PARAMETRIZING RELATIVISTIC EFFECTS ON GRAVITATIONAL WAVEFORMS

NPACT 2022, Stavanger

By Vegard Undheim, PhD student at UiS

18th of August

- Outline

Who am I?

What is the inspiral?

Orbital mechanics at 1PN - short scale

GW production at 1PN - long scale

GW waveform at 1PN

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  - From Undheim, here in Rogaland *(close to Bryne)*.
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  - Thesis was presented at "Fysikermøtet" in 2021.
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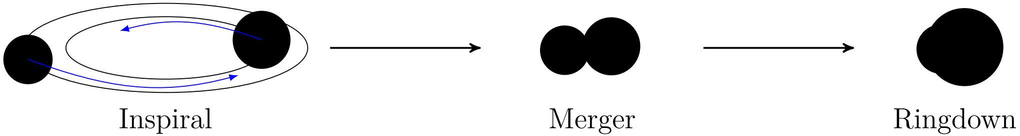
What is the inspiral? - A phase of compact binary evolution



- The inspiral - the first phase.
  - The orbit is to a good approximation circular, and Keplerian, with some small corrections scaling as $\left(\frac{v}{c}\right)^{2n}$ corresponding to the nth post-Newtonian correction (nPN).

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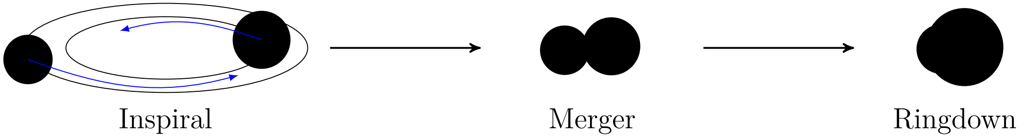

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What is the inspiral? - How to determine the orbit as a function of time

Seperate into two different scales (of both time and distance):

Short scale ($t \sim \frac{2\pi}{\omega}$, $L \sim r \gg R_s$)

- The effects of gravitational wave energy dissepation can be neglected.
- Expand GR action to desired order in $\left(\frac{v}{c}\right)^2$ and $\frac{GM}{rc^2} \simeq \left(\frac{v}{c}\right)^2$, and compute the Lagrangian.
- From the Lagrangian, compute the Hamiltonian to obtain energy as a function of orbital frequency, $E(\omega)$.

Long scale ($t \gg \frac{2\pi}{\omega}$, $L \sim \lambda_{GW} \gg r$)

- The binary can be approximated as a point source with multipole structure for gravitational waves.
- We can use the EoM obtained at short scale to determine the motion of the source (the effect of radiation on the binary is *still* not included).
- Obtain the energy flux as a function of orbital frequency, $\mathcal{F}(\omega)$.

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What is the inspiral? - Adding effects of radiation to the orbit: inspiral

How?- Demand energy conservation!

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}. \tag{1}$$

Then the phase of the binary can be analytically expressed as

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \omega \quad \Rightarrow \quad \mathrm{d}\Phi = \omega\,\mathrm{d}t = -\frac{\omega}{\mathcal{F}}\mathrm{d}E = -\frac{\omega}{\mathcal{F}}\frac{\mathrm{d}E}{\mathrm{d}\omega}\mathrm{d}\omega\,. \tag{2}$$

There are a lot of approximations being preform to obtain $E(\omega)$ and $\mathcal{F}(\omega)$, so next these wil be presented and kept track of throughout the computation until $\Phi(t)$.

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Orbital mechanics at 1PN - short scale - Two categories of expansion

Because of Keplers third law $\omega^2 = GM/r^3$ and $v^2 \simeq r^2\omega^2 = GM/r$, expansions in the coupling constant belongs in the PN expansion. We can thus use strategies from QFT!

Thus I have organized the short scale expansion into 2 categories:

Relativistic velocity expansions (mostly SR)

- C_τ Expansion of proper time $\mathrm{d}\tau = \gamma^{-1}\,\mathrm{d}t$
- C_E The energy-momentum tensor $T^{00}_{pp} = \gamma mc^2, \quad T^{ij}_{pp} = \gamma m v^i v^j$
- C_s Simultanity of the propegator/Greens function $\Box = -\partial_0^2 + \nabla^2 = \nabla^2\left(1 - \frac{\partial_0^2}{\nabla^2}\right)$ *(makes more sense in Fourier space).*

Gravitational expansions (mostly static)

- C_{prop} Expansion of the graviton action/EoM.
- C_{int} Expansion in the interaction term.
- C_{ten} Velocity coupling in the interaction term (dependent on tensor structure of graviton action).

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Orbital mechanics at 1PN - short scale - Expanding the graviton action

Expanding the EH action to quadratic order in metric pertubations ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) and imposing the linearized version of coordinate invariance produces an equivalent Lagrangian density to the *masless Fierz-Pauli*

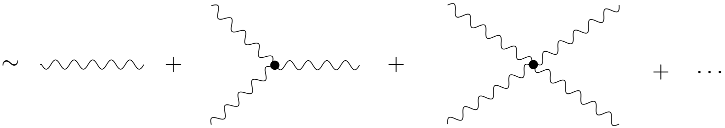
$$\mathcal{L}_{FP} = -\frac{1}{2} h_{\mu\nu,\rho} h^{\mu\nu,\rho} + \frac{1}{4} h_{,\sigma} h^{,\sigma} + \frac{\lambda}{2} h_{\mu\nu} T^{\mu\nu}, \tag{3}$$

which reproduses linearized gravity (which must further be PN expanded, with the leading order term being Newtonian gravity).

Orbital mechanics at 1PN - short scale - Expanding the action

Terms with higher powers in $h_{\mu\nu}$ will produce couplings of the graviton field with itself in the EoM:

$$P^{\mu\nu:\alpha\beta}\Box h_{\mu\nu} = -\frac{\lambda}{2}\left(T^{\alpha\beta} + t^{\alpha\beta}_{(2)} + \frac{\lambda}{2}t^{\alpha\beta}_{(3)} + \dots\right) \quad (4)$$



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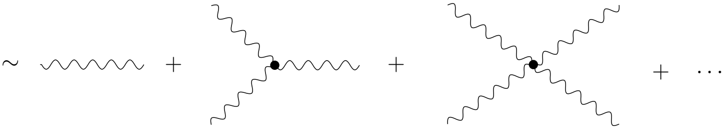

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Orbital mechanics at 1PN - short scale - Expanding the interaction term

In full GR the motion of point particles are determined by geodesics, derived from extrimizing the path through space-time

$$S_{pp} = -mc \int \sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}\,\mathrm{d}\tau = -mc^2 \int \sqrt{1 - \lambda h_{\mu\nu}\frac{\dot{x}^{\mu}}{c}\frac{\dot{x}^{\nu}}{c}} \cdot \gamma^{-1}\,\mathrm{d}t\,. \quad (5)$$

Taylor expanding the square root produces a power series in the coupling constant as well, and can be described diagramatically as

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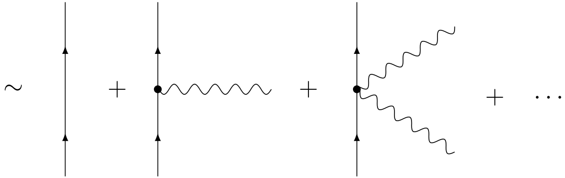

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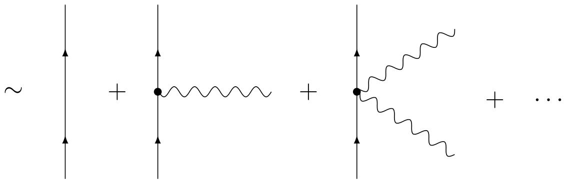

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Orbital mechanics at 1PN - short scale - Velocity dependent coupling

In the EoM, and the Greens function, there is a tensor structure

$$P_{\mu\nu:\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right) \quad (6)$$

$$D_{\mu\nu:\alpha\beta}(x^\mu - y^\mu) = P_{\mu\nu:\alpha\beta}\frac{\delta\left(x^0 - y^0\right)}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} \quad (7)$$

These are written graphically in Feynman diagrams as

These contributions will be given a factor of $1 + C_{\text{ten}}$.

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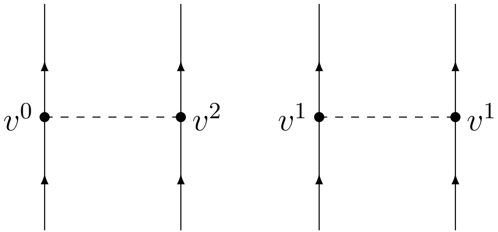

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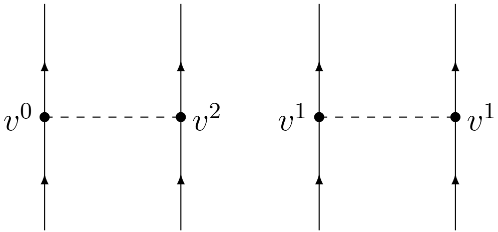

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Orbital mechanics at 1PN - short scale - The final Lagrangian

Obtained by integrating out the volum of the point particle action, substituting $h_{\mu\nu}$ for EoM obtained by the graviton action.

$$L_{0\mathrm{PN}} = \frac{\mu \boldsymbol{v}^2}{2} + \frac{GM\mu}{r}, \tag{8}$$

$$L_{1\mathrm{PN}} = \begin{cases} \frac{\mu \boldsymbol{v}^2}{2} \frac{1+C_\tau}{4} (1-3\eta) \frac{\boldsymbol{v}^2}{c^2} - \frac{GM\mu}{r} \frac{GM}{rc^2} \left(\frac{1}{2} + C_{\mathrm{prop}} - \frac{C_{\mathrm{int}}}{2} \right) \\ + \frac{GM\mu}{r} \frac{\boldsymbol{v}^2}{c^2} \left[\frac{3}{2} + C_{\mathrm{ten}} + \frac{C_E}{2} + \eta \left(\frac{1+(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{r}})^2}{2} + 2C_{\mathrm{ten}} - C_E - \frac{C_s}{2} (1 - (\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{r}})^2) \right) \right] . \end{cases} \tag{9}$$

Solving the equation of motion for ω, and imposing circular motion yields

$$\omega^2 = \frac{GM}{r^3}\left[1 - \frac{GM}{rc^2}\left(3 + 2C_{\text{prop}} - C_{\text{int}} + C_{\text{ten}} + \frac{C_E + C_\tau}{2}\right.\right.$$
$$\left.\left. + \eta\left\{-1 + 2C_{\text{ten}} - C_E - \frac{C_s + 3C_\tau}{2}\right\}\right)\right]. \quad (10)$$

Next up: preforming the Legendre transform of the Lagrangian, we obtain the Hamiltonian!

Orbital mechanics at 1PN - short scale - The orbital equation of motion

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Orbital mechanics at 1PN - short scale - Finaly, the energy

$$H = \frac{\mu \boldsymbol{v}^2}{2}\left[1 + \frac{\boldsymbol{v}^2}{c^2}\frac{3(1-3\eta)}{4}(1+C_\tau) + \frac{GM}{rc^2}\left(3 + 2C_{\rm ten} + C_E + \eta\left\{2 + 4C_{\rm ten} - 2C_E\right\}\right)\right]$$
$$- \frac{\mu r^2\omega^2}{2}\frac{GM\eta}{rc^2}(1+C_s) - \frac{GM\mu}{r}\left[1 - \frac{GM}{rc^2}\left(\frac{1}{2} + C_{\rm prop} - \frac{C_{\rm int}}{2}\right)\right]. \quad (11)$$

Solving v^2 and $\dfrac{GM}{r}$ for ω using the EoM yields the final result:

$$E_{\rm orb} = -\frac{\mu(GM\omega)^{2/3}}{2}\left[1 - \left(\frac{GM\omega}{c^3}\right)^{2/3}\left\{\frac{3}{4} + \frac{1}{12}C_\tau + \frac{1}{3}C_E + \frac{2}{3}C_{\rm ten} - \frac{2}{3}C_{\rm prop}\right.\right. \quad (12)$$
$$\left.\left. + \frac{1}{3}C_{\rm int} + \eta\left(\frac{1}{12} - \frac{1}{4}C_\tau - \frac{1}{3}C_s - \frac{2}{3}C_E + \frac{4}{3}C_{\rm ten}\right)\right\}\right].$$

GW production at 1PN - long scale - The multipole structure GW sources

To leading order

$$h_{ij}(t, \boldsymbol{R}) = \frac{\lambda}{16\pi R}\ddot{Q}_{ij}(t_{\text{ret}}) \tag{13}$$

$$Q^{ij}_{\text{quad}}(t) = \int T^{00}(t, \boldsymbol{x}) x^i x^j \, \mathrm{d}^3 x \tag{14}$$

And at higher orders there are source contributions from higher order multipoles (from energy), and current multipoles (from momentum). These has been given there own correction constants: C_{quad}, C_{oct}, and $C_{\text{curr.quad}}$.

GW production at 1PN - long scale - The energy flux up to 1PN

$$\mathcal{F}_{1\text{PN}} = \frac{32\eta^2}{5G}\left(\frac{GM\omega}{c^{3/2}}\right)^{\frac{10}{3}}\Bigg[(1+C_{\text{quad}}) + \left(\frac{GM\omega}{c^3}\right)^{\frac{2}{3}}\frac{1}{1008}\Bigg((1+C_{\text{quad}})\Big[-3741$$
$$-672\Big\{4C_{\text{prop}} - 2C_{\text{int}} + 2C_{\text{ten}} + C_E + C_\tau\Big\}\Big] + 1367\cdot C_{\text{oct}}$$
$$+28\cdot C_{\text{curr.quad}} - \eta\Bigg\{(1+C_{\text{quad}})\Big(-2940 + 672\Big[4C_{\text{ten}} - 2C_E - C_s - 3C_\tau\Big]\Big)$$
$$+5468\cdot C_{\text{oct}} + 112 C_{\text{curr.quad}}\Bigg\}\Bigg)\Bigg]. \tag{15}$$

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GW waveform at 1PN - Putting it all together

Using conservation of energy between the scales, the phase of the binary is given by

$$\mathrm{d}\Phi = -\frac{\omega}{\mathcal{F}}\frac{\mathrm{d}E}{\mathrm{d}\omega}\mathrm{d}\omega = \frac{5}{64\eta C_{\text{quad}}}x^{-\frac{7}{2}}\Bigg[1 + x\Bigg(\frac{743}{336} + 4C_{\text{prop}} - 2C_{\text{int}} + \frac{1}{2}C_\tau - \frac{1367}{1008}\frac{C_{\text{oct}}}{C_{\text{quad}}} - \frac{1}{36}\frac{C_{\text{curr.quad}}}{C_{\text{quad}}} + \eta\Bigg\{\frac{11}{4} - \frac{3}{2}C_\tau + \frac{1367}{252}\frac{C_{\text{oct}}}{C_{\text{quad}}} + \frac{1}{9}\frac{C_{\text{curr.quad}}}{C_{\text{quad}}}\Bigg\}\Bigg)\Bigg]\mathrm{d}x \tag{16}$$

Here, $x \equiv (GM\omega/c^3)^{2/3} \simeq v^2/c^2 \simeq GM/rc^2$.

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GW waveform at 1PN - The phase as a function of frequency

$$\Phi(x) = \Phi_0 - \frac{1}{32\eta(1+C_{\text{quad}})} x^{-\frac{5}{2}} \Bigg[1 + x \Bigg(\frac{3715}{1008} + \frac{20}{3} C_{\text{prop}} - \frac{10}{3} C_{\text{int}} + \frac{5}{6} C_\tau - \frac{6835}{3024} \frac{C_{\text{oct}}}{(1+C_{\text{quad}})} - \frac{5}{108} \frac{C_{\text{curr.quad}}}{(1+C_{\text{quad}})} + \eta \left\{ \frac{55}{12} - \frac{5}{2} C_\tau + \frac{6835}{756} \frac{C_{\text{oct}}}{(1+C_{\text{quad}})} + \frac{5}{27} \frac{C_{\text{curr.quad}}}{(1+C_{\text{quad}})} \right\} \Bigg) \Bigg]. \tag{17}$$

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GW waveform at 1PN - Solving for time

$$(t-t_c) \equiv \tau = \frac{5GM}{265\eta c^3(1+C_{\rm quad})} x^{-4}\left[1 + x\frac{3}{4}\left(\frac{23}{21} + 4C_{\rm prop} - 2C_{\rm int} + \frac{1}{2}C_\tau \right.\right. \tag{18}$$

$$\left.\left. - \frac{1367}{1008}\frac{C_{\rm oct}}{(1+C_{\rm quad})} - \frac{1}{36}\frac{C_{\rm curr.quad}}{(1+C_{\rm quad})} + \eta\left\{-\frac{9}{7} - \frac{3}{2}C_\tau + \frac{1367}{252}\frac{C_{\rm oct}}{(1+C_{\rm quad})} + \frac{1}{9}\frac{C_{\rm curr.quad}}{(1+C_{\rm quad})}\right\}\right)\right]$$

Defining $\Theta \equiv \frac{\eta c^3(1+C_{\rm quad})}{5GM}\tau$ and inverting to solv for $x(\Theta)$ gives

$$x(\Theta) = \frac{\Theta^{-\frac{1}{4}}}{4}\left[1 + \Theta^{-\frac{1}{4}}\left(\frac{743}{4032} + \frac{1}{3}C_{\rm prop} - \frac{1}{6}C_{\rm int} + \frac{1}{24}C_\tau - \frac{1367}{12096}\frac{C_{\rm oct}}{(1+C_{\rm quad})}\right.\right.$$

$$\left.\left. - \frac{1}{432}\frac{C_{\rm curr.quad}}{(1+C_{\rm quad})} + \eta\left\{\frac{11}{48} - \frac{1}{8}C_\tau + \frac{1367}{3024}\frac{C_{\rm oct}}{(1+C_{\rm quad})} + \frac{1}{108}\frac{C_{\rm curr.quad}}{(1+C_{\rm quad})}\right\}\right)\right] \tag{19}$$

GW waveform at 1PN - Phase as a function of time

$$\Theta \equiv \frac{\eta c^3(1+C_{\text{quad}})}{5GM}\tau$$

$$\begin{aligned}\Phi(\Theta) = &\Phi(x(\Theta)) = \Phi_0 - \frac{\Theta^{\frac{5}{8}}}{\eta(1+C_{\text{quad}})}\Bigg[1+\Theta^{-\frac{1}{4}}\Bigg(\frac{3715}{8064}+\frac{5}{6}C_{\text{prop}}-\frac{5}{12}C_{\text{int}} \\ &+\frac{5}{48}C_\tau - \frac{6835}{24192}\frac{C_{\text{oct}}}{(1+C_{\text{quad}})} - \frac{5}{864}\frac{C_{\text{curr.quad}}}{(1+C_{\text{quad}})} \\ &+\eta\Bigg\{\frac{55}{96}-\frac{5}{16}C_\tau+\frac{6835}{6048}\frac{C_{\text{oct}}}{(1+C_{\text{quad}})}+\frac{5}{216}\frac{C_{\text{curr.quad}}}{(1+C_{\text{quad}})}\Bigg\}\Bigg)\Bigg]. \qquad (20)\end{aligned}$$

GW waveform at 1PN - The spatial distance over time

$$\Theta \equiv \frac{\eta c^3(1+C_{\text{quad}})}{5GM}\tau$$

$$\gamma^{-1}(\Theta) = \frac{c^2}{GM}r(\Theta) = 4\Theta^{\frac{1}{4}}\left[1 - \Theta^{-\frac{1}{4}}\left(\frac{743}{48384} + \frac{7}{18}C_{\text{prop}} - \frac{7}{36}C_{\text{int}} + \frac{1}{18}C_{\tau} + \frac{1}{36}C_{\text{ten}} + \frac{1}{72}C_E - \frac{1367}{12096}\frac{C_{\text{oct}}}{(1+C_{\text{quad}})} - \frac{1}{432}\frac{C_{\text{curr.quad}}}{(1+C_{\text{quad}})} + \eta\left\{\frac{29}{144} - \frac{1}{6}C_{\tau} + \frac{1}{18}C_{\text{ten}} - \frac{1}{36}C_E - \frac{1}{72}C_s + \frac{1367}{3024}\frac{C_{\text{oct}}}{(1+C_{\text{quad}})} + \frac{1}{108}\frac{C_{\text{curr.quad}}}{(1+C_{\text{quad}})}\right\}\right)\right] \quad (21)$$

Thank you for your attention.