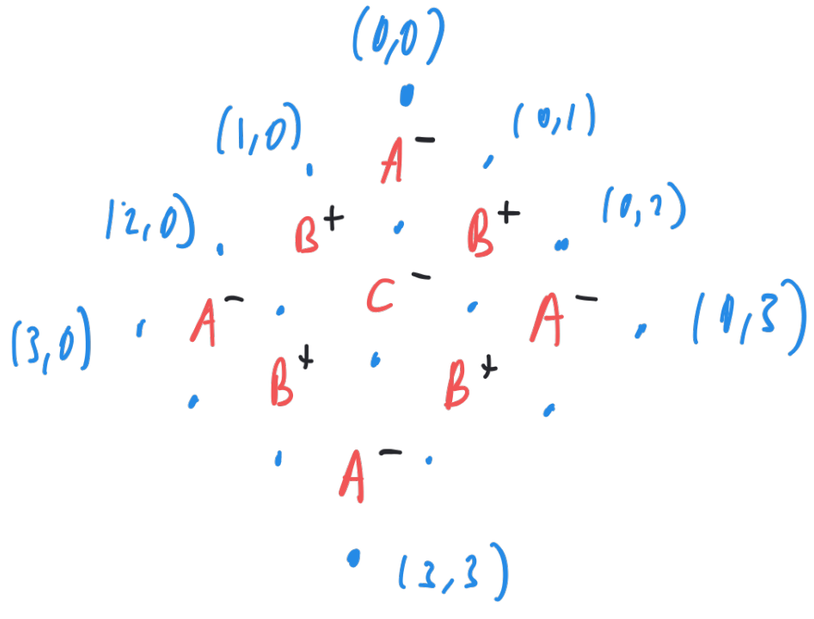
← [BCOV '93]

$$H_{L,R} = \{Q_{L,R}, Q^{\dagger}_{L,R}\}$$

A-model: $H = H_L + H_R \rightsquigarrow$ Laplacian $\Delta^{p,q}$ on $\Omega^{p,q}$

$$\rightsquigarrow Z_{1,A} = \left(\prod_{p,q} (\det'\Delta^{p,q})^{(-1)^{p+q}pq}\right)^{-\frac{1}{2}}$$

$$= (A^{-4}B^{4}C^{-1})^{\frac{1}{2}}$$



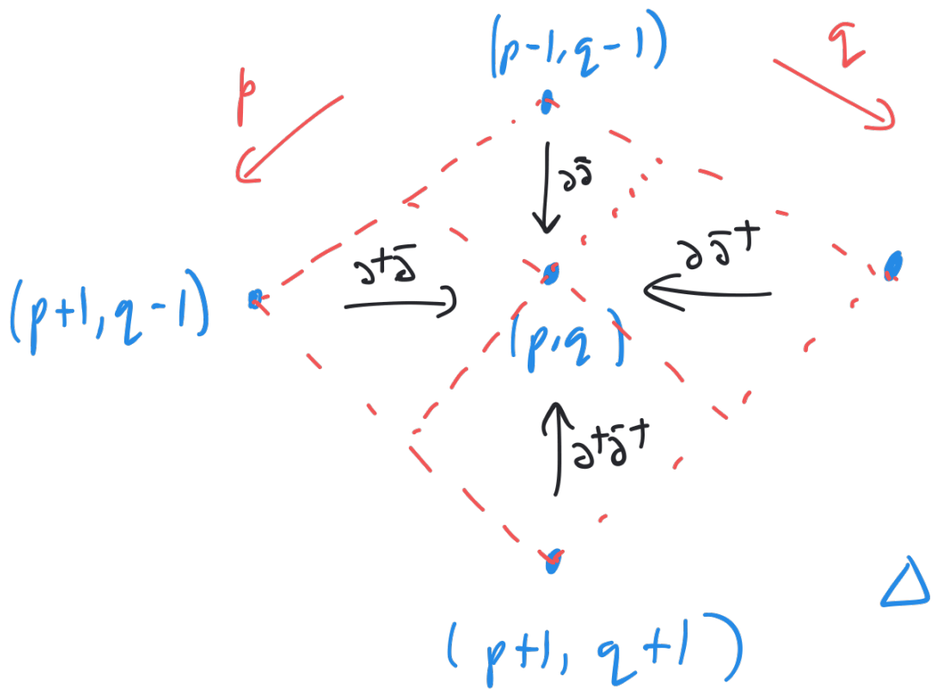
~ "Analytic torsion" of double complex

Checkerboard pattern : [Pestun & Witten]

•) For $H^{\bullet,\bullet} = \{0\}$ have decompositions :

$$\Omega^{p,q} = \partial\,\Omega^{p-1,q} \oplus \partial^\dagger \Omega^{p+1,q}$$

$$= \partial\bar\partial\,\Omega^{p-1,q-1} \oplus \partial\bar\partial^\dagger \Omega^{p-1,q+1} \oplus \partial^\dagger\bar\partial\,\Omega^{p+1,q-1} \oplus \partial^\dagger\bar\partial^\dagger\,\Omega^{p+1,q+1}$$



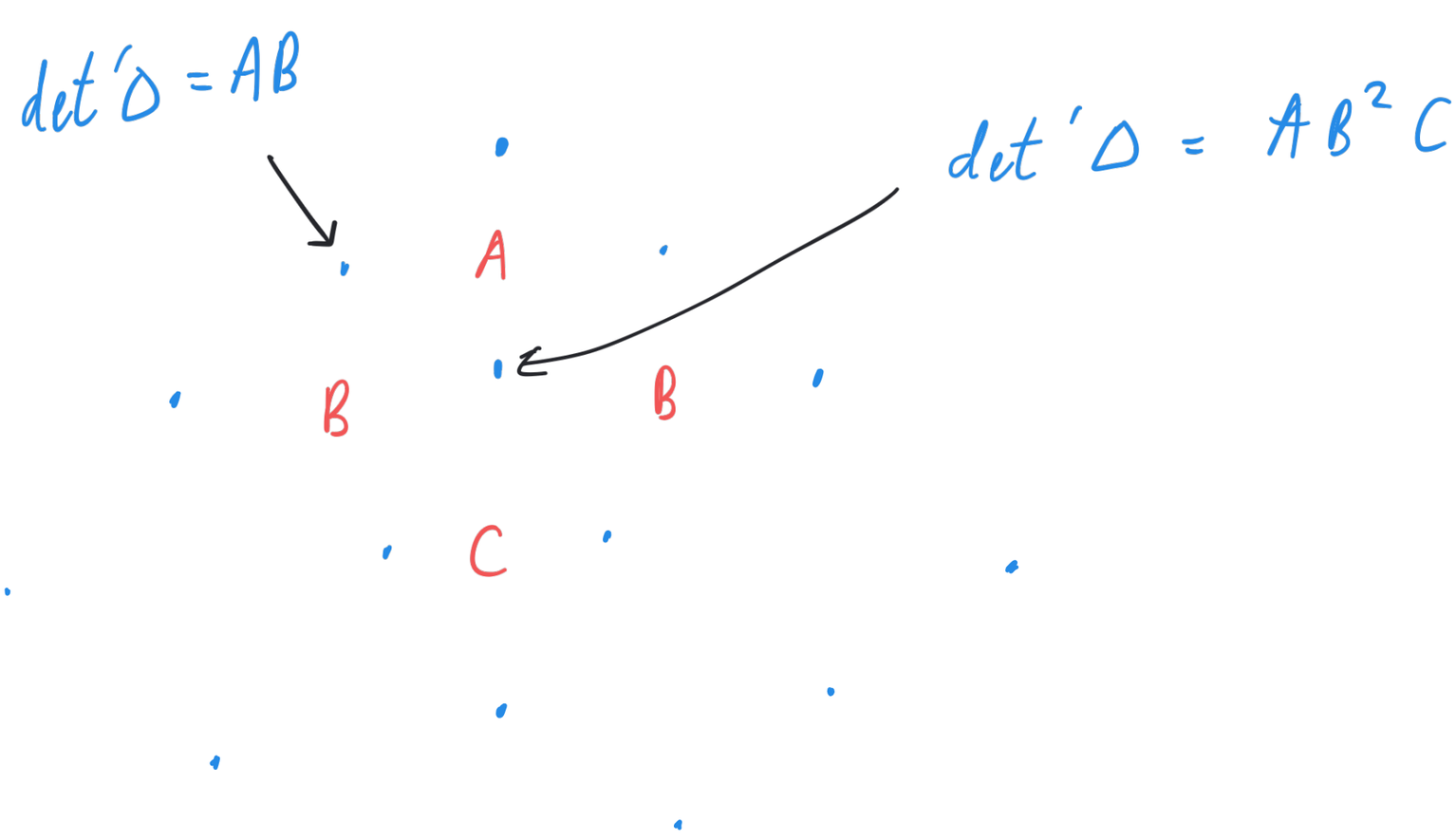
Laplacian preserves summands :

$$\Delta = \left(\overset{\bullet}{\Delta}\right) \oplus \left(\Delta\,\bullet\right) \oplus \left(\bullet\,\Delta\right) \oplus \left(\underset{\bullet}{\Delta}\right)$$

Checkerboard 2 :

1) Identities $\Rightarrow$ only a few independent $\det'\Delta$

$\hookrightarrow$ attached to squares :



Partition fⁿs (Cont.)

B-model gives same $Z_{1,B} = (A^{-4} B^4 C^{-1})^{\frac{1}{2}}$ [BCOV '93]

$\sim$ Ray-Singer torsion

Target space action:

Pestun & Witten '05 $\Rightarrow$ BV quantisation of:

$$S = \int b_{00} \wedge \partial\bar{\partial} b_{22} + b_{11} \wedge \partial\bar{\partial} b_{11}$$

gives $Z_{1,B}$ as partition fⁿ.

Quadratic order term of Hitchin functional for generalised CY

G_2 string [de Boer, Naqvi & Shomer]

- ∃ complicated CFT argument (via spectral flow etc.)
  that G_2 σ-model has "twist"

- L/R-moving physical ops:

$$f(x) \longrightarrow A_m(x)\,\psi^m \longrightarrow \underbrace{B_{mn}^{(7)}(x)}\,\psi^m\psi^n \longrightarrow \underbrace{C_{mnp}^{(1)}(x)}\,\psi^m\psi^n\psi^p$$

G_2 Dolbeault ↗

restricted to $\underline{7}$ & $\underline{1}$ of G_2 respectively

- BRST operator $G_{-\frac{1}{2}}$ ↓ as particular conformal block of $G_{-\frac{1}{2}}$

- Full BRST double complex is product of two of these.
  (L/R-moving)

Questions :

- B model looks different to A-model ? (A : $Q_L \sim \bar{\partial}$; B : $Q_L \sim \bar{\partial} + \partial^+$)
- spacetime picture of G_2 BRST (double) complex ?
- target space action for G_2 string @ 1-loop ?

Generalised Geometry

- Generalised tangent bundle $E \cong T \oplus T^*$
- Split signature metric $\langle v+\lambda, v+\lambda \rangle = i_v \lambda$

  $\rightsquigarrow$ Action of $O(d,d)$ on E
- Lie algebra $so(d,d) \simeq \Lambda^2 E \simeq (T \oplus T^*) \oplus \Lambda^2 T \oplus \Lambda^2 T^*$
- Dorfman derivative

$$\mathbb{L}_{(v+\lambda)} = \mathcal{L}_v - (d\lambda)\cdot$$

$\mathcal{L}_v$: diffeo

$(d\lambda)\cdot$: gauge transf^{n} $\delta B_2 = d\lambda$

Connections, Torsion & Metric

•) Connection = 1st order operator $D : Q \longrightarrow E^* \otimes Q$

e.g. $D_M V^A = \partial_M V^A + \Omega_M{}^A{}_B V^B$

$O(d,d)$ indices ↗

↖ $\partial_M = \begin{cases} \partial_m \\ 0 \end{cases}$ No "extended spacetime"

•) Torsion : $T(V) := \Omega_V^{(D)} - \Omega_V$

•) Metric : $(G_{AB}) = \frac{1}{2} \begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}$

•) Levi-Civita connection ? $T = 0$ $DG = 0$?

↳ Exists but not unique !

Gen Levi-Civita:

•) Gen metric $\rightsquigarrow$ $E \cong C_+ \oplus C_-$ (Each $C_\pm \cong TM$)

↖ $O(d) \times O(d)$ structure

In basis $\{ \hat{E}^+_a, \hat{E}^-_{\bar{a}} \}$

$$(G_{AB}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{\bar{a}\bar{b}} \end{pmatrix} \qquad (\eta_{AB}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & -g_{\bar{a}\bar{b}} \end{pmatrix}$$

•) Write gen Levi-Civita as 4 tangent bundle connections

$$D_a v^b = \nabla_a v^b - \frac{1}{6} H_a{}^b{}_c v^c + \frac{2}{9}\left(v_a \partial^b \phi - S_a{}^b v(\phi) \right) + A^{+\,b}_{a\ c}$$

$$D_a v^{\bar{b}} = \nabla_a v^{\bar{b}} - \frac{1}{2} H_a{}^{\bar{b}}{}_{\bar{c}} v^{\bar{c}}$$

$$D_{\bar{a}} v^b = \nabla_{\bar{a}} v^b + \frac{1}{2} H_{\bar{a}}{}^b{}_c v^c$$

$$D_{\bar{a}} v^{\bar{b}} = \nabla_{\bar{a}} v^{\bar{b}} + \frac{1}{6} H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} v^{\bar{c}} + \frac{2}{9}\left(v_{\bar{a}} \partial^{\bar{b}} \phi - S_{\bar{a}}{}^{\bar{b}} v(\phi) \right) + A^{-\,\bar{b}}_{\bar{a}\ \bar{c}}$$

(Undetermined in ☐ rep of $O(d)$)

Why do this ??

(Take RR = 0 here)

Type II - sugra : $G \sim (g, B_2, \phi)$ $\quad E \sim$ diffeo + gauge

$\exists$ curvatures $R_{a\bar{b}}$, R $\sim$ Bosonic e.o.m.

Fermions : $\rho^{\pm} \in S(C_{\pm})$ $\quad \begin{cases} \psi^{+}_{\bar{a}} \in C_- \otimes S(C_+) \\ \psi^{-}_{a} \in C_+ \otimes S(C_-) \end{cases}$

SUSY : $\delta \psi^{+}_{\bar{a}} = D_{\bar{a}} \epsilon^{+}$ $\quad \delta \rho^{+} = \gamma^a D_a \epsilon^{-}$

Fermion e.o.m. : $\gamma^a D_a \psi^{+}_{\bar{a}} - D_{\bar{a}} \rho^{+} = 0$ $\quad$ etc.

[Coimbra, CSC & Waldram '11]

Also: string worldsheets: E.g [Blair, Malek & Routh '13] [CSC '21]

For $x : \Sigma_2 \longrightarrow M$ set $v^m = \frac{\partial x^m}{\partial \tau}$ $\tilde{v}^m = \frac{\partial x^m}{\partial \sigma}$

& $v^{\pm} = v \pm \tilde{v} \in C_{\pm} \rightsquigarrow V \in E$

Set $P_M = G_{MN} V^N$ "conjugate momentum"

& string e.o.m. is $\mathbb{L}_V P = 0$ ← (gauge fixed $h_{\alpha\beta} = \eta_{\alpha\beta}$)

Virasoro constraints $G(V,V) = \eta(V,V) = 0$

Moral of story: $C_{\pm}$ are "left/right moving directions along string worldsheets"

SUSY backgrounds :

Bosonic background is SUSY if $\exists$ non-vanishing spinors $\epsilon^{\pm}$

$$\text{s.t.} \quad \delta_{\epsilon^{\pm}} \psi^{\pm} = \delta_{\epsilon^{\pm}} \rho^{\pm} = 0$$

In gen geom : [Coimbra, CSC & Waldram '14, Coimbra & CSC '16]

SUSY $\Longleftrightarrow$ Vanishing intrinsic torsion of gen. G-str defined by $\epsilon^{\pm}$

$\Longleftrightarrow$ $\exists$ choice of gen. Levi-Civita connection s.t.

$$D \epsilon^{\pm} = 0$$

$G \times G$ Dolbeault complex:

•) $\epsilon^{\pm}$ defines:

$$\begin{cases} SU(3) \times SU(3) \subset SO(6) \times SO(6) & \text{in } d = 6 \\ G_2 \times G_2 \subset SO(7) \times SO(7) & \text{in } d = 7 \\ Spin(7) \times Spin(7) \subset SO(8) \times SO(8) & \text{in } d = 8 \end{cases}$$

•) Build double complex

$$A^{p,q} = \bigwedge^p_r C^*_+ \otimes \bigwedge^q_s C^*_-$$

project onto relevant representations of $G \times G$

so tensor product of instanton complexes

Differentials :

$$d_+ : \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p+1,q} \qquad d_+ = P \circ (D_+ \wedge_{(+)})$$

$$d_- : \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p,q+1} \qquad d_- = P \circ (D_- \wedge_{(-)})$$

i.e. $P_{a_1 \cdots a_{p+1}}{}^{c_1 \cdots c_{p+1}} \left((p+1)\, D_{[c_1} X_{c_2 \cdots c_{p+1}]} \bar{\theta}_1 \cdots \bar{\theta}_q \right)$

Torsion-free (i.e. SUSY) $G \times G$ structure

$$\Rightarrow \quad d_+^2 = d_-^2 = \{ d_+, d_- \} = 0$$

↳ (Non-trivial !)

A- & B-models revisited

$SU(3) \times SU(3)$ splits $C_\pm = C_\pm^{1,0} \oplus C_\pm^{0,1}$ $\quad m \to (a, \tilde{a})$ $\quad \bar{m} \to (\bar{a}, \tilde{\bar{a}})$

A-model ops : $\mathcal{O}_A = A_{a_1 \dots a_p \bar{b}_1 \dots \bar{b}_q} \underbrace{\psi_-^{a_1} \dots \psi_-^{a_p}}_{\text{L-moving}} \underbrace{\overline{\psi}_+^{\bar{b}_1} \dots \overline{\psi}_+^{\bar{b}_q}}_{\text{R-moving}}$

$\therefore$ Take $\mathcal{A}^{p,q} = \Lambda^{p,0} C_+^* \otimes \Lambda^{0,q} C_-^* \cong \Lambda^{p,q} T^*$

$\mathcal{O}_A \longleftrightarrow A_{a_1 \dots a_p \bar{b}_1 \dots \bar{b}_q} \left(E_+^{a_1} \wedge \dots \wedge E_+^{a_p} \right) \otimes \left(E_-^{\tilde{\bar{b}}_1} \wedge \dots \wedge E_-^{\tilde{\bar{b}}_q} \right) \in \mathcal{A}^{p,q}$

$Q_L \longleftrightarrow d_+$ $\quad$ $Q_R \longleftrightarrow d_-$

$\curvearrowleft$ "∂" $\quad$ $\curvearrowleft$ "$\bar{\partial}$" in $\Omega^{p,q}$ picture

B-model: write $\mathcal{O}_B = B_{\bar{a}_1 \dots \bar{a}_p, \bar{b}_1 \dots \bar{b}_q} \overline{\psi}_-^{\bar{a}_1} \dots \overline{\psi}_-^{\bar{a}_p} \overline{\psi}_+^{\bar{b}_1} \dots \overline{\psi}_+^{\bar{b}_q}$

$\Omega^{0,p} \otimes \Omega^{0,q}$ (Levi-Civita)

$\overline{\psi}_-^{\bar{a}_1} \dots \overline{\psi}_-^{\bar{a}_p}$: L-moving; $\overline{\psi}_+^{\bar{b}_1} \dots \overline{\psi}_+^{\bar{b}_q}$: R-moving

Find:

$$Q_L \mathcal{O}_B \longrightarrow \nabla_{[\bar{a}_1} B_{\bar{a}_2 \dots \bar{a}_{p+1}]\bar{b}_1 \dots \bar{b}_q}$$

$$Q_R \mathcal{O}_B \longrightarrow (-1)^p \nabla_{[\bar{b}_1} B_{|\bar{a}_1 \dots \bar{a}_p|\bar{b}_2 \dots \bar{b}_{q+1}]}$$

I.e. $\mathcal{A}^{p,q} = \Lambda^{0,p} C_+^* \otimes \Lambda^{0,q} C_-^*$

$\begin{cases} Q_L \leftrightarrow d_+ \\ Q_R \leftrightarrow d_- \end{cases}$

$$\mathcal{O}_B \leftrightarrow B_{\tilde{a}_1 \dots \tilde{a}_p \tilde{\bar{b}}_1 \dots \tilde{\bar{b}}_q} \left(E_+^{\tilde{a}_1} \wedge \dots \wedge E_+^{\tilde{a}_p} \right) \otimes \left(E_-^{\tilde{\bar{b}}_1} \wedge \dots \wedge E_-^{\tilde{\bar{b}}_q} \right)$$

A & B-models with H-flux [Kapustin & Li '04]

$N=(2,2)$ target geometry $\longleftrightarrow$ Generalised Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$

$$\begin{cases} \mathcal{J}_i \in End(E) & \mathcal{J}_i^2 = -\mathbb{1} \qquad E = L_i \oplus \bar{L}_i \leftarrow (\pm i \text{ eigenspcs}) \\ G = -\mathcal{J}_1\mathcal{J}_2 \text{ +ve def} & \end{cases} \qquad ([L_i, L_i] \subset L_i)$$

$$[\mathcal{J}_1, \mathcal{J}_2] = 0 \Rightarrow E = \underbrace{L_{++}}_{C_+^{1,0}} \oplus \underbrace{L_{+-}}_{C_-^{1,0}} \oplus \underbrace{L_{-+}}_{C_-^{0,1}} \oplus \underbrace{L_{--}}_{C_+^{0,1}}$$

A-model:

$$\mathcal{A}_A^{\bullet,\bullet} = \Lambda^\bullet\left(C_+^{*1,0} \oplus C_-^{*0,1}\right) = \Lambda^\bullet \bar{L}_2$$

B-model:

$$\mathcal{A}_B^{\bullet,\bullet} = \Lambda^\bullet\left(C_+^{*0,1} \oplus C_-^{*0,1}\right) = \Lambda^\bullet \bar{L}_1$$

Differential d_L dual to $[\,,\,]|_L \rightsquigarrow d_L = d_+ + d_-$

"Kähler" identities : $(G_2 \times G_2)$

•) Generalised metric $\rightsquigarrow$ +ve def inner product on $\mathcal{A}^{\bullet,\bullet}$

$$\langle X, Y \rangle = K_p K_q \int \sqrt{g}\, e^{-2\phi}\, X^{a\ldots,\bar{b}\ldots}\, Y_{a\ldots,\bar{b}\ldots}$$

$K_p K_q$ → As in (single) G_2 Dolbeault complex

•) Define $\langle d_\pm^\dagger X, Y \rangle = \langle X, d_\pm Y \rangle$

Explicitly e.g. $(d_+^\dagger \alpha)_{a_2\ldots a_p,\bar{b}_1\ldots\bar{b}_q} \sim D^{a_1} \alpha_{a_1\ldots a_p,\bar{b}_1\ldots\bar{b}_q}$

•) Single complex identities e.g. $\check{d}^\dagger|_1 = \theta_1 \check{d} \theta_7$

$\Rightarrow$ $\boxed{\{d_+^\dagger, d_-\} = \{d_-^\dagger, d_+\} = 0}$

Hodge theory

-) Define $\Delta_{\pm} = d_{\pm}^{\dagger} d_{\pm} + d_{\pm} d_{\pm}^{\dagger}$

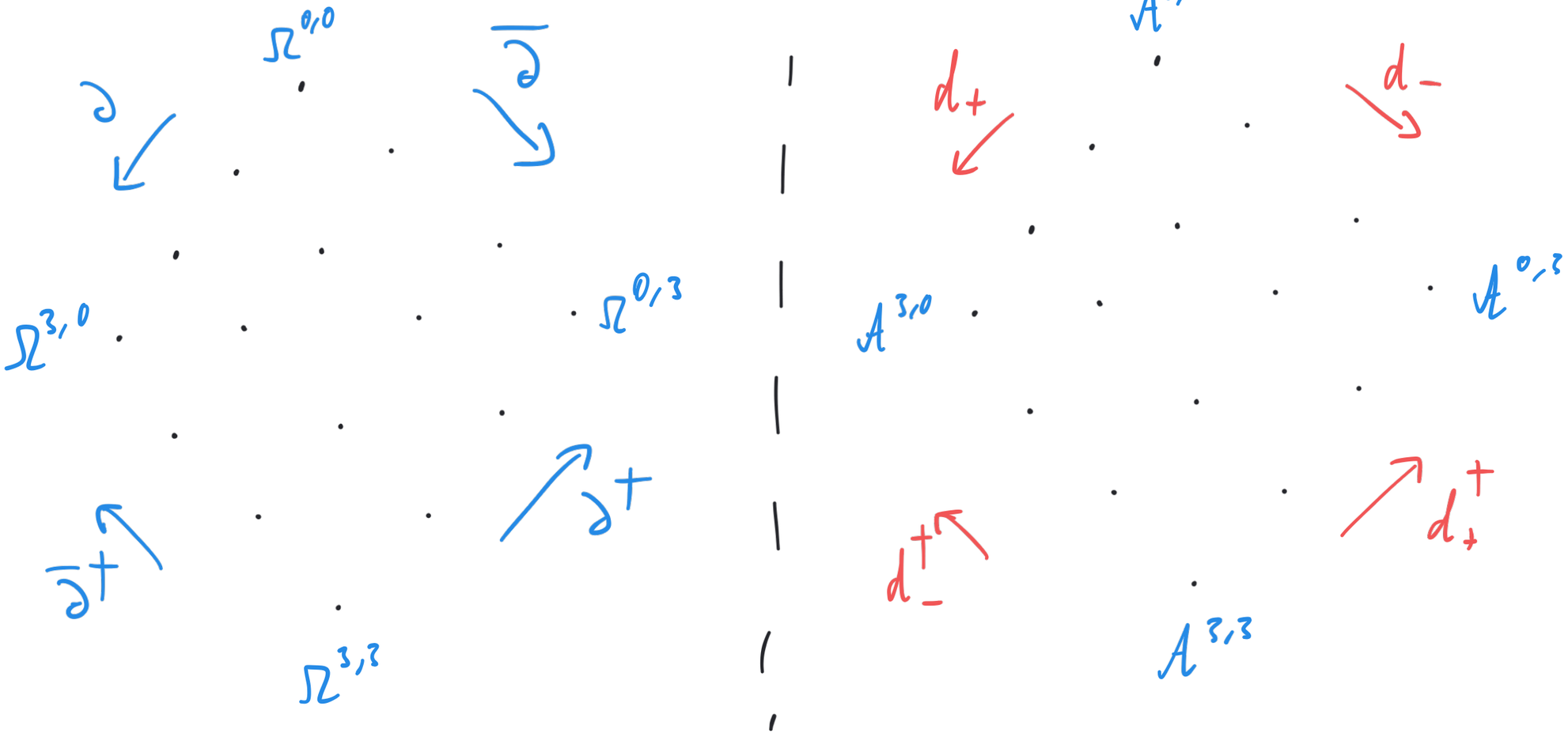
-) Depends only on $G_2 \times G_2$ rep $\underline{n}$

e.g. $\Delta_+^{(2,3)} = (\theta_+)_{\underline{7}} (\theta_-)_{\underline{1}} \Delta_+^{(1,0)}$

-) In fact $\Delta_+ = \Delta_- = \frac{1}{2}\Delta$

Proof : Direct calc$^{\underline{n}}$ for $(0,0), (1,0), (1,1)$ cases. □

Summary :



Usual $\Omega^{p,q}$ on $dim_{\mathbb{C}} = 3$

Kähler manifold

$G_2 \times G_2$ double complex $A^{p,q}$

on G_2 manifold

Same symbolic algebra

G_2 string BRST complex

•) L/R - moving operators each form G_2 Dolbeault single complex

$$C^{\bullet}: \quad 0 \to \Omega^0 \xrightarrow{Q_{L/R}} \Omega^1 \xrightarrow{Q_{L/R}} \Omega^2_{(7)} \xrightarrow{Q_{L/R}} \Omega^3_{(1)} \longrightarrow 0$$

$$\text{with} \quad Q_{L/R} = \check{d} = P \circ d$$

•) Our $A^{p,q} \equiv C_+^p \otimes C_-^q$ is full BRST complex

$$Q_L \sim d_+ \qquad Q_R \sim d_-$$

•) Physical states $\longleftrightarrow H^{p,q}(A)$

$\longleftrightarrow$ Harmonic forms of $\Delta_+^{p,q} = \Delta_-^{p,q}$

1-loop partition fn

Using standard worldsheet formula & identify:

$$H_L \to \Delta_+ \qquad H_R \to \Delta_- \qquad H_L + H_R \to \Delta$$

$$Z_1 = \left[\prod_{p,q} (\det{}' \Delta^{p,q})^{(-1)^{p+q} pq} \right]^{-\frac{1}{2}}$$ ~ analytic torsion

$(G_2 \times G_2)$

$$= (\det{}' \Delta_{(1,1)})^{-\frac{9}{2}} (\det{}' \Delta_{(7,1)})^{3/2} (\det{}' \Delta_{(1,7)})^{3/2} (\det{}' \Delta_{(7,7)})^{-\frac{1}{2}}$$

$(G_2 \text{ holonomy})$

$$= (\det{}' \Delta_1)^{-5} (\det{}' \Delta_7)^{5/2} (\det{}' \Delta_{14})^{-\frac{1}{2}} (\det{}' \Delta_{27})^{-\frac{1}{2}}$$

[de Boer, de Medeiros, EL-Showk, Sinkowics]

Is there a target space action which gives this?

Target space action

-) Pestun & Witten $\Rightarrow$ $S = \int_M \left(b^{1,1} \partial\bar{\partial} b^{1,1} + a^{0,0} \partial\bar{\partial} c^{2,2} \right)$

  gives analytic torsion of usual $\Omega^{p,q}$ complex.

-) $\therefore$ Guess take $b^{1,1} \in \mathcal{A}^{1,1}$ $a^{0,0} \in \mathcal{A}^{0,0}$ $c^{2,2} \in \mathcal{A}^{2,2}$

  & $S = \int_M \sqrt{g} e^{-2\phi} (\theta_!^+)^{-1} (\theta_!^-)^{-1} \left[b^{1,1} \wedge d_+ d_- b^{1,1} + a^{0,0} \wedge d_+ d_- c^{2,2} \right]$

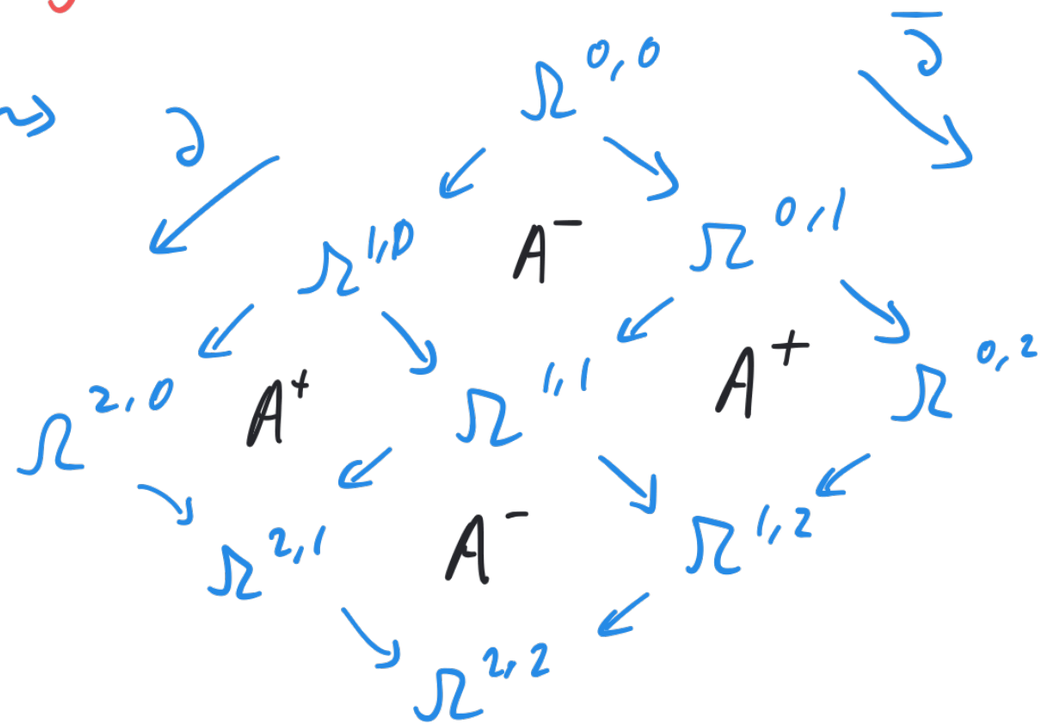
Then quantise in BV formalism ...

$\rightsquigarrow$ Reproduces $Z_{1\text{-loop}}$

K3 : Have $(\Omega^{0,\bullet}, \bar{\partial})$ on left & right

(for any of S^2 of $\mathbb{C}$-sts !)

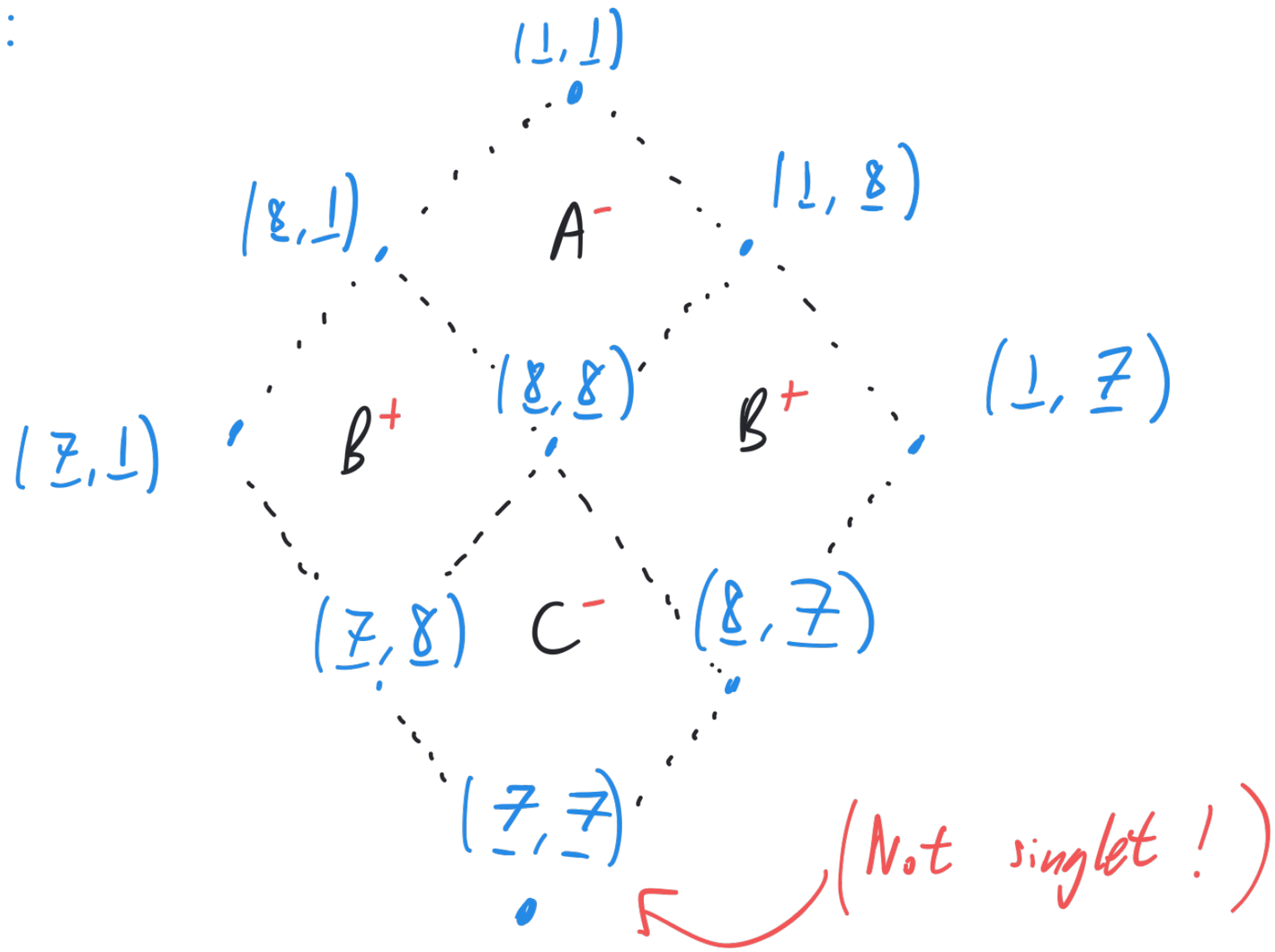
∴ A- & B-models both ⇝



⇝ $Z = 1$ as expected.

(Could write all this without considering worldsheet !)

Spin(7) :



•) Naive $Z_1 = (A^{-1} B^2 C^{-1})^{\frac{1}{2}}$

•) ∃ analogue of G_2 worldsheet argument for this.

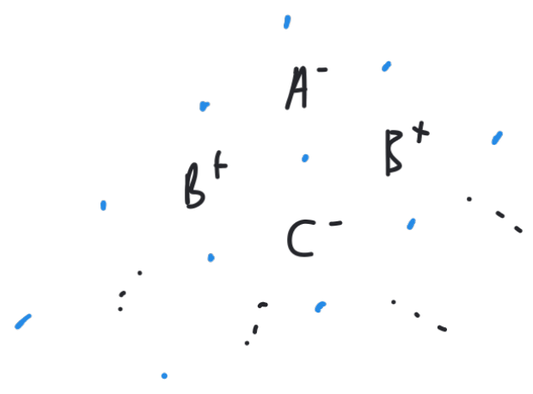
Summary :

•) All SUSY Minkowski backgrounds w/ NS-NS fluxes have a Dolbeault double complex

$$A^{\bullet,\bullet} = \mathcal{P}\left(\Lambda^\bullet C_+^* \otimes \Lambda^\bullet C_-^*\right) \qquad \begin{cases} d_+ \sim \mathcal{P}(D_+ \wedge) \\ d_- \sim \mathcal{P}(D_- \wedge) \end{cases}$$

•) This provides local ops & L/R-moving BRST operators for corresponding twisted strings

•) Kähler identities & Hodge theory



"Analytic torsion"

$$Z_{1\text{-loop}} = \left(\prod_{p,q} \left(\det' \Delta^{p,q}\right)^{(-1)^{p+q} pq}\right)^{-\frac{1}{2}}$$

$$= \left(\prod_{\text{squares}} \left(\det \text{ in square}\right)^{\text{"sign"}}\right)^{\frac{1}{2}}$$

Outlook :

- Defined $\mathcal{A}$ as tensor product of gauge / open string complexes

  $\rightsquigarrow$ other quantities from double copy relations here ?

- Pestun & Witten :

$$S_{quadratic} = \int b'' \partial\bar{\partial} b'' + b^{00} \partial\bar{\partial} b^{22}$$

$\Phi \in \Lambda^{\bullet} T^*_{\mathbb{C}}$ defining GCY structure

$$= \delta^2 S_{Hitchin}[\Phi]$$

Variation under $\Phi \to \Phi + d\psi$

- $G_2 \times G_2$ Hitchin functional does not work. What does ?

Outlook :

1.) Construction also generalises gravitational instantons :

Take $0 \to T \to T \otimes T^* \to T \otimes \Lambda^2 T^* \to T \otimes \Lambda^3 T^* \to$ (No d exists)

(diff) ↑ $\sim \mathfrak{gl}(d,\mathbb{R})$ ↑ (torsion)

Quotient by $\mathfrak{g} \wedge \Lambda^0 T^*$

$\rightsquigarrow$ $0 \to T \xrightarrow{d_\nabla} \mathfrak{g}^\perp \xrightarrow{d_\nabla} T^{(int)} \to \dots$

↑ $M^1 =$ "moduli of torsion-free G-structures"

•) Take $0 \to \Lambda^0 E \xrightarrow{d} \Lambda^1 E \to \Lambda^2 E \xrightarrow{(\text{No } d)} \Lambda^3 E \to$

(degree -1 ghost) (GDiff) $\sim so(d,d)$ (generalised torsion)

Quotient by $(\mathfrak{g}_L \oplus \mathfrak{g}_R) \curvearrowright \Lambda^0 E$

$\rightsquigarrow \quad 0 \to \Lambda^0 E \to E \to (\mathfrak{g}_L \oplus \mathfrak{g}_R)^+ \to T^{(\text{int})} \to \dots$

$H^1 =$ "moduli of $G_L \times G_R$ structures in $O(d,d)$"

$\sim$ SUSY backgrounds

Total complex of $A^{\bullet,\bullet}$!

Starting point complex $\sim$ tensor hierarchy

•) M-theory case ?

- Bracket structures? Interactions?

  Link with twisted supergravity? [Costello & Li '16]

- Calculated det's assuming no cohomology.

  Also need to wonder about gravitational anomalies.

  - G_2 partition function not topological (base metric dependence)

  If take everything into account, find true invariants?

Thanks for listening!